**(Title Page Submitted by a Candidate from a GSAS Department)**

# On $\Lambda$-adic Saito-Kurokawa Lifting and its Application

**Zhi Li**

**Submitted in partial fulfillment of the**

**Requirements for the degree**

**of Doctor of Philosophy**

**in the Graduate School of Arts and Sciences**

**COLUMBIA UNIVERSITY**

**2009**

**Abstract**

This thesis studies the $p$-adic nature of the Saito-Kurokawa lifting from a classic modular form to a Siegel modular form of degree 2, and its application on the algebraicity of central values. Applying Stevens' result on $\Lambda$-adic Shintani lifting, a $\Lambda$-adic Saito-Kurokawa lifting is constructed analogous to the construction of the classic $\Lambda$-adic Eisenstein Series. It's applied to construct a $p$-adic $L$-function on $\mathrm{Sp}_2 \times \mathrm{GL}_2$. A conjecture on the specialization of this $p$-adic $L$-function is stated.

# Contents

**Acknowledgements** **iii**

**1 Introducation** **1**

**2 Modular Forms** **9**

2.1 Integral Weight Modular Forms . . . . . . . . . . . . . . . . . . . . 9

2.2 Half-Integral Weight Modular Forms . . . . . . . . . . . . . . . . 15

2.3 Jacobi Forms . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17

2.4 Siegel Modular Forms . . . . . . . . . . . . . . . . . . . . . . . . 19

**3 Saito-Kurokawa Lifting** **25**

3.1 Shimura Lifting . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

3.2 Shintani Lifting . . . . . . . . . . . . . . . . . . . . . . . . . . . 26

3.3 Half-Integral Weight Forms to Jacobi Forms . . . . . . . . . . . . 28

3.4 Jacobi Forms to Siegel Forms . . . . . . . . . . . . . . . . . . . . 30

3.5 A pullback formula of Saito-Kurokawa lifting . . . . . . . . . . . . 31

**4 A $\Lambda$-adic Saito-Kurokawa Lifting** **33**

4.1 $p$-adic Modular Forms . . . . . . . . . . . . . . . . . . . . . . . 33

4.2 $p$-adic Families of Cusp Forms . . . . . . . . . . . . . . . . . . . 37

4.3 Modular Symbols . . . . . . . . . . . . . . . . . . . . . . . . . . 41

4.4 A $\Lambda$-adic Shintani lifting . . . . . . . . . . . . . . . . . . . . . . . . 43

4.5 A $\Lambda$-adic Saito-Kurokawa Lifting . . . . . . . . . . . . . . . . . . . . 46

**5 One Variable $p$-adic $L$-Functions** **52**

5.1 $p$-adic measures attached to modular symbols . . . . . . . . . . . . 52

5.2 Linear form . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 54

5.3 One variable $p$-adic $L$-functions for $\mathrm{Sp}_2 \times \mathrm{GL}_2$ . . . . . . . . . . . . 55

## Acknowledgements

My adisor Eric Urban deserves many thanks for a lot of things. Not only did he help me pick an interesting problem, without his very helpful guidance and patience throughout the whole process, this thesis is far from what it is today. I would also take this chance to thank Shou-Wu Zhang and Dorian Goldfeld for their influence in shaping my knowledge of mathematics, espcially number theory. Their pleasant lectures make difficult mathematics so much elegant and accessible to all.

Here I thank the other members of my Committee, as well as other professors in mathematics department for their instruction and support. In particular, I thank Terrance Cope for being the greatest department administrator I've never imagined, he made everything so easy for me in the past few years.

I would like to express my gratitude to my parents and brother for their support from the very beginning.

For Ying-Xue and Chang-Zhao

# 1 Introducation

Suppose $G$ is a reductive group over a number field $F$. We may define $L$-functions attached to automorphic forms on $G$. In algebraic number theory, the arithmetic of critical values of $L$-functions plays a key role in Iwasawa-Greenberg Main Conjectures, which connect it to the size of Selmer groups. And the special values of automorphic $L$-functions are closely related to Eisenstein series via Rankin's method [Ran]. Garrett [Ga1], [Ga2], Böcherer [Bo1], [Bo2] and Heim [Hei] applied pullback formulae of Siegel Eisenstein series to prove the algebraicity of critical values of certain automorphic $L$-functions. More generally, there are conjectures on the relationship between pullbacks of Siegel cusp forms and central critical values of $L$-functions, such as the Gross-Prasad conjecture in [GP1, GP2]. In this work, we are concerned with a pullback formula of Saito-Kurokawa lifts by Ichino [Ich], which shows the algebraicity of critical values of certain $L$-functions for $\mathrm{Sp}_2 \times \mathrm{GL}_2$. In this work, we will construct a $\Lambda$-adic Saito-Kurokawa liftings, and make a conjecture on a one-variable $p$-adic $L$-function interpolating those algebraic critical values.

Precisely, let us fix a prime number $p \geq 5$. Let $k$ be an odd positive integer, and let $N$ be an odd positive integer prime to $p$. Let $f \in S_{2k}(\Gamma_0(N))$ be a normalized Hecke eigenform of weight $2k$ and level $N$. Put $h \in S^+_{k+1/2}(\Gamma_0(4N))$ to be a Hecke eigenform associated to $f$ by the Shimura correspondence. Let

$F \in S_{k+1}(\Gamma_0^2(N))$ be the Saito-Kurokawa lift of $h$. For each normalized Hecke eigenform $g \in S_{k+1}(\Gamma_0(N))$, we know that the numbers

$$\frac{A(2k, \mathrm{Sym}^2(g) \otimes f)}{\langle g, g\rangle^2 \Omega_f^+} \tag{1}$$

are algebraic by modified Ichino's pullback formula on Saito-Kurokawa lifting. Here $\Omega_f^+$ is the period of $f$ as in [Sh3], and $A(s, \mathrm{Sym}^2(g) \otimes f)$ is the completed $L$-function given by

$$A(s, \mathrm{Sym}^2(g) \otimes f) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k)\Gamma_{\mathbb{C}}(s-2k+1)L(s, \mathrm{Sym}^2(g) \otimes f),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Let $f$ and $g$ vary through families of cusp forms, we collect families of algebraic critical values. We shall construct a one-variable $p$-adic $L$-function to interpolate these algebraic numbers in (1).

Let us fix a fundamental discriminant $-D < 0$, with $-D \equiv 1 \bmod 8$, such that $A(k, f, \chi_{-D}) = D^k\Gamma_{\mathbb{C}}(k)L(k, f, \chi_{-D}) \neq 0$, where $\chi_{-D}$ is the Dirichlet character associated to $\mathbb{Q}(\sqrt{-D})/\mathbb{Q}$. Such a discriminant exists by [BFH, Wal]. Put $\mathcal{K} = \mathbb{Q}(\sqrt{-D})$, the quadratic imaginary field. Let $\pi$ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to $g$, and let $\pi_{\mathcal{K}}$ be the base change of $\pi$ to $\mathcal{K}$. Let $\sigma$ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to $f$. Let $\nu(N)$ denote the number of prime divisors of $N$. Ichino proved three seesaw identities in [Ich]. By the local integral representation of $L(s, \pi_{\mathcal{K}} \otimes \sigma)$ and the generalized Kohnen-Zagier formula [KZ, Ko3]:

$$A(k, f, \chi_{-D}) = 2^{1-k-\nu(N)}D^{1/2}|c_h(D)|^2\frac{\langle f, f\rangle}{\langle h, h\rangle}, \tag{2}$$

he got a pullback formula on Saito-Kurokawa lifting for full level cusp forms. Here'a a modified version of the pullback formula for higher level forms [Li]:

$$A(2k, \mathrm{Sym}^2(g) \otimes f) = 2^{k+1-\nu(N)} \xi_N \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{|\langle F|_{\mathfrak{H} \times \mathfrak{H}}, g \times g \rangle|^2}{\langle g, g \rangle^2}, \tag{3}$$

Where, $\xi_N$ is an alebraic number given by:

$$\xi_N = N^2 \prod_{p|N} \epsilon_p (1+p)^5 (1-p)^2 (\epsilon_p - p)^{-2},$$

and $\epsilon_p = -a_f(p)$, $\nu(N)$ is the number of prime divisors of $N$. Hence we may decompose the algebraic numbers in formula (1) into:

$$\begin{aligned} &\frac{A(2k, \mathrm{Sym}^2(g) \otimes f)}{\langle g, g \rangle^2 \Omega_f^+} \\ &= \frac{2^{2k} \xi_N}{\sqrt{D}} \cdot \left\{ |c_h(D)|^{-2} \right\}_{\mathrm{I}} \cdot \left\{ \frac{A(k, \chi_{-D}, f)}{\Omega_f^+} \right\}_{\mathrm{II}} \cdot \left\{ \frac{|\langle F|_{\mathfrak{H} \times \mathfrak{H}}, g \times g \rangle|^2}{\langle g, g \rangle^4} \right\}_{\mathrm{III}}. \end{aligned} \tag{4}$$

Here, $c_h(D)$ is the $D$-th Fourier coefficient of $h$. Our goal in this work is to study the $p$-adic nature of these algebraic numbers.

We now outline our interpolation argument. Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}_K$ denote its $p$-adic integer ring. Put $\Lambda_K = \mathcal{O}_K[[X]]$ be the Iwasawa algebra. Let $\mathcal{L}_K$ be the fractional field of $\Lambda_K$. Suppose $\mathcal{K}$ is a finite extension of $\mathcal{L}_K$ and let $\mathcal{I}$ denote the integral closure of $\Lambda_K$ in $\mathcal{K}$. Suppose $h^0(N, \mathcal{O}_K)$ is the universal ordinary $p$-adic Hecke algebra of level $N$. Suppose $\lambda : h^0(N, \mathcal{O}_K) \to \mathcal{I}$ is a homomorphism of $\Lambda_K$-algebras. Let $\mathbf{f}$ be the $\Lambda$-adic cusp form corresponding to $\lambda$. It's a Hida family of $p$-adic cusp forms. Greenberg and Stevens attatch a

$\Lambda$-adic modular symbol $\Phi_{\mathbf{f}}$ to $\mathbf{f}$. And they define a two-variable $p$-adic $L$-function $L_p(\Phi_{\mathbf{f}})$ on $\mathrm{Spec}(\mathcal{I}) \times \mathrm{Spec}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$, which interpolates the critical values of the $L$-function associated to $\mathbf{f}$ and twisted by a character. Our second factor in (4) is the central value of the $L$-function.

Next, we consider the first factor in (4), involving the $D$-th Fourier coefficient of $h$. By Stevens [Ste], there exists a $\Lambda$-adic half-integral cusp form $\mathbf{h} = \Theta(\Phi_{\mathbf{f}})$, which is the $\Lambda$-adic Shintani lifting of $\mathbf{f}$. Let $\alpha_D$ be its $D$-th Fourier coefficient. Up to a period scale, we may use the $\alpha_D$ to interpolate $c_h(D)$.

Before we interpolate the third factor in (4), we construct a $\Lambda$-adic Saito-Kurokawa lifting $\mathbf{SK}(\mathbf{f})$, which is a $\Lambda$-adic Siegel cusp form interpolating the Saito-Kurokawa lifting $F = \mathrm{SK}(f)$. Skinner and Urban developed a more general $\Lambda$-adic Saito-Kurokawa lifting in [SU]. In this work, our method is different from theirs. This construction is a generalization of the classical construction of $\Lambda$-adic Eisenstein series.

Put $A_2$ be the semigroup of symmetric, semi-definite positive, half-integral matrices of size 2. For every $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in A_2$, we denote $A(T) = A(n, r, m)$ and we say $d|T$ if $d|(n, r, m)$ for any integer $d$. Suppose the Saito-Kurokawa lifting $F$ has a formal $q$-expansion:

$$F(Z) = \sum_{T \in A_2} A(T) q^{\mathrm{Tr}(TZ)},$$

for any $Z \in \mathfrak{H}^2$. Then the coefficients satisfying the following relation:

$$A(n,r,m) = \sum_{0<d|(n,r,m)} d^k c(\frac{4nm-r^2}{d^2}),$$

where $c(l)$ are the coefficients of $q$-expansion of $h$. The expression of $A(n,r,m)$ is similar to that of the coefficient

$$\sigma_k(n) = \sum_{0<d|n} d^k$$

of Eisenstein series. Based on this observation and Stevens' result, we construct a $\Lambda$-adic Saito-Kurokawa lifting.

Put $\psi = \omega^a$ be an even Dirichlet character, where $\omega : \mathbb{Z}_p^\times \to \mu_{p-1}$ is the Teichmuller character. For each power series $\Theta = \sum \alpha(n)q^n \in \mathcal{I}[[q]]$, we define a formal power series:

$$\mathbf{SK}(\Theta,\psi) := \sum_{T\in A_2}(\sum_{\substack{0<d|T\\(d,p)=1}} \psi(d)A_d(X)\alpha(\frac{\det(2T)}{d^2}))q^T,$$

where $A_d(X) = d^{-1}(1+X)^{s(\langle d\rangle)}$ is an element of $\Lambda = \mathbb{Z}_p[[X]]$.

Denote $\mathbf{SK}(\mathbf{f},\psi) = \mathbf{SK}(\Theta(\Phi_{\mathbf{f}}),\psi)$. Put $\delta_p = diag(p,p,1,1) \in \mathrm{M}_4(\mathbb{Z})$. Here's the main theorem:

**Theorem 1.1** *For each arithmetic point $P \in Spec(\mathcal{I})$ with signature $(2k, id)$, satisfying $a \equiv k+1 \mod p-1$, we have*

$$\boldsymbol{SK}(\mathbf{f},\psi)(P) = \frac{\Omega_P}{\Omega_{\mathbf{f}_P^-}}(SK(\mathbf{f}_P) - SK(\mathbf{f}_P)|\delta_p) \in S_{k+1}(\Gamma_0^2(Np);\mathcal{O}_K). \quad (5)$$

*Here, $\Omega_P$ is a p-adic period given by Greenberg and Stevens, and $\Omega_{\mathbf{f}_P^-}$ is a complex period defined by Shimura.*

Denote $\varpi(\mathbf{SK}(\mathbf{f},\omega^a))$ the pullback of $\mathbf{SK}(\mathbf{f},\omega^a)$ on $\mathfrak{H}\times\mathfrak{H}$. Let $\mathcal{M}$ be a finite extension of $\mathcal{L}_K$ defined over $K$, and $\mathcal{J}$ be the integral closure of $\Lambda_K$ in $\mathcal{M}$. Suppose $\lambda' : h^o(N;\mathcal{O}_K)\to\mathcal{J}$ is a homomorphism of $\Lambda_K$-algebras. Let $\mathbf{g}$ be the $\Lambda$-adic cusp form corresponding to $\lambda'$. Then there's an isomorphism $h^o(N;\mathcal{O}_K)\otimes_{\Lambda_K}\mathcal{M}\cong\mathcal{M}\oplus\mathcal{B}$. Let $\mathbf{1}_{\mathbf{g}}$ be the idempotent corresponding to the first factor of the decomposition. Following Hida, we define a congruence module $\mathcal{C}$, and fix $H_{\mathbf{g}}$ to be a generator of the annihilator of $\mathcal{C}$.

Let $\mathbf{g}_i\in S^o(N;\Lambda_K)$ $(i=0,1,2,\cdots,m)$ be a basis, and suppose $\mathbf{g}=\mathbf{g}_0$. Linear forms $l^i : S^o(N;\Lambda_K)\to\Lambda_K$ are defined by $l^{(i)}(\mathbf{g}_j)=\delta_{i,j}$. We now define a linear form

$$\langle\varpi(\mathbf{SK}(\mathbf{f}),\omega^a),\mathbf{g}\times\mathbf{g}\rangle := l^{(0)}\times l^{(0)}(\varpi(\mathbf{SK}(\mathbf{f}),\omega^a)|H_{\mathbf{g}}\cdot 1_{\mathbf{g}_Q}\times H_{\mathbf{g}}\cdot 1_{\mathbf{g}_Q}). \quad (6)$$

Suppose the arithmetic point $P'\in\mathrm{Spec}(\mathcal{I}_\delta)$ lies above $P\in\mathrm{Spec}(\mathcal{I})$. We define a $p$-adic function $\alpha_D(\mathbf{f},P)=\alpha_D(P')$, then we have

$$\alpha_D(\mathbf{f},P)=\frac{\Omega_P}{\Omega_{\mathbf{f}_P}^-}\cdot c_h(D).$$

Here, $\Omega_P$ and $\Omega_{\mathbf{f}_P}^-$ are periods defined in Theorem 1. Put $\tau_N=\begin{pmatrix}0&-1\\N&0\end{pmatrix}$. Now we define a one-variable $p$-adic $L$-function $L_p(\mathrm{Sym}^2(\mathbf{g})\otimes\mathbf{f})$ for $\mathrm{Sp}_2\times\mathrm{GL}_2$ on

$\mathrm{Spec}(\mathcal{J}) \times \mathrm{Spec}(\mathcal{I})$ by

$$L_p(\mathrm{Sym}^2(\mathbf{g}) \otimes \mathbf{f})(Q,P) := \alpha_D(\mathbf{f},P)^{-1} \cdot \alpha_D(\mathbf{f}|\tau_N,P)^{-1} \cdot L_p(\Phi_{\mathbf{f}})(P)$$
$$\times \langle \omega(\mathbf{SK}(\mathbf{f},\omega^a)), \mathbf{g} \times \mathbf{g}\rangle(P,Q) \cdot \langle \mathbf{SK}(\mathbf{f}|\tau_N,\omega^a), \mathbf{g}|\tau_N \times \mathbf{g}|\tau_N\rangle(P,Q). \qquad (7)$$

Suppose $P$ and $Q$ have matching weights such that $k(P)+2 = 2k(Q)$, $\epsilon_P = \epsilon_Q = \mathrm{id}$. Assume $a \equiv k(Q) \bmod p-1$.

**Conjecture 1** *The p-adic L-function $L_p(Sym^2(\mathbf{g}) \otimes \mathbf{f})$ interpolates the central values by :*

$$L_p(Sym^2(\mathbf{g}) \otimes \mathbf{f})(Q,P) = \boldsymbol{t} \cdot H(Q)^4 \Omega_P \frac{A(2k, Sym^2(\mathbf{g}_Q) \otimes \mathbf{f}_P)}{\langle \mathbf{g}_Q, \mathbf{g}_Q\rangle^2 \Omega^+_{\mathbf{f}_P}}, \qquad (8)$$

*where*

$$\boldsymbol{t} = \frac{\tau(\chi_{-D})(-1)^{\frac{k-1}{2}}}{2^{2k}\sqrt{D}\xi_N} \cdot W(\mathbf{g}_Q)^2 a(p,\mathbf{f}_P)^{-r}(1-\chi_{-D}(p)a(p,\mathbf{f}_P)^{-1}p^{k-1})(1-a(p,\mathbf{g}_Q)^2p^{-2})^2$$

*and $W(\mathbf{g}_Q)$ is the root number of $\mathbf{g}_Q$.*

Here's the outline of the content in this work. In section 2, we give a review of various types of modular forms needed for Saito-Kurokawa lifting. In section 3 we recall the correspondences between spaces of modular forms, including Shimura correspondence and Shintani correspondence between integral weight and half-integral weight modular forms, the maps from half-integral weight forms to Jacobi forms, and the maps from Jacobi forms to Siegel modular forms of degree 2. In section 4, after reviewing Stevens work on $\Lambda$-adic Shintani lifting, we construct a $\Lambda$-adic Saito-Kurokawa lifting. In the last section 5, applying the $p$-adic $L$-function of a $\Lambda$-adic form by Greenberg and Stevens, we get a conjecture on the specialization of the one variable $p$-adic $L$-function for $\mathrm{Sp}_2 \times \mathrm{GL}_2$.

# 2 Modular Forms

Since we work on various types of modular forms, in this section, we present basic definitions and facts on modular forms and Hecke algebras if necessary to understand Saito-Kurokawa lifting. For more details, please refer to standard textbooks and papers, e.g. [Sh1, Sh2, EZ].

## 2.1 Integral Weight Modular Forms

**2.1.1.** Let $\mathfrak{H} = \mathfrak{H}^1 = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$ denote the complex upper half plane. And let $\mathrm{GL}_2^+(\mathbb{R})$ be the group of $2 \times 2$ real matrices with positive determinant. We let $\mathrm{GL}_2^+(\mathbb{R})$ act on $\mathfrak{H}$ by linear fractional transformation.

Let $k$ be a positive integer. For any function $f$ on $\mathfrak{H}$, and for any $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, we define $f|_k\gamma$ on $\mathfrak{H}^1$ by

$$f|_k\gamma(z) = \det(\gamma)^{k/2} f(\gamma(z)) j(\gamma, z)^{-k}, \tag{9}$$

where $j(\gamma, z) = c_\gamma \cdot z + d_\gamma$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on functions $f : \mathfrak{H} \to \mathbb{C}$.

For a positive integer $N$, we define subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); a \equiv d \equiv 1 \mod N, b \equiv c \equiv 0 \mod N\},$$

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); c \equiv 0 \mod N\},$$

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); a \equiv d \equiv 1 \mod N, c \equiv 0 \mod N\}.$$

Let $\Phi$ be a congruent subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$. We consider those holomorphic functions on $\mathfrak{H}$ satifying $f|_k\gamma = f$ for all $\gamma \in \Gamma(N)$. Such a function has Fourier expansion of the form:

$$\sum_{n\in\mathbb{Z}} a(\frac{n}{N})e(\frac{nz}{N})$$

where $e(z) = e^{2\pi iz}$.

**Definion**: Let $k$ a positive integer with $k \geq 2$, and $\Phi$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A holomorphic function $f$ on $\mathfrak{H}$ is said to be a modular form of weight $k$ with respect to $\Phi$, if it satisfies:

1. $f|_k\gamma = f$ for all $\gamma \in \Phi$,

2. $a(\frac{n}{N}, f|_k\alpha) = 0$ if $n < 0$ for each $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

We denote by $M_k(\Phi)$ the vector space of modular forms of weight $k$ with respect to $\Phi$.

Let $\mathbf{U}$ be the standard unipotent subgroup of $\mathrm{SL}_2(\mathbb{Z})$. When the congruence subgroup $\Phi$ equals $\Gamma_0(N)$ (or $\Gamma_1(N)$), we call $f$ a modular form of level $N$. Since $\Phi$ contains $\mathbf{U}$, $f \in M_k(\Phi)$ has Fourier expansion of the form:

$$f(z) = \sum_{n=0}^{\infty} a(n,f) q^n$$

where $q = e(z)$. It's called the $q$-expansion of $f$, by which we may view $M_k(\Phi)$ as a subspace of $\mathbb{C}[[q]]$.

**Definition**: $f \in M_k(\Phi)$ is called a cusp form if $a(0,f) = 0$. We denote by $S_k(\Phi)$ the vector space of cusp forms of weight $k$ and with respect to $\Phi$.

**2.1.2.** For any $\mathbb{Z}$-subalgebra $A$ of $\mathbb{C}$, we put

$$M_k(\Phi; A) = M_k(\Phi) \cap A[[q]],$$

$$S_k(\Phi; A) = S_k(\Phi) \cap A[[q]].$$

Then we have the following isomorphisms [Sh1]:

$$M_k(\Phi; A) = M_k(\Phi; \mathbb{Z}) \otimes A, \quad S_k(\Phi; A) = S_k(\Phi; \mathbb{Z}) \otimes A. \tag{10}$$

For any commutative algebra $A$, we take the above equations as the definitions of the spaces of integral modular forms and cusp forms with coefficients in $A$. When $\Phi = \Gamma_1(N)$, we simply write

$$M_k(N; A) = M_k(\Gamma_1(N); A), \quad S_k(N; A) = S_k(\Gamma_1(N); A).$$

For each character $\chi : \Phi \to A^\times$ of finite order, we put

$$M_k(\Phi, \chi; A) = \{f \in M_k(\ker(\chi); A); f|_k\gamma = \chi(\gamma)f, \forall \gamma \in \Phi\},$$

$$S_k(\Phi, \chi; A) = M_k(\Phi, \chi; A) \cap S_k(\ker(\chi); A).$$

**2.1.3.** Let $G$ be a multiplicative group, and $\Phi$ be a subgroup of $G$. We define the commensurator of $\Phi$ in $G$ to be the subgroup:

$$\tilde{\Phi} = \{\gamma \in G; \gamma\Phi\gamma^{-1} \text{ and } \Phi \text{ are commensurable.}\}$$

For any semi-group $\Delta$ such that $\Phi \subset \Delta \subset \tilde{\Phi}$, we denote by $R(\Phi, \Delta)$ the $\mathbb{Z}$-module of all formal finite sums of double cosets $[\Phi\gamma\Phi]$ with $\gamma \in \Delta$. We introduce a multiplication on $R(\Phi, \Delta)$ by disjoint coset decomposition [Sh1]. Then $R(\Phi, \Delta)$ becomes an associative ring, we call it the Hecke ring with respect to $\Phi$ and $\Delta$.

Fix a positive integer $N$ prime to $p$. For any two integers $s$ and $r$ such that $s \geq 0, r \geq 1$ and $s \leq r$, we put

$$\Phi_r^s = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); a \equiv d \equiv 1(\mod Np^s), c \equiv 0(\mod Np^r)\},$$

$$\Delta_r^s = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}); a \equiv 1(\mod Np^s), c \equiv 0(\mod Np^r), ad - bc > 0\}.$$

For each prime $\ell$, with $\ell \nmid Np$, we take an element $\sigma_\ell$ of $\mathrm{SL}_2(\mathbb{Z})$ satisfying

$$\sigma_\ell \equiv \begin{pmatrix} * & * \\ 0 & \ell \end{pmatrix} (\mod Np^r).$$

Let $\Phi = \Phi_r^s, \Delta = \Delta_r^s$, and we put

$$T(\ell) = [\Phi \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \Phi], \qquad T(\ell, \ell) = [\Phi(\ell \sigma_\ell)\Phi].$$

It's well-known that $R(\Phi, \Delta)$ is the free $\mathbb{Z}$-module generated by $T(\ell)$ for all primes $\ell$ and $T(\ell, \ell)$ for $\ell \nmid Np$.

Note that $\Phi_r^r = \Gamma_1(Np^r)$. The action of $R(\Phi_r^r, \Delta_r^r)$ on $M_k(Np^r)$ is defined by

$$f|[\Phi_r^r \alpha \Phi_r^r] = \det(\alpha)^{k/2} \sum_i f|_k \alpha_i, \tag{11}$$

where $\Phi_r^r \alpha \Phi_r^r = \sqcup \Phi_r^r \alpha_i$. It's independent of the choice of the representatives $\{\alpha_i\}$. The space of modular forms $M_k(Np^r)$ is stable under this action, so do the subspaces $S_k(Np^r)$, $M_k(\Phi, \chi; \mathbb{C})$ and $S_k(\Phi, \chi; \mathbb{C})$. For any $R(\Phi_r^r, \Delta_r^r)$-module $L$, we define the Hecke algebra $h(L)$ as the image of $R(\Phi_r^r, \Delta_r^r)$ in $End_{\mathbb{Z}}(L)$. We denote

$$\mathcal{H}_k(Np^r; \mathbb{Z}) = h(M_k(Np^r)), \quad h_k(Np^r; \mathbb{Z}) = h(S_k(Np^r)),$$

$$\mathcal{H}_k(\Phi_r^s, \chi; \mathbb{Z}) = h(M_k(\Phi_r^s, \chi)), \quad h_k(\Phi_r^s, \chi; \mathbb{Z}) = h(S_k(\Phi_r^s, \chi)).$$

When $k \geq 2$, the pairing $<,>$: $S_k(\Phi_r^s, \chi; \mathbb{Z}) \times h_k(\Phi_r^s, \chi; \mathbb{Z}) \to \mathbb{Z}$ defined by

$$< f, T > \mapsto a(1, f|T) \tag{12}$$

is a perfect pairing.

$M_k(Np^r; \mathbb{Z})$ is also stable under the action of Hecke operators by Hida, [Hi4]. Then the Hecke algebras $\mathcal{H}_k(Np^r; \mathbb{Z})$, $h_k(Np^r; \mathbb{Z})$ are free of finite rank over $\mathbb{Z}$.

For any commutative algebra $A$, we define

$$\mathcal{H}_k(Np^r;A) = \mathcal{H}_k(Np^r;\mathbb{Z}) \otimes A, \quad h_k(Np^r;A) = h_k(Np^r;\mathbb{Z}) \otimes A.$$

**2.1.4.** Put $\overline{\mathrm{SL}_2(\mathbb{Z})} = \mathrm{SL}_2(\mathbb{Z})/\pm I_2$. Let $\overline{\Phi}$ be the image of $\Phi$ in $\overline{\mathrm{SL}_2(\mathbb{Z})}$. For $f, g \in S_k(\Phi)$, we define the Petersson inner product of $f$ and $g$ by

$$\langle f, g\rangle = \frac{1}{[\overline{\mathrm{SL}_2(\mathbb{Z})} : \overline{\Phi}]} \int_{\Phi\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k-2}dxdy. \tag{13}$$

The Hecke operators are self-adjoint with respect to the Petersson product:

$$\langle T(n)f, g\rangle = \langle f, T(n)g\rangle,$$

where $\Phi = \Gamma_0(N)$ and $\gcd(n, N) = 1$.

Suppose $f \in S_k(\Gamma_0(N))$. The $L$-function associated to $f$ is defined by

$$L(s, f) = \sum_{n=1}^{\infty} a(n, f)n^{-s}.$$

It has analytic continuation to the entire complex plane, and it also satisfies a functional equation. Suppose $f$ is a newform, then $L(s, f)$ has an Euler product.

Let $\chi$ be a Dirichlet character, the twisted $L$-function is defined by

$$L(s, f, \chi) = \sum_{n=1}^{\infty} \chi(n)a(n, f)n^{-s}.$$

This $L$-function also has analytic continuation, functional equation and Euler product expansion.

## 2.2 Half-Integral Weight Modular Forms

**2.2.1.** In this section, we review the definitions and facts about half-integral weight modular forms by Shimura [Sh2] and refined by Kohnen [Ko1, Ko2, Ko3].

Let $\sqrt{z}$ be the branch of the square root function taking its argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$. For any positive integer $m$, put $z^{m/2} = (\sqrt{z})^m$.

For a positive integer $k$, and for any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$, we define a complex-valued holomorphic function $\phi_\gamma$ on $\mathfrak{H}$ by:

$$|\phi_\gamma(z)| = \det(\gamma)^{k/2+1/4}|c_\gamma z + d_\gamma|^{k-1/2}.$$

Denote by $\mathfrak{G}_{k-1/2}$ the set of pairs $(\gamma, \phi(z))$ for $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. We introduce a multiplication law on $\mathfrak{G}_{k-1/2}$ by

$$(\gamma, \phi(z)) \cdot (\alpha, \psi(z)) = (\gamma\alpha, \phi(\alpha z)\psi(z)).$$

Then $\mathfrak{G}_{k-1/2}$ becomes a group. Let $\pi_{k-1/2} : \mathfrak{G}_{k-1/2} \to \mathrm{GL}_2^+(\mathbb{Q})$ be the projection onto the first factor. The group $\mathfrak{G}_{k-1/2}$ acts on functions $g : \mathfrak{H} \to \mathbb{C}$ by

$$g|(\gamma, \phi_\gamma)(z) = \phi_\gamma(z)^{-1} g(\gamma z).$$

We directly copy the definition of the quadratic residue symbol $\left(\frac{a}{b}\right)$ by Shimura in [Sh2]. For any element $\gamma$ of $\Gamma_0(4)$, we define a function

$$j(\gamma, z) = \left(\frac{c_\gamma}{d_\gamma}\right)\left(\frac{-4}{d_\gamma}\right)^{-k-3/2}(c_\gamma z + d_\gamma)^{k-1/2}.$$

Now we define an inverse map of $\pi_{k-1/2}$. Let $\Phi \subset \Gamma_0(4)$ be a congruence subgroup. Put $\tilde{\Phi} = \{\tilde{\gamma} = (\gamma, j(\gamma, z)) : \gamma \in \Phi\}$. The map $\gamma \mapsto \tilde{\gamma} = (\gamma, j(\gamma, z))$ is a left inverse of the map $\pi_{k-1/2}$, so we can define an action of any congruence subgroup of $\Gamma_0(4)$ on the set of functions $g : \mathfrak{H} \to \mathbb{C}$ through the action of $\mathfrak{G}_{k-1/2}$. Suppose that $\Phi$ is a congruence group of level $4N$. The action of $\Phi$ on functions $g : \mathfrak{H} \to \mathbb{C}$ is given by

$$g|_{k-1/2}\gamma(z) = j(\gamma, z)^{-1} g(\gamma z).$$

**Definition**: A complex-valued function $g$ on $\mathfrak{H}$ is called a modular form of weight $k - 1/2$ for $\Phi$ if $g|_{k-1/2}\gamma = g$ for all $\gamma \in \Phi$ and that $g$ is holomorphic at all the cusps. If $g$ vanishes at all the cusps, we say $g$ is a cusp form.

We denote the space of weight $k - 1/2$ modular forms by $M_{k-1/2}(\Phi)$, and denote its subspace of cusp forms by $S_{k-1/2}(\Phi)$.

For any commutative algebra $A$, we put

$$M_{k-1/2}(\Phi; A) = M_{k-1/2}(\Phi) \cap A[[q]],$$

$$S_{k-1/2}(\Phi; A) = S_{k-1/2}(\Phi) \cap M_{k-1/2}(\Phi; A).$$

**2.2.2.** Since we will not make use of the Hecke operators on half-integral weight modular forms, here we skip its definition. One may consult [Sh2, Ko2] for further details.

Suppose $\Phi$ is a congruence subgroup of $\Gamma_0(4)$.Let $f, g \in S_{k-1/2}(\Phi)$, we define the Petersson inner product of $f$ and $g$ by:

$$\langle f, g\rangle = \frac{1}{6[\Gamma_0(4):\Phi]}\int_{\Phi\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k-5/2}dxdy.$$

We now recall a subspace of $S_{k-1/2}(\Gamma_0(4N))$, called Kohnen's +-space. Let $S^+_{k-1/2}(\Gamma_0(4N))$ denote this subspace consisting of forms with expansion:

$$h(z) = \sum_{\substack{n\geq 1 \\ (-1)^{k-1}n\equiv 0,1(\bmod 4)}} c(n,h)q^n.$$

There's a Hecke-equivariant isomorphism between the space $S^{+,\mathrm{new}}_{k-1/2}(\Gamma_0(4N))$ and the space $S^{\mathrm{new}}_{2k-2}(\Gamma_0(N))$.

## 2.3 Jacobi Forms

**2.3.1** In this section, we give a review of Jacobi forms based on the standard reference [EZ].

Firstly, we introduce the Jacobi group for a congruence subgroup $\Gamma$. Let $\Gamma^J$ denote the semi-direct product $\Gamma \ltimes \mathbb{Z}^2$, its mulitplication laws are given by

$$(M, X)(M', X') = (MM', XM' + X') \in \mathrm{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2.$$

Put $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, and we call $\Gamma_1^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ the full Jacobi Group.

The actions of $\Gamma$ and $\mathbb{Z}^2$ on $\mathfrak{H} \times \mathbb{C}$ are given by

$$\gamma\cdot(\tau, z) = (\frac{a_\gamma\tau + b_\gamma}{j(\gamma,\tau)}, \frac{z}{j(\gamma,\tau)}),$$

$$(\lambda,\mu)\cdot(\tau, z) = (\tau, z + \lambda\tau + \mu).$$

It's easy to check that these actions are compatible with the group structure of the Jacobi group $\Gamma^J$. Thus, it defines an action of $\Gamma^J$ on $\mathfrak{H} \times \mathbb{C}$.

Let $k$ and $m$ be positive integers. Let $\phi$ be a complex-valued function defined on $\mathfrak{H} \times \mathbb{C}$, we define an action of $\Gamma^J$ on $\phi$ by

$$\phi|_{k,m}\gamma(\tau, z) = j(\gamma, \tau)^{-k} e_m(-\frac{c_\gamma z^2}{j(\gamma, \tau)})\phi(\gamma \cdot (\tau, z)),$$

$$\phi|_m(\lambda, \mu)(\tau, z) = e_m(\lambda^2\tau + 2\lambda z)\phi(\tau, z + \lambda\tau + \mu)$$

for $\gamma \in \Gamma, (\lambda, \mu) \in \mathbb{Z}^2$, and $e_m(z) = e^{2\pi i m z}$.

**Definition**: A holomorphic function $\phi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ is called a Jacobi form of weight $k$ and index $m$ on a subgroup $\Gamma \subset \Gamma_1$ of finite index, if it satisfies:

1. $\phi|_{k,m}\gamma = \phi$ for every $\gamma \in \Gamma$,

2. $\phi|_m(\lambda, \mu) = \phi$ for every $(\lambda, \mu) \in \mathbb{Z}^2$,

3. for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\phi|_{k,m}\gamma$ has a Fourier expansion of the form

$$\sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4mn}} c_\phi(n, r) q^n \zeta^r,$$

   where $q = e(\tau)$ and $\zeta = e(z)$.

If $\phi$ satisfies the stronger condition

$$c_\phi(n, r) \neq 0 \Rightarrow r^2 < 4mn,$$

it is called a cusp form. The vector space of all such Jacobi forms (resp. cusp forms) is denoted $J_{k,m}(\Gamma)$ (resp. $J_{k,m}^{\text{cusp}}(\Gamma)$).

Note that it's also convenient to write the Fourier expansion of a Jacobi form in terms of a discriminant $D = r^2 - 4nm$:

$$\phi(\tau, z) = \sum_{\substack{D \le 0, r \in \mathbb{Z} \\ D \equiv r^2 (\bmod 4m)}} c_\phi(D, r) e(\frac{r^2 - D}{4m}\tau + rz).$$

**2.3.2.** Now we recall the definition of the operator $V_n$ ($n > 0$) on functions $\phi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$, which is important to understand the Saito-Kurokawa lifting.

For any positive integer $n \in \mathbb{N}$, we define a linear operator $V_n$ on $J_{k,m}(\Gamma_0(N))$ by

$$\phi | V_n = \sum_{\substack{D \le 0, r \in \mathbb{Z} \\ D \equiv r^2 (\bmod 4mn)}} \left( \sum_{d | \gcd(\frac{r^2 - D}{4mn}, n, r)} d^{k-1} c_\phi(\frac{D}{d^2}, \frac{r}{d}) \right) q^{\frac{r^2 - D}{4mn}} \zeta^r. \tag{14}$$

**Theorem 2.1 ([EZ, MR1])** *The operator $V_n$ is an index changing operator mapping $J_{k,m}(\Gamma_0(N))$ to $J_{k,mn}(\Gamma_0(N))$.*

## 2.4 Siegel Modular Forms

**2.4.1** In this section, we make a review on the definitions of Siegel modular forms of degree 2.

The Siegel upper half-space of degree 2 is defined as the set

$$\mathfrak{H}^2 = \{Z \in \mathrm{M}_4(\mathbb{C}); {}^tZ = Z, \mathrm{Im}(Z) > 0\}$$

of complex symmetric $n \times n$ matrices with positive definite imaginary part. Let $G = \mathrm{GSp}_4$ be the symplectic similitude group defined by

$$\mathrm{GSp}_4 = \{M \in \mathrm{GL}_4; {}^tMJ_4M = \nu(M)J_4, \nu(M) \in \mathbb{G}_m\},$$

where $J_4 = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$ and $\nu : \mathrm{GSp}_4 \to \mathbb{G}_m$ the scale map. Let $\mathrm{Sp}_4 = \ker(\nu)$ denote the sympletic group. Let $G^+(\mathbb{R})$ be the positive real symplectic group, it acts transitively on $\mathfrak{H}^2$ by the rule:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}.$$

Let $k$ be a positive integer. For a holomorphic function $F : \mathfrak{H}^2 \to \mathbb{C}$, we define an action of $G^+(\mathbb{R})$ on $F$ by:

$$F|_k\gamma(Z) = \det(\gamma)^k \det(C_\gamma Z + D_\gamma)^{-k} F(\gamma(Z)),$$

for any $\gamma \in G^+(\mathbb{R})$. Let $A_2$ denote the lattice of all half integral symetric matrices in the vector space of $2 \times 2$ symetric matrices over $\mathbb{R}$. Let $B_2 \subset A_2$ denote the subset of all positive definite matrices. Let $\Phi \subset G^+(\mathbb{Q})$ be a congruence subgroup commensurable with $\mathrm{Sp}_4(\mathbb{Z})$.

**Definition**: A holomorphic function $F : \mathfrak{H}^2 \to \mathbb{C}$ is called a Siegel modular form of degree 2 and weight $k$ with respect to $\Phi$ if $F|_k\gamma = F$ for all $\gamma \in \Phi$. (Since the degree is 2, the regularity at cusps is automatically satisfied by Koecher.)

Any Siegel modular form $F$ of degree 2 has a Fourier expansion of the form:

$$F(Z) = \sum_{T \in A_2, T \geq 0} A(T) e(\mathrm{Tr}(TZ)).$$

If $A(T) = 0$ unless $T \in B_2$, we say $F$ is a Siegel cusp form. The vector space of all such Siegel modular forms (resp. Siegel cusp forms) is denoted $\mathcal{M}_k^s(\Phi)$ (resp. $\mathcal{S}_k^s(\Phi)$). We drop the index $s$ for simplicity if there's no confusion.

For a positive integer $N$, we define the congruence subgroups $\Gamma^2(N), \Gamma_0^2(N)$ and $\Gamma_1^2(N)$ of $\mathrm{Sp}_4(\mathbb{Z})$ as:

$$\Gamma^2(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}); A \equiv D \equiv 1_2(\bmod N), B \equiv C \equiv 0_2(\bmod N)\},$$

$$\Gamma_0^2(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}); C \equiv 0_2(\bmod N)\},$$

$$\Gamma_1^2(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^2(\mathbb{Z}); A \equiv 1_2(\bmod N)\}$$

with the convention that the congruences regarding the matrices are with respect to their entries. If $\Phi \subset \mathrm{Sp}_4(\mathbb{Z})$ is a subgroup of finite index that contains $\Gamma^2(N)$ for some $N$, we say that $\Phi$ is a congruence subgroup of level $N$.

Let $\chi$ be a Dirichlet character. $F$ is called a classical Siegel modular form of weight $k$ and character $\chi$ for $\Gamma_0^2(N)$ if

$$F(\gamma(Z)) = \chi(\det(D_\gamma))\det(C_\gamma Z + D_\gamma)^k F(Z)$$

for all $\gamma \in \Gamma_0^2(N)$. The vector space of all such Siegel forms is denoted $\mathcal{M}_k^s(N, \chi)$.

**2.4.2.** For any $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ in the Siegel upper half-space $\mathfrak{H}^2$, we may write it as a row vector $(\tau, z, \tau)$, here $\tau, \tau' \in \mathfrak{H}$, $z \in \mathbb{C}$ and $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\mathrm{Im}(\tau')$. And we write $F(\tau, z, \tau')$ instead of $F(Z)$. Similarly, we write $A(n, r, m)$ for $A(T)$, where $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in A_2$ with $n, r, m \in \mathbb{Z}$, $n, m \geq 0$ and $r^2 \leq 4mn$. Thus, the Fourier expansion of $F$ has the form:

$$F(\tau, z, \tau') = \sum_{\substack{n,r,m\in\mathbb{Z} \\ n,m,4mn-r^2\geq 0}} A(n, r, m)e(n\tau + rz + m\tau'). \tag{15}$$

Now we introduce a subspace of $\mathcal{M}_k^s(\Phi)$ called Maass 'Spezialschar'. A Siegel modular form $F \in \mathcal{M}_k^s(\Phi)$ is called a Maass modular form if its Fourier coefficients satisfy the relation

$$A(n, r, m) = \sum_{d|\mathrm{gcd(n,r,m)}} d^{k-1} A(\frac{nm}{d^2}, \frac{r}{d}, 1) \tag{16}$$

for every $n, r, m \in \mathbb{Z}$ with $n, m, 4nm - r^2 \geq 0$. The space of Maass modular forms (resp. Maass cusp forms) is denoted $\mathcal{M}_k^*(\Phi)$ (resp. $\mathcal{S}_k^*(\Phi) := \mathcal{S}_k^s(\Phi) \cap \mathcal{M}_k^*(\Phi)$).

The relation of Siegel modular forms to Jacobi forms is given by the following result.

**Theorem 2.2** *Let $F$ be a Siegel modular form of weight $k$ and degree 2 with respect to $\Phi$, we write the Fourier expansion of $F$ in the form*

$$F(\tau, z\tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z)e(m\tau'). \tag{17}$$

*Then $\phi_m(\tau, z)$ is a Jacobi form of weight $k$ and index $m$.*

Following Piatetski-Shapiro, we call (14) the Fourier-Jacobi expansion of the Siegel modular form $F$. Note that $\phi(\tau, 0)$ the restriction of a Jacobi form $\phi$ on $\mathfrak{H}$ is an ordinary modular form of weight $k$.

Let's fix an embedding $\pi_d : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}^2$ by

$$(\tau, \tau') \mapsto \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}.$$

Then we have the pullback of a Siegel modular form $F$ via the embedding $\pi_d$, $F|_{\mathfrak{H}\times\mathfrak{H}} \in S_k(\Phi) \otimes S_k(\Phi)$. And by Lemma 1.1 in [Ich],

$$(T(p) \otimes \mathrm{id})F|_{\mathfrak{H}\times\mathfrak{H}} = (\mathrm{id} \otimes T(p))F|_{\mathfrak{H}\times\mathfrak{H}}$$

for all primes $p$. Here $T(p)$ is the Hecke operator on $S_k(\Phi)$.

**2.4.3.** We give a short review of the definitions of the Hecke operators. Put $\Sigma = G^+(Q) \cap \mathrm{M}_4(\mathbb{Z})$, and $\Sigma_n = \{\gamma \in \Sigma; \nu(\gamma) = n\}$. Note that $\mathrm{Sp}_4(\mathbb{Z}) = \Sigma_1$. Put

$$\widetilde{\Sigma} = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix}; C \equiv 0(\mod N), A \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\}.$$

Let $\widetilde{\Sigma}_n = \Sigma_n \cap \widetilde{\Sigma}$. For each positive integer $n$, define the $n^{\text{th}}$ Hecke operator $T_S(n)$ by

$$T_S(n) = \sum_{\alpha} \Gamma_0^2(N)\alpha\Gamma_0^2(N)$$

where the sum is taken over all $\alpha \in \Gamma_0^2(N) \setminus \widetilde{\Sigma}_n/\Gamma_0^2(N)$.

As the case of integer weight modular forms, we call $F$ a Heck eigenform, if there exists $\lambda_F(n) \in \mathbb{C}$ such that $T_S(n)F = \lambda_F(n)F$ for all $n \geq 1$. Suppose $F$ is a non-zero Hecke eigenform of weight $k$ and level $N$ with eigenvalues $\lambda_F(n)$. We define the spinor zeta function associated to $F$ by

$$L_{\text{spin}}(s,F) = \zeta(2s-2k+4)\sum_{n=1}^{\infty}\lambda_F(n)n^{-s}. \tag{18}$$

It has meromorphic continuation, functional equation and Euler product as well by Andrianov [An1, An3].

# 3 Saito-Kurokawa Lifting

Based on numerical evidence, Saito and Kurokawa in 1977 made a conjecture that for a Hecke eigenform $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, there exists a Siegel eigenform $F$ of degree 2 and weight $k+1$ such that

$$L_{\mathrm{spin}}(s, F) = \zeta(s-k)\zeta(s-k+1)L(s, f).$$

Most of this conjecture was proved by Maass in his series paper, another part by Andrianov [An2] and the remaining part by Zagier. In 1993 Manickham, Ramakrishnan and Vasudevan [MRV] extended this result to odd square-free level. Later Manickham and Ramakrishnan improved it to all positive levels [MR1, MR2].

## 3.1 Shimura Lifting

Shimura in 1973 [Sh2] firstly pointed out the correspondence of integral weight elliptic modular forms of weight $2k$ to half-integral weight modular forms of weight $k+1/2$. Kohnen later provided more precise results on determining the level of half-integral weight modular forms.

Let $M$ be a positive integer. Suppose $D$ is a fundamental discriminant with $(-1)^k D > 0$. For any $g$ in Kohnen's $+$-space $S^+_{k+1/2}(\Gamma_0(4M))$ with Fourier expansion:

$$g(z) = \sum_{n=1}^{\infty} c(n;g)q^n,$$

where $c(n;g) = 0$ whenever $(-1)^k n \equiv 2,3 (\mathrm{mod}\ 4)$, The Shimura lifting $\zeta_D^*$ : $S^+_{k+1/2}(\Gamma_0(4M)) \to M_{2k}(\Gamma_0(M))$ is given by:

$$\zeta_D^* g(z) = \sum_{n=1}^{\infty} \left( \sum_{\substack{d|n \\ \gcd(d,M)=1}} \left(\frac{D}{d}\right) d^{k-1} c(\frac{|D|n^2}{d^2}; g) \right) q^n.$$

## 3.2 Shintani Lifting

**3.2.1.** Fix a positive integer $M$, and denote by $\mathcal{F}_M$ the set of all integral indefinite binary quadratic forms, $Q(X,Y) = aX^2 + bXY + cY^2$, satisfying the conditions:

$$b^2 - 4ac > 0; \quad \text{and } (a, M) = 1, \quad b \equiv c \equiv 0 \bmod M. \tag{19}$$

The action of $\Gamma_0(M)$ on $\mathcal{F}_M$ is defined as:

$$(Q|\gamma)(X,Y) := Q((X,Y)\gamma^{-1}),$$

for $Q \in \mathcal{F}_M$ and $\gamma \in \Gamma_0(M)$.

To each quadratic form $Q(X,Y)$ in $\mathcal{F}_M$, Shintani associated the following data: the discriminant $\delta_Q = b^2 - 4ac$, the pair $(\omega_Q, \omega'_Q)$ of points in $\mathbb{R} \cup \{i\infty\}$, and the oriented geodesic path $C_Q$ in the upper half plane. For further details of the definitions of $\omega_Q$, $\omega'_Q$ and $C_Q$, please refer to [Shi].

Fix an interger $k \geq 0$. Suppose $M$ is odd. For a given Dirichlet character $\chi$ of conductor $M$, we define a new Dirichlet character $\chi' : (\mathbb{Z}/4M\mathbb{Z})^\times \to \mathbb{C}^\times$ by

$$\chi'(d) := \chi(d) \cdot \left( \frac{(-1)^k M}{d} \right), \quad d \in (\mathbb{Z}/4M\mathbb{Z})^\times. \tag{20}$$

For each $Q \in \mathcal{F}_M$, we put $\chi(Q) = \chi(a)$. For each $f \in S_{2k}(\Gamma_0(M), \chi^2)$, we define an integral

$$I_{k,\chi}(f, Q) := \chi(Q) \cdot \int_{C_Q} f(\tau) Q(1, -\tau)^{k-1} d\tau. \tag{21}$$

For any $z \in \mathfrak{H}$, we define

$$\zeta_{k,\chi}(f)(z) := \sum_{Q \in \mathcal{F}_M / \Gamma_0(M)} I_{k,\chi}(f, Q) q^{\delta_Q / M}. \tag{22}$$

When $\chi$ is trivial, we write $\zeta_k$ instead of $\zeta_{k,\chi}$.

**Theorem 3.1 ([Shi])** *For any $f \in S_{2k}(\Gamma_0(M), \chi^2)$, the series $\zeta_{k,\chi}(f)(z)$ is the $q$-expansion of a cusp form in $S_{k+1/2}(\Gamma_0(M), \chi')$. Moreover, the map*

$$\zeta_{k,\chi} : S_{2k}(\Gamma_0(M), \chi^2) \to S_{k+1/2}(\Gamma_0(4M), \chi')$$

*is a Hecke equivariant linear function. Precisely, for any prime $\ell$,*

$$\zeta_{k,\chi}(f|T_\ell) = \zeta_{k,\chi}(f)|T_{\ell^2}.$$

**3.2.2.** As above, let $D$ be a fundamental discriminant with $(-1)^k D > 0$, and $M$ a positive integer such that $\gcd(D, M) = 1$.

**Theorem 3.2** *The Shimura lifting $\zeta_D^*$ and the Shintani lifting $\zeta_k$ give a Hecke-equivariant isomorphism between $S_{k+1/2}^{+,new}(\Gamma_0(4M))$ and $S_{2k}^{new}(\Gamma_0(M))$*

Let $\mathcal{O}_f$ be the ring generated by the Hecke eigenvalues of $f$. Suppose an embedding of $\mathcal{O}_f$ into $\mathbb{C}$ exists. Choose and fix such an embedding. We identify $\mathcal{O}_f$ with its image in $\mathbb{C}$ via the embedding.

**Theorem 3.3 ([Ste])** *For any Hecke eigenform $f \in S_{2k}(\Gamma_0(M), \chi^2)$, there is a non-zero complex number $\Omega_f^- \in \mathbb{C}^\times$ such that*

$$\frac{1}{\Omega_f^-} \cdot \zeta_{k,\chi}(f) \in S_{k+1/2}(\Gamma_0(4M), \chi'; \mathcal{O}_f).$$

Fix such a period $\Omega_f^-$ once for all for each Hecke eigenform $f \in S_{2k}(\Gamma_0(M), \chi^2)$ and define:

$$\theta_{k,\chi}(f) := \frac{1}{\Omega_f^-} \cdot \zeta_{k,\chi}(f) \in S_{k+1/2}(\Gamma_0(4M), \chi'; \mathcal{O}_f). \tag{23}$$

We call $\theta_{k,\chi}(f)$ the algebraic part of $\zeta_{k,\chi}(f)$.

## 3.3 Half-Integral Weight Forms to Jacobi Forms

The case of full level is presented in [EZ], here we recall the correspondence for arbitrary levels $M$ by Manickham and Ramakrishnan in [MR1] and [MR2].

For a negative fundamental discriminant $D$, let $P_D^+$ be the $D$-th Poincare series in Kohnen's $+$-space $S_{k+1/2}^+(\Gamma_0(4M))$. We put $S_{k+1/2}^*(\Gamma_0(4M))$ for the subspaces of cusp forms, where $*$ means $+$ if $2 \nmid M$ and nothing if $2|M$. Let $\mathcal{P}$ denote the subspace of $S_{k+1/2}^*(\Gamma_0(4M))$ generated by the Poincare series $P_{|D|}^*$, with $D$ running over all discriminants satisfying $D \equiv 0, 1 (\text{mod } 4), (-1)^k D > 0$.

Denote by $S^{\mathrm{new}}_{k+1/2}(4M, f)$ the subspace of cusp forms $g \in S^{*}_{k+1/2}(\Gamma_0(4M))$ satisfying $T(p^2)g = a(p; f)g$ for all primes $p \nmid M$. Since the index of Jacobi forms in this section is 1, we drop it from the notations for simplicity. Denote by $J^{c,n}_{k+1}(M, f)$ the subspace of Jacobi forms $\phi \in J^{\mathrm{cusp}}_{k+1}(M)$ satisfying $T_J(p)\phi = a(p; f)\phi$ for all primes $p \nmid M$. Put

$$S^{\mathrm{new}}_{k+1/2}(4M) = \mathcal{P} \cap \bigoplus_{f \in S^{\mathrm{new}}_{2k}(\Gamma_0(M))} S^{\mathrm{new}}_{k+1/2}(4M, f),$$

$$J^{c,n}_{k+1}(M) = \bigoplus_{f \in S^{\mathrm{new}}_{2k}(\Gamma_0(M))} J^{c,n}_{k+1}(M, f).$$

We define a map $\mu : S^{\mathrm{new}}_{k+1/2}(4M) \to J^{c,n}_{k+1}(M)$ by

$$\sum_{\substack{N>0 \\ N \equiv 0,3(\bmod 4)}} c(N)q^N \mapsto \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq n}} c(4n - r^2)q^n \zeta^r. \tag{24}$$

**Theorem 3.4 ([MR1])** *The map $\mu : S^{new}_{k+1/2}(4M) \to J^{c,n}_{k+1}(M)$ is a cononical isomorphism compatible with the action of Hecke operators.*

For any integer ring $\mathcal{O}$, by (21) we have the following result.

**Corollary 3.1** *For any $g \in S^{new}_{k-1/2}(4M; \mathcal{O})$, the corresponding Jacobi form $\phi = \mu(g)$ has Fourier coefficients in $\mathcal{O}$ and vice versa.*

## 3.4 Jacobi Forms to Siegel Forms

Fix positive integers $k$ and $M$. Introduced in section 2.4, we know that a Siegel modular form $F \in \mathcal{M}^*_{k+1}(\Gamma^2_0(M))$ has Fourier-Jacobi expansion of the form:

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e(m\tau').$$

Here, $\phi_1(\tau, z)$ is a Jacobi form of weight $k+1$ and index 1 with respect to $\Gamma^J_0(M)$ by Theorem 2.2. Assigning $\phi_1$ in its Fourier-Jacobi expansion to each Siegel modular form $F$, we define a map

$$\omega : \mathcal{M}^*_{k+1}(\Gamma_0(M)) \longrightarrow J_{k+1,1}(\Gamma_0(M)).$$

$$F(\tau, z, \tau') \longmapsto \phi_1(\tau, z)$$

**Theorem 3.5 ([MR2])** *The map $\omega : \mathcal{M}^*_{k+1}(\Gamma^2_0(M)) \to J_{k+1,1}(\Gamma_0(M))$ is an isomorphism commuting with the action of Hecke operators.*

Let $\phi(\tau, z)$ be a Jacobi form of weight $k+1$ and index 1 with Fourier expansion:

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4n}} c_\phi(n, r) q^n \zeta^r.$$

Recall the index changing operator $V_m : J_{k+1,n}(\Gamma_0(M)) \to J_{k+1,mn}(\Gamma_0(M))$, we define a map on $J_{k+1,1}(\Gamma_0(M))$,

$$\mathcal{V} : \phi(\tau, z) \mapsto F(\tau, z, \tau') = \sum_{m \geq 0} (\phi | V_m)(\tau, z) e(m\tau'). \tag{25}$$

**Theorem 3.6 ([EZ],[MRV])** *$\mathcal{V}(\phi) \in \mathcal{M}_k^*(\Gamma_0^2(M))$ is a Siegel modular form in the Maass 'Spezialschar'. And $\mathcal{V} : J_{k+1,1}(\Gamma_0(M)) \to \mathcal{M}_{k+1}^*(\Gamma_0^2(M))$ is a Hecke equivariant isomorphism. $\mathcal{V}$ is the inverse of $\omega$.*

The Fourier coefficients of $F(\tau, z, \tau') = \mathcal{V}(\phi)$ and the Fourier coefficients of $\phi$ satisfy the relation:

$$A(n, r, m) = \sum_{d|(n,r,m)} d^k c(\frac{4nm - r^2}{d^2}), \tag{26}$$

for $(n, r, m) \neq (0, 0, 0)$.

**Corollary 3.2** *Suppose $M$ is odd. Let $h(\tau) = \sum c(N)q^N$ be a cusp form in Kohnen's $+$-space $S_{k+1/2}^+(4M)$, then $A(n, r, m)$ defined by equation (23) are the coefficients of a Maass form $F \in \mathcal{S}_{k+1}^*(\Gamma_0^2(M))$. The map $h \mapsto F$ is an isomorphism between this two spaces.*

## 3.5 A pullback formula of Saito-Kurokawa lifting

Let $f \in S_{2k}(\Gamma_0(N))$ be a normalized Hecke eigenform. Suppose $F \in \mathcal{S}_{k+1}(\Gamma_0^2(N))$ is the Siegel cusp form corresponding to $f$ by the composition of $\theta_{k,\chi}$, $\mu$ and $\mathcal{V}$ defined by (20), (21) and (22) respectively.

**Theorem 3.7 ([MRV])** *Assume $N$ is a positive odd square free integer. The space $\mathcal{S}_{k+1}^{*,new}(\Gamma_0^2(N))$ is 1-1 correspondence with $S_{2k}^{new}(\Gamma_0(N))$. For a Siegel Hecke* *eigenform $F$ and a nomalized Hecke eigenform $f$, the correspondence is given by:*

$$L_{spin}(s,F)=\zeta(s-k)\zeta(s-k+1)L(s,f).$$

Suppose $h\in S^{+}_{k+1/2}(\Gamma_0(4N))$ is a Hecke eigenform associated to $f$ by Shimura correspondence. For each normalized eigenform $g\in S_{k+1}(\Gamma_0(N))$, we consider the period integral

$$\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g\times g\rangle$$

$$=\frac{1}{[\overline{\mathrm{SL}_2(\mathbb{Z})}:\overline{\Gamma_0(N)}]^2}\int_{\Gamma_0(N)\backslash\mathfrak{H}}\int_{\Gamma_0(N)\backslash\mathfrak{H}}F\left(\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}\right)\overline{g(\tau_1)g(\tau_2)}y_1^{k-1}y_2^{k-1}d\tau_1 d\tau_2.$$

Let $A(s,\mathrm{Sym}^2(g)\otimes f)$ be the complete $L$-function.

**Theorem 3.8 ([Ich, Li])** *Assume $N$ is a positive odd square free integer, and $k$ is odd with $k\geq 3$. Then we have*

$$A(2k,Sym^2(g)\otimes f)=2^{k+1-\nu(N)}\xi_N\frac{\langle f,f\rangle}{\langle h,h\rangle}\frac{|\langle F|_{\mathfrak{H}\times\mathfrak{H}},g\times g\rangle|^2}{\langle g,g\rangle^2},$$

*where $\xi_N$ is an algebraic number given by*

$$\xi_N=N^2\prod_{p|N}\epsilon_p(1+p)^5(1-p)^2(\epsilon_p-p)^{-2},$$

*and $\epsilon_p=-a_f(p)$, $\nu(N)$ is the number of prime divisors of $N$.*

Let $\Omega_f^{+}$ be the period of $f$ as in [Sh3]. By the generalized Kohnen-Zagier formula [KZ, Ko3]:

$$A(k,f,\chi_{-D})=2^{1-k-v(N)}D^{1/2}|c_h(D)|^2\frac{\langle f,f\rangle}{\langle h,h\rangle},$$

we have the following corollary.

**Corollary 3.3** *The quotients*

$$\frac{A(2k, Sym^2(g) \otimes f)}{\langle g, g\rangle^2 \cdot \Omega_f^+} \in \overline{\mathbb{Q}}$$

*are algebraic numbers.*

# 4 A $\Lambda$-adic Saito-Kurokawa Lifting

We develope a $\Lambda$-adic analog of Saito-Kurokawa lifting in this chapter, which is a generalization of $\Lambda$-adic Eisenstein series.

## 4.1 $p$-adic Modular Forms

**4.1.1.** In this section, we recall $p$-adic theory on classic modular forms by Hida. We fix a prime $p \geq 5$ once for all. Let $K_0$ be a finite extension over $\mathbb{Q}$ in $\overline{\mathbb{Q}}$, and let $K$ be the topological closure of $K_0$ in $\mathbb{C}_p$. Let $\mathcal{O}_K$ be the $p$-adic integer ring of $K$. Suppose $\Phi$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and $\chi : \Phi \to K^\times$ is a finite order character. By results in section 2.1, we may put

$$M_k(\Phi; K) = M_k(\Phi; K_0) \otimes_{K_0} K, M_k(\Phi, \chi; K) = M_k(\Phi, \chi; K_0) \otimes_{K_0} K,$$

$$S_k(\Phi; K) = S_k(\Phi; K_0) \otimes_{K_0} K, S_k(\Phi, \chi; K) = S_k(\Phi, \chi; K_0) \otimes_{K_0} K,$$

If the congruence group $\Phi$ is of level $N$, we may view these spaces inside the formal power series ring $K[[q]]$ by using their $q$-expansions.

For each positive integer $j$, we put

$$M^j(\Phi;K) = \bigoplus_{k=0}^{j} M_k(\Phi;K), S^j(\Phi;K) = \bigoplus_{k=0}^{j} S_k(\Phi;K).$$

We take inductive limits inside $K[[q]]$:

$$M(\Phi;K) = M^\infty(\Phi;K) = \varinjlim_j M^j(\Phi;K) \simeq \bigoplus_{k=0}^{\infty} M_k(\Phi;K),$$

$$S(\Phi;K) = S^\infty(\Phi;K) = \varinjlim_j S^j(\Phi;K) \simeq \bigoplus_{k=0}^{\infty} S_k(\Phi;K).$$

We define a $p$-adic norm on these spaces by

$$|f| = |f|_p = \sup_n |a(n,f)|_p.$$

for a power series $f = \sum a(n,f)q^n \in K[[q]]$. We take the $p$-adic completion of these space under this norm $|\cdot|_p$ inside $K[[q]]$, and denote the completion of $M(\Phi;K)$ (resp. $S(\Phi;K)$) by $\overline{M}(\Phi;K)$ (resp. $\overline{S}(\Phi;K)$).

We now consider the forms with coefficients in $\mathcal{O}_K$. Put

$$M_k(\Phi;\mathcal{O}_K) = M_k(\Phi;K) \cap \mathcal{O}_K[[q]], S_k(\Phi;\mathcal{O}_K) = S_k(\Phi;K) \cap \mathcal{O}_K[[q]],$$

$$\overline{M}_k(\Phi;\mathcal{O}_K) = \overline{M}_k(\Phi;K) \cap \mathcal{O}_K[[q]], \overline{S}_k(\Phi;\mathcal{O}_K) = \overline{S}_k(\Phi;K) \cap \mathcal{O}_K[[q]].$$

The space $\overline{M}_k(\Phi;\mathcal{O}_K)$ (resp. $\overline{S}_k(\Phi;\mathcal{O}_K)$) is the completion of $M_k^\infty(\Phi;\mathcal{O}_K)$ (resp. $S_k^\infty(\Phi;\mathcal{O}_K)$) under the norm $|\cdot|_p$. When $\Phi = \Gamma_1(N)$, we write $\overline{M}_k(N;\mathcal{O}_K)$ and $\overline{S}_k(N;\mathcal{O}_K)$ for $\overline{M}_k(\Phi;\mathcal{O}_K)$ and $\overline{S}_k(\Phi;\mathcal{O}_K)$ respectively.

For $A = K, \mathcal{O}_K$ or any commutative algebra $A \subset \mathbb{C}$, we put

$$M_k(Np^\infty; A) = \varinjlim_r M_k(\Gamma_1(Np^r); A),$$

$$S_k(Np^\infty; A) = \varinjlim_r S_k(\Gamma_1(Np^r); A).$$

We have natural inclusions:

$$M_k(Np^\infty; \mathcal{O}_K) \subset \overline{M}(N; \mathcal{O}_K), \quad S_k(Np^\infty; \mathcal{O}_K) \subset \overline{S}(N; \mathcal{O}_K).$$

**4.1.2.** For a subspace $V$ of $\overline{M}(N; \mathcal{O}_K)$, we define the Hecke algebra $h(V)$ of $V$ as in section 2.1. We write $h_k(\Phi, \chi; \mathcal{O}_K)$ for $h(S_k(\Phi, \chi; \mathcal{O}_K))$ and $h^j(\Gamma_1(Np^r); \mathcal{O}_K)$ for $h(S^j(\Gamma_1(Np^r); \mathcal{O}_K))$ with $\Gamma_1(Np^r) \subset \Phi \subset \Gamma_1(N)$. For any pair $i > j$, the restriction of $h^i(\Gamma_1(Np^r); \mathcal{O}_K)$ to the subspace $S^j(\Gamma_1(Np^r); \mathcal{O}_K)$ of $h^j(\Gamma_1(Np^r); \mathcal{O}_K)$ gives a surjective $\mathcal{O}_K$-algebra homomorphism:

$$h^i(\Gamma_1(Np^r); \mathcal{O}_K) \to h^j(\Gamma_1(Np^r); \mathcal{O}_K).$$

By taking the projectiv limit, we define

$$h(N; \mathcal{O}_K) = \varprojlim_j h^j(\Gamma_1(Np); \mathcal{O}_K).$$

We extend the action of $h(N; \mathcal{O}_K)$ to $\overline{S}(N; \mathcal{O}_K)$ by uniform continuity.

Denote $\Gamma = 1 + p\mathbb{Z}_p$. Let $\Lambda_K = \mathcal{O}_K[[\Gamma]]$. For each positive integer $N$ prime to $p$, we put

$$Z_N = \varprojlim (\mathbb{Z}/Np^r\mathbb{Z})^\times = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times.$$

Write $(z_p, z_0)$ for the projection of an element $z \in Z_N$. And we have an isomorphism $\mathcal{O}_K[[Z_N]] \cong \Lambda_K \otimes_{\mathcal{O}_K} \mathcal{O}_K[(\mathbb{Z}/Np\mathbb{Z})^\times]$. For any $f \in M_k(Np^r; \mathcal{O}_K)$, we define a slash operater by

$$f|z = z_p^k f|_k[\sigma_z],$$

where $\sigma_z \in \mathrm{SL}_2(\mathbb{Z})$ is a matrix defined as in Section 2.1. This operator induces an endomorphism of $\overline{S}(N; \mathcal{O}_K)$ which belongs to $h(N; \mathcal{O}_K)$, hence it induces a continuous character : $Z_N \to h(N; \mathcal{O}_K)^\times$. Then we have an $\mathcal{O}_K$-algebra homomorphism from $\mathcal{O}_K[[Z_N]]$ to $h(N; \mathcal{O}_K)$. And $h(N; \mathcal{O}_K)$ is a $\Lambda_K$-algebra.

Recall the pairing we define by equation (9), we extend it to

$$\begin{aligned} <,>: h(N; \mathcal{O}_K) \times \overline{S}(N; \mathcal{O}_K) &\longrightarrow \mathcal{O}_K \\ (h, f) &\longmapsto a(1, f|h). \end{aligned} \tag{27}$$

**Theorem 4.1 ([Hi3])** *The pairing defined by (23) is perfect. It induces isomorphisms:*

$$h(N; \mathcal{O}_K) \cong Hom_{\mathcal{O}_K}(\overline{S}(N; \mathcal{O}_K), \mathcal{O}_K), \overline{S}(N; \mathcal{O}_K) \cong Hom_{\mathcal{O}_K}(h(N; \mathcal{O}_K), \mathcal{O}_K).$$

**4.1.3.** The Hecke algebra $h(N; \mathcal{O}_K)$ is a pro-artinian ring, we may write it as the direct sum of local rings:

$$h(N; \mathcal{O}_K) = \oplus R.$$

Let $m(R)$ be the unique maximal ideal of $R$. We call the local component $R$ ordinary if $T(p) \notin m(R)$, i.e. $T(p)$ is invertible in $R$. We define the ordinary part of $h(N;\mathcal{O}_K)$ by the direct sum of all the ordinary local components. We denote by $h^o(N;\mathcal{O}_K)$ the ordinary part, and denote by $e$ the idempotent corresponding to the ordinary part. Then we have:

$$h^o(Np^\infty;\mathcal{O}_K) = eh(Np^\infty;\mathcal{O}_K).$$

For any $h(N;\mathcal{O}_K)$-module $M$, we define the ordinary part of $M$ by $M^o = eM$. Similarly, we define the ordinary parts $h_k^o(\Gamma_1(Np^r);\mathcal{O}_K)$ and $h_k^o(\Gamma_0(Np^r),\chi;\mathcal{O}_K)$ by the biggest direct factor of the original Hecke algebra on which the image of $T(p)$ is a unit.

**Theorem 4.2** *([Hi3]) $h^o(N;\mathcal{O}_K)$ is free of finite rank over $\Lambda_K$.*

## 4.2 $p$-adic Families of Cusp Forms

**4.2.1.** Denote $\Gamma = 1+p\mathbb{Z}_p$, and $\Lambda = \mathbb{Z}_p[[\Gamma]]$. For any finite flat $\Lambda$-algebra $\mathcal{R}$, we put $\mathcal{X}(\mathcal{R}) = \mathrm{Hom}_c(\mathcal{R},\mathbb{C}_p)$. The elements of $\mathcal{X}(\mathcal{R})$ are called points of $\mathcal{R}$. The restriction to $\Lambda$ induces a surjective finite-to-one mapping

$$\pi : \mathcal{X}(\mathcal{R}) \to \mathcal{X}(\Lambda). \tag{28}$$

We identify $\mathcal{X}(\Lambda)$ with the group of continuous characters $P : \Gamma \to \mathbb{C}_p^\times$. For any $P \in \mathcal{X}(\mathcal{R})$ unramified over $\Lambda$, there exists a natural local section $S_P$ of $\pi$,

$$S_P : U \subseteq \mathcal{X}(\Lambda) \to \mathcal{X}(\mathcal{R}) \tag{29}$$

defined on a neighborhood $U$ of $\pi(P) \in \mathcal{X}_\Lambda$ and $S_P \circ \pi(P) = P$. The image of any local chart $S_P$ about an unramified point $P$ is called an analytic neighborhood of $P$. A function $f : U \subseteq \mathcal{X}(\mathcal{R}) \to \mathbb{C}_p$ defined on an analytic neighborhood $U$ of $P$ is called analytic if $f \circ S_P$ is analytic.

**4.2.2.** Choose and fix a topological generator $u$ of $1 + p\mathbb{Z}_p$. For each finite order character $\epsilon : \Gamma \to \mathbb{C}_p^\times$, and for each integer $k$, the continuous character $u \mapsto u^k\epsilon(u)$ of $\Gamma$ induces a $\mathbb{Z}_p$-algebra homomorphism $P_{k,\epsilon} : \Lambda \to \mathbb{C}_p$. Such a character is called arithmetic. A point $P \in \mathcal{X}(\Lambda)$ is said to be arithmetic if the associated character of $\Gamma$ is arithmetic. We put

$$\mathcal{X}_{\text{alg}}(\Lambda) = \{P_{k,\epsilon}; k \geq 2, \epsilon : \Gamma \to \mathbb{C}_p^\times, [\Gamma : \ker(\epsilon)] < \infty\}.$$

For a finite flat $\Lambda$-algebra $\mathcal{R}$, a point $P \in \mathcal{X}(\mathcal{R})$ is said to be arithmetic if it lies over an arithmetic point on $\mathcal{X}(\Lambda)$. We put

$$\mathcal{X}_{\text{alg}}(\mathcal{R}) = \{P \in \mathcal{X}(\mathcal{R}); P|_\Lambda \in \mathcal{X}_{\text{alg}}(\Lambda)\}.$$

A point $P \in \mathcal{X}_{\text{alg}}(\mathcal{R})$ is called an arithmetic point of type $(k, \epsilon)$, if $P|_\Lambda = P_{k,\epsilon}$.

**Theorem 4.3 ([Hi3])** *For each integer $k \geq 2$, and for each finite order character $\epsilon : \Gamma \to \mathcal{O}_K^\times$ of conductor $p^r$ with $r \geq 1$, there is an isomorphism:*

$$h^o(N; \mathcal{O}_K)/P_{k,\epsilon}h^o(N; \mathcal{O}_K) \cong h_k^o(\Phi_r^1, \epsilon; \mathcal{O}_K),$$

*which takes $T(n)$ of $h^o(N; \mathcal{O}_K)$ to $T(n)$ of the right-hand side.*

**4.2.3.** Let $\mathcal{L}_K$ denote the quotient field of $\Lambda_K$, and let $\mathcal{K}$ be a finite extension of $\mathcal{L}_K$. Let $\mathcal{I}$ be the integral closure of $\Lambda_K$ in $\mathcal{K}$. Suppose that the algebraic closure of $\mathbb{Q}_p$ inside $\mathcal{K}$ coincides with $K$.

For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$, there are an integer $k \geq 0$ and a finite order character $\epsilon$ on $\Gamma$, such that $P|_{\Lambda_K} = P_{k,\epsilon}$. The integer $k$ is called the weight of $P$, denoted by $k(P)$. The character $\epsilon$ is called the character of $P$, denoted by $\epsilon_P$. The exponent in $p$ of the conductor of $\epsilon_P$ will be denoted by $r(P)$. (When $\epsilon_P$ is trivial, we put $r(P) = 1$.)

Let $\lambda : h^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ be an $\mathcal{I}$-algebra homomorphism. The $\mathcal{O}_K[[Z_N]]$-algebra structure on $h^o(N; \mathcal{O}_K)$ induces a character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathcal{O}_K^\times$, it's called the character of $\lambda$. For $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$, we consider the reduction of $\lambda \bmod P$:

$$\lambda_P : h^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} (\mathcal{I}/P\mathcal{I}) \longrightarrow \mathcal{I}/P\mathcal{I} \cong \mathcal{O}_K.$$

If the weight $k(P) \geq 2$, by Theorem 4.3 we have:

$$h^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} (\mathcal{I}/P\mathcal{I}) \cong h^o_{k(P)}(\Phi^1_{r(P)}, \epsilon_P; \mathcal{O}_K).$$

Hence we get an $\mathcal{O}_K$-algebra homomorphism:

$$\lambda_P : h^o_{k(P)}(\Phi^1_{r(P)}, \epsilon_P; \mathcal{O}_K) \longrightarrow \mathcal{O}_K.$$

By duality theorem, there's a unique normalized eigenform

$$f_P \in S_{k(P)}(\Gamma_0(Np^{r(P)}, \epsilon_P \psi \omega^{-k(P)})),$$

such that $f_P|T(n) = \lambda_P(T(n))f_P$ for all $n \geq 0$. We say $\lambda$ is primitive if it satisfies one of the following two equivalent conditions:

1. there exists $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ with $k(P) \geq 2$, such that $f_P$ is primitive of conductor $Np^{r(P)}$;

2. $f_P$ is primitive for every $P$ with $k(P) \geq 2$ such that the $p$-part of $\epsilon_P\psi\omega^{-k(P)}$ is non-trivial.

**Theorem 4.4 ([Hi2])** *Suppose $\lambda : h^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I} \to \mathcal{I}$ is a primitive $\mathcal{I}$-algebra homomorphism. Then $\lambda$ induces a decomposition of $\mathcal{K}$-algebras:*

$$h^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{K} \cong \mathcal{K} \oplus \mathcal{A} \tag{30}$$

*such that the projection of $h^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ into $\mathcal{K}$ coincides with $\lambda$.*

Let $1_{\mathcal{K}}$ be the idempotent corresponding to the first factor. Let $\mathcal{R}_N$ and $\mathcal{A}_N$ denote the images of $h^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ in $\mathcal{K}$ and $\mathcal{A}$ respectively. We may identify the elements in $h^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I}$ and their images in $\mathcal{R}_N$. Define a congruence module $\mathcal{C}(\lambda;\mathcal{I}) := (\mathcal{R}_N \oplus \mathcal{A}_N)/(h^o(N;\mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{I})$.

The universal $p$-stabilized ordinary newform of tame conductor $N$ is defined to be the formal $q$-expansion $\mathbf{f}_N := \sum_{n=1}^{\infty} \alpha_n q^n \in \mathcal{R}_N[[q]]$, where $\alpha_n = T(n) \in \mathcal{R}_N$. We may regard $\mathbf{f}_N$ as an analytic function on $\mathcal{X}(\mathcal{R}_N)$ interpolating the $q$-expansions of ordinary newforms at arithmetic points.

## 4.3 Modular Symbols

Let $D$ be the free abelian group generated by the rational cusps $\mathbb{Q} \cup \{i\infty\} = \mathbb{P}^1(\mathbb{Q})$ of the upper half plane $\mathfrak{H}$. Put $\mathfrak{H}_0 = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$. Let $D_0$ be the subgroup of $D$ of divisors of degree 0. The congruence group $\Phi \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $D_0$ by fractional linear transformations. Suppose $A$ is a right $\mathbb{Z}[\frac{1}{6}][\Phi]$-module. The group of $A$-valued modular symbols is defined to be

$$\mathrm{Symb}(\Phi; A) := \mathrm{Hom}_\Phi(D_0, A).$$

**Theorem 4.5** *There is a canonical isomorphism between the cohomology group with compact support and modular symbols $H^1_c(\Phi, A) \cong Symb(\Phi; A)$.*

The matrix $\iota = \mathrm{diag}(1, -1) \in \mathrm{SL}_2(\mathbb{Z})$ induces natural involutions on the cohomology groups $H^1(\Phi, A)$ and $H^1_c(\Phi, A)$. On modular symbols, this involution is given by $\varphi \mapsto \varphi|\iota$, where $\varphi|\iota : d \mapsto \varphi(\iota d)|\iota$ for $d \in D_0$. Through this involution, the cohomology groups can be decomposed into $\pm$ eigenspaces: $H = H^+ \oplus H^-$. Each cohomology class $\varphi$ decomposes as $\varphi = \varphi^+ + \varphi^-$, where $\varphi^\pm := \frac{1}{2}(\varphi \pm \varphi|\iota)$. Simple computation shows $\varphi^\pm|\iota = \pm\varphi^\pm$.

Suppose $\Phi = \Gamma_0(M)$ and that $A$ is a commutative ring. For a non-negative integer $n$, we put $L_n(\mathbb{Z}) = \mathbb{Z}^{n+1}$ and $L_n(A) = L_n(\mathbb{Z}) \otimes A$. For $\begin{bmatrix} X \\ Y \end{bmatrix} \in L_1(\mathbb{Z})$, put

$$F_n(X, Y) = \begin{bmatrix} X \\ Y \end{bmatrix}^n = {}^t(X^n, X^{n-1}Y, \cdots, Y^n) \in L_n(\mathbb{Z}).$$

For each $\gamma \in \mathrm{M}_2(\mathbb{Z})$, we define an action on $L_n(\mathbb{Z})$ by

$$(F_n|\gamma)(X,Y) = F_n(\gamma(X,Y)).$$

This action extends to $L_n(A)$. Let $\epsilon$ be a $A$-valued Dirichlet character modulo $M$. If the action of $\Phi$ on the same underlying module is twisted by $\epsilon$:

$$(F_n|\gamma)(X,Y) = \epsilon(\gamma) \cdot F_n(\gamma(X,Y)), \tag{31}$$

we denote this $A[\Phi]$-module by $L_{n,\epsilon}(A)$.

**Theorem 4.6 (Eichler, Shimura)** *For either choice of sign $\pm$, there is a Hecke equivariant isomorphism*

$$\mathcal{E} : S_{n+2}(\Gamma_0(M), \epsilon) \to H_{par}(\Gamma_0(M), L_{n,\epsilon}(\mathbb{C}))^{\pm}.$$

For each $f \in S_{n+2}(\Gamma_0(M), \epsilon)$, we define $\psi_f : D_0 \to L_{n,\epsilon}(\mathbb{C})$ by

$$\psi_f(\{c_2\} - \{c_1\}) = \int_{c_1}^{c_2} f(z) F_n(z,1) dz.$$

The integral is taken over the oriented geodesic path joining $c_1$ to $c_2$ in $\mathfrak{H}$. By Theorem 4.5, we identify $\psi_f$ as a compactly supported cohomology class in $H^1_c(\Gamma_0(M), L_{n,\epsilon}(\mathbb{C}))$. And $\mathcal{E}(f)$ is the image of $\psi_f$ in $H^1_{par}(\Gamma_0(M), L_{n,\epsilon}(\mathbb{C}))$.

Let $\mathcal{O}_f$ be the ring generated by the Hecke eigenvalues of $f$. It's well known that there are two complex numbers $\Omega_f^{\pm} \in \mathbb{C}^{\times}$ such that $(\Omega_f^{\pm})^{-1} \cdot \psi_f^{\pm}$ is defined over $\mathcal{O}_f$. Put $\varphi_f^{\pm} = (\Omega_f^{\pm})^{-1} \cdot \psi_f^{\pm}$, we may refer to it as the algebraic part of $\psi_f^{\pm}$.

## 4.4 A $\Lambda$-adic Shintani lifting

**4.4.1.** We call a pair $(a, b) \in \mathbb{Z}_p^2$ primitive if $p \nmid (a, b)$ as a vector. Let $(\mathbb{Z}_p^2)'$ denote the set of primitive vectors in $\mathbb{Z}_p^2$. Let $D = \mathrm{Meas}((\mathbb{Z}_p^2)'; \mathbb{Z}_p)$ be the group of $\mathbb{Z}_p$-valued measures on $(\mathbb{Z}_p^2)'$. The scalar action of $\mathbb{Z}_p^\times$ on $(\mathbb{Z}_p^2)'$ induces a natural action of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ on $D$. For $\mu \in D$, $\gamma \in \Gamma_0(N)$, and $f \in \mathrm{Cont}((\mathbb{Z}_p^2)')$, we define an action of $\Gamma_0(N)$ on $D$ by $((\mu|\gamma)f)(x, y) = \mu(f|\gamma)(x, y)$. Thus we regard $D$ as a $\mathbb{Z}_p[[\mathbb{Z}_p^\times]][\Gamma_0(N)]$-module.

Put $\Gamma = 1 + p\mathbb{Z}_p$, and $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}_K$ denote its $p$-adic integer ring. Let $\Lambda_K$ be the iwasawa algebra $\mathcal{O}_K[[\Gamma]]$ and $\mathcal{L}_K$ be the fractional field of $\Lambda_K$. Let $\mathcal{K}$ be a finite extension of $\mathcal{L}_K$ and $\mathcal{I}$ be the integral closure of $\Lambda_K$ in $\mathcal{K}$. Define

$$\mathcal{D} := D \otimes_{\mathbb{Z}_p[[\mathbb{Z}_p^\times]]} \mathcal{I}.$$

Let $\Gamma_0(N)$ act on $\mathcal{D}$ through $D$. Then $\mathcal{D}$ is a $\mathcal{I}[\Gamma_0(N)]$-module.

For an arithmetic point $P$ of signature $(n, \epsilon)$ in $\mathcal{X}(\mathcal{I})$, we factor $\epsilon = \epsilon_p \epsilon_N$, where $\epsilon_N$ is defined modulo $N$ and $\epsilon_p$ is defined modulo a power of $p$. And we denote $\mathcal{O}_P = \mathcal{I}/P$. Define a map $\phi_P : \mathcal{D} \to L_{n,\epsilon}(\mathcal{O}_P)$ by

$$\phi_P(\mu \otimes \alpha) = P(\alpha) \cdot \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \epsilon_p(x) \cdot F_n(x, y) d\mu(x, y). \tag{32}$$

It induces a map on cohomologies:

$$\phi_{P,*} : H_c^1(\Gamma_0(N), \mathcal{D}) \to H_c^1(\Gamma_0(Np^r), L_{n,\epsilon}(\mathcal{O}_P)). \tag{33}$$

Where $r$ is the integer such that $\epsilon_p$ is defined modulo $p^r$.

Here $H^1_c(\Gamma_0(N), \mathcal{D}) \cong \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{D})$ is a space $\Lambda$-adic modular symbols.

Suppose $h^0(N, \mathcal{O}_K)$ is the universal ordinary $p$-adic Hecke algebra of level $N$. and $\lambda : h^0(N, \mathcal{O}_K) \to \mathcal{I}$ is a homomorphism of $\Lambda_K$-algebras. Let $\mathbf{f}$ be the $\Lambda$-adic cusp form corresponding to $\lambda$. It's a Hida family of $p$-adic cusp forms. For each arithmatic point $P \in \mathcal{X}(\mathcal{I})$, the associated cohomology class $\varphi_{\mathbf{f}_P}$ is defined over $\mathcal{O}_{\mathbf{f}_P}$.

**Theorem 4.7 ([GS])** *For each arithmetic point $P \in \mathcal{X}(\mathcal{I})$, there is a nontrivial cohomology class $\Phi_{\mathbf{f}} \in H^1_c(\Gamma_0(N), \mathcal{D})$ and a p-adic period $\Omega_P \in \mathcal{O}_P$, such that $\Phi_{\mathbf{f}}(P) = \Omega_P \cdot \varphi^-_{\mathbf{f}_P} = \frac{\Omega_P}{\Omega^-_{\mathbf{f}_P}} \cdot \psi^-_{\mathbf{f}}$.*

**4.4.2.** Let $\sigma : \mathcal{O}_K[[\Gamma]] \to \mathcal{O}_K[[\Gamma]]$ be the ring homomorphism associated to the group homomorphism $t \mapsto t^2$ on $\Gamma$. For each quadratic form $Q \in \mathcal{F}_{Np}$, we put $[Q]_N := [a]_N \in \Delta_N$. For each $\mu \in D$, we define a $\mathbb{Z}_p$-linear map

$$j_Q : D \to \mathrm{Meas}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$$

$$\mu \mapsto j_Q(\mu)(f) = \mu(f \circ Q)$$

for any continuous function $f : \mathbb{Z}_p^\times \to \mathbb{Z}_p$. For each $t \in \mathbb{Z}_p^\times$, by definition we have

$$j_Q([t]_N \cdot \mu) = [t^2]_N \cdot j_Q(\mu).$$

We extend $j_Q$ to $J_Q : \mathcal{D} \to \Lambda_K \otimes_\sigma \mathcal{I}$ by :

$$J_Q(\mu \otimes \alpha \otimes \beta) = j_Q(\mu) \cdot \sigma(\alpha) \otimes_\sigma \beta.$$

Put $\mathcal{I}_\sigma = \Lambda_K \otimes_\sigma \mathcal{I}$. For each $\Phi \in H_c^1(\Gamma_0(N), \mathcal{D})$, we define

$$J(\Phi, Q) := J_Q(\Phi(\partial C_Q)) \in \mathcal{I}_\sigma.$$

It's well defined, since $\partial C_Q \in D_0$, the subgroup of $D$ consisting of divisors of degree 0. And $J(\Phi, Q)$ only depends on the $\Gamma_0(Np)$-equivalent class of $Q$.

The ring homomorphism $\mathcal{I} \to \mathcal{I}_\sigma$ by $\alpha \mapsto \alpha \otimes 1$ is not a homomorphism of $\Lambda_K$-algebras. If $P' \in \mathcal{X}(\mathcal{I}_\sigma)$ lies over $P \in \mathcal{X}(\mathcal{I})$, the signature $(2k, \epsilon^2)$ of $P$ is twice the signature $(k, \epsilon)$ of $P'$.

We define $\Theta : H_c^1(\Gamma_0(N), \mathcal{D}) \to \mathcal{I}_\sigma[[q]]$ by :

$$\Theta(\Phi) = \sum_{Q \in \mathcal{F}_{Np}/\Gamma_0(Np)} J(\Phi, Q) q^{\delta_Q/Np}.$$

**Theorem 4.8 ([Ste])** *Suppose $\Theta(\Phi_{\mathbf{f}}) = \sum_{n\geq 1} \alpha_n q^n \in \mathcal{I}_\sigma[[q]]$. For each arithmetic point $P' \in Spec(\mathcal{I}_\sigma)$ with signature $(k, \epsilon)$, we have*

$$\Theta(\Phi_{\mathbf{f}})(P') = \sum_{n=1}^{\infty} \alpha_n(P') q^n \in S_{k+\frac{1}{2}}(\Gamma_0(4Np^r), \epsilon'; \overline{\mathbb{Q}}_p).$$

*Moreover, if $P$ is the image of $P'$ in $Spec(\mathcal{I})$, then there is a $p$-adic period $\Omega_P \in \overline{\mathbb{Q}}_p$, such that*

$$\Theta(\Phi_{\mathbf{f}})(P')|T_p^{r-1} = \frac{\Omega_P}{\Omega^-_{\mathbf{f}_P}} \cdot \theta_{k,\epsilon}(\mathbf{f}_P).$$

*Where $r$ is the smallest positive integer for which $\epsilon$ is defined modulo $p^r$.*

By Stevens, $\Theta(\Phi)$ is called a $\Lambda$-adic Shintani lifting of $\Phi$.

## 4.5 A $\Lambda$-adic Saito-Kurokawa Lifting

**4.5.1** Let's first recall the classical construction of $\Lambda$-adic Eisenstein series. For each integer $k > 2$, we put

$$E_k(z) = 2^{-1}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_m(n) = \sum_{0<d|n} d^m$ is the sum of $m$-th powers of divisors of $n$. We remove the powers of the divisors divisible by $p$ to get the modified coefficient :

$$\sigma'_m(n) = \sum_{0<d|n,(d,p)=1} d^m,$$

which viewed as a function on the weight $m$ is $p$-adic continuous.

Fix a topological generator $u$ in $\Gamma$. Let log be the $p$-adic logarithm function. We define a function $s : \Gamma \to \mathbb{Z}_p$ by $s(z) = \log(z)/\log(u)$. For any element $z \in \Gamma$, we may write $z = u^{s(z)}$. Let $\langle x \rangle = \omega(x)^{-1}x$ donte the projection from $\mathbb{Z}_p^\times$ to $\Gamma$, where $\omega : \mathbb{Z}_p^\times \to \mu_{p-1}$ is the Teichmuller character. For any integer $d$ prime to $p$, we define a function by

$$A_d(X) = d^{-1}(1+X)^{s(\langle d \rangle)}.$$

Easy to check that $A_d(u^k - 1) = d^{-1}\langle d \rangle^k = \omega(d)^{-k} d^{k-1}$. We regard $A_d(X)$ as an element of $\Lambda$ by identifying $\Lambda = \mathbb{Z}_p[[X]]$.

Suppose $\psi = \omega^a$ is an even Dirichlet character. There exists a power series $\Phi_\psi(X)$ in $\mathbb{Z}_p[[X]]$, such that for all integer $k > 1$,

$$\Phi_\psi(u^k - 1) = (1 - \psi\omega^{-k}(p)p^{k-1}) L(1-k, \psi\omega^{-k})$$

if $\psi \neq \mathrm{id}$; and

$$\Phi_{\mathrm{id}}(u^k - 1) = (u^k - 1)(1 - \omega^{-k}(p)p^{k-1})L(1 - k, \omega^{-k}).$$

We put $A_\psi(0; X) = \Phi_\psi(X)/2$ if $\psi \neq \mathrm{id}$ and $A_{\mathrm{id}} = \Phi_{\mathrm{id}}(X)/2X$. For every positive integer $n$, we define

$$A_\psi(n; X) = \sum_{0<d|n,(d,p)=1} \psi(d) A_d(X).$$

For each even character $\psi = \omega^a$, we define the $\Lambda$-adic Eisenstein series $E(\psi)(X) \in \Lambda[[q]]$ by

$$E(\psi)(X) = \sum_{n=0}^{\infty} A_\psi(n; X) q^n.$$

**Proposition 4.1** *For each positive even integer $k \geq 2$ with $k \equiv a \bmod p-1$, we have*

$$E(\psi)(u^k - 1) = E_k(z) - p^{k-1}E_k(pz) \in M_k(\Gamma_0(p))$$

*in* $\mathbb{Q}[[q]]$.

**4.5.2** Let $A_2$ be the semigroup of symmetric, semi-definite positive, half-integral matrices of size 2. Let $\Lambda$ be the one variable power series ring $\mathcal{O}[[X]]$ with coefficients in $\mathcal{O}$. We fix a character $\chi = \omega^a$ of $(\mathbb{Z}/p\mathbb{Z})^\times$, where $a$ is some integer with $0 \leq a \leq p-1$, and $\omega : \mathbb{Z}_p^\times \to \mu_{p-1}$ is the Teichmuller character. Following Panchishkin and Kitagawa [Pan], we call a formal power series $F$ of $\Lambda[[q^{A_2}]]$ a $\Lambda$-adic Siegel modular form with character $\chi$ if $F(P)$ gives a $q$-expansion of

Siegel modular forms in $\mathcal{M}_{k(P)}(\Gamma_0^2(p^{r(P)}), \epsilon_P \chi \omega^{-k(P)}; \mathcal{O})$ for all $P \in \mathcal{X}_{\mathrm{alg}}(\Lambda)$ with $k(P) \geq 5$. We denote by $\mathcal{M}^s(\chi; \Lambda)$ the $\Lambda$-submodule of $\Lambda[[q^{A_2}]]$ generated by $\Lambda$-adic Siegel modular forms. A $\Lambda$-adic Siegel modular form $F$ is called a $\Lambda$-adic Siegel cusp form if $F(P)$ is a cusp form for any $P \in \mathcal{X}_{alg}(\Lambda)$ with $k(P) \geq 5$. We may regard it as a power series in $\Lambda[[q^{A_2}]]$.

For every $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in A_2$ with $a, b, c \in \mathbb{Z}$, we say $d|T$ if $d|(a, b, c)$ for any integer $d$. Suppose $\psi = \omega^a$ is an even Dirichlet character. For each power series $\Theta = \sum_n \alpha(n) q^n \in \mathcal{I}_\sigma[[q]]$, and for each $T \in A_2$, we define an element

$$\omega_\psi(\Theta, T) := \sum_{\substack{0<d|T \\ (d,p)=1}} \psi(d) A_d(X) \alpha(\frac{\det(2T)}{d^2}) \in \mathcal{I}_\sigma.$$

Now we define a map $\mathbf{SK} : \mathcal{I}_\sigma[[q]] \to \mathcal{I}_\sigma[[q^{A_2}]]$ by

$$\mathbf{SK}(\Theta, \psi) := \sum_{T \in A_2} \omega_\psi(\Theta, T) q^T. \tag{34}$$

Let $\mathbf{f}$ be the $\Lambda$-adic cusp form associated to $\lambda$ as in section 4.4. Let $\Theta = \Theta(\Phi_{\mathbf{f}})$ be the $\Lambda$-adic Shintani lifting of $\mathbf{f}$. We denote $\mathbf{SK}(\mathbf{f}, \psi) = \mathbf{SK}(\Theta_{\mathbf{f}}, \psi)$, and put $\delta_p = \mathrm{diag}(p, p, 1, 1) \in \mathrm{M}_4(\mathbb{Z})$. For each arithmetic point $P \in \mathcal{X}_{alg}(\mathcal{I})$ with signature $(2k, \mathrm{id})$, suppose $P^1 \in \mathcal{X}(\mathcal{I}_\sigma)$ and $P^2 \in \mathcal{X}(\mathcal{I}_\sigma \otimes_\Lambda \Lambda)$ lie over $P$, then $P^1$ has signature $(k, \mathrm{id})$ and $P^2$ has signature $(k+1, \mathrm{id})$.

**Theorem 4.9** *Let $\psi = \omega^a$ be an even Dirichlet character with $0 \leq a \leq p-1$. For each arithmetic point $P \in \mathcal{X}(\mathcal{I})$ with signature $(2k, id)$, satisfying $a \equiv k+1$*

mod $p-1$, *we have*

$$\boldsymbol{SK}(\mathbf{f},\psi)(P^2)=\frac{\Omega_P}{\Omega^-_{\mathbf{f}_P}}(SK(\mathbf{f}_P)-SK(\mathbf{f}_P)|\delta_p)\in S_{k+1}(\Gamma_0^2(Np);\mathcal{O}_K) \tag{35}$$

We call $\mathbf{SK}(\mathbf{f},\psi)$ a $\Lambda$-adic Saito-Kurokawa lifting of $\mathbf{f}$.

**Proof**. Since $\Theta(P^1)$ is a half-integral cusp form in Kohnen's +-space, we may write its $q$-expansion as

$$\Theta(P^1)=\sum_{0<n\equiv 0,3 \mod 4} c(n)q^n\in S^+_{k+1/2}(\Gamma_0(4Np^r);\mathcal{O}_K).$$

By Theorem 3.4, the power series defined by

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z}\\ r^2<4n}} c(4n-r^2)q^n\zeta^r$$

is a Jacobi cusp form in $J_{k+1,1}(Np^r;\mathcal{O}_K)$. For each positive integer $m$, recall the index changing operator $V_m : J_{k+1,1}\to J_{k+1,m}$ defined by equation (11) in scetion 2.3. Applying Theorem 3.6, we have

$$\begin{aligned}F_{k+1}(Z):=\mathcal{V}\phi(Z)&=\sum_{m\geq 0}(\phi|V_m)(\tau,z)e(m\tau')\\ &=\sum_{n,r,m}\left(\sum_{d|(n,r,m)} d^{k-1}c(\frac{4mn-r^2}{d^2})\right)e(n\tau+rz+m\tau')\end{aligned}$$

is a Siegel cusp form in $S_{k+1}(\Gamma_0(Np^r);\mathcal{O}_K)$.

On the other side, for $T=\begin{pmatrix} n & r/2\\ r/2 & m\end{pmatrix}\in A_2$, we have

$$\omega_\psi(\Theta,T)(P^1)=\sum_{\substack{0<d|T\\ (d,p)=1}} d^k c(\frac{4mn-r^2}{d^2}).$$

Therefore,

$$\mathbf{SK}(\mathbf{f},\psi)(P^2) = \sum_{T\in A_2} \omega(\Theta,T)(P^1)q^T = F_{k+1}(Z) - p^k F_{k+1}(pZ).$$

Note that $p^k F_{k+1}(pZ) = F_{k+1}|_{k+1}\delta_p(Z)$. Thus $F_{k+1}(Z) - p^k F_{k+1}(pZ)$ is a Siegel modular form for

$$\Phi = \Gamma_0^2(Np^r) \cap \delta_p^{-1}\Gamma_0^2(Np^r)\delta_p.$$

Easy to see that $\Phi$ contains $\Gamma_0^2(Np^{r+1})$. This completes the proof.

**4.5.3** Let $\pi : A_2 \to \mathbb{N}\times\mathbb{N}$ be the diagonal projection map:

$$\pi : \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \mapsto (n,m).$$

Note that for each pair $(n,m) \in \mathbb{N}\times\mathbb{N}$, the fiber $\pi^{-1}(n,m)$ is a finite set:

$$\{r \in \mathbb{Z}; r^2 \leq 4mn\}.$$

For each $\Lambda$-adic form $F = \sum A(T,F)q^T \in \Lambda[[q^{A_2}]]$, we define a power series $\varpi(F) \in \Lambda[[q^{\mathbb{N}\times\mathbb{N}}]]$ by

$$\sum_{n,m\in\mathbb{N}} a(n,m,\varpi(F))q_1^n q_2^m = \sum_{n,m\in\mathbb{N}} \left( \sum_{r\in\pi^{-1}(n,m)} A((n,r,m),F) \right) q_1^n q_2^m. \qquad (36)$$

Then $\varpi$ is a $\Lambda$-module homomorphism.

**Proposition 4.2** *If $F$ is an ordinary Siegel cusp form, then*

$$\varpi(F) \in S^o(N;\Lambda) \otimes S^o(N;\Lambda).$$

**Proof**. Following Panchishkin's argument in [Pan], and applying the patching lemma by Hida and Wiles, we have $\varpi(F) \in S^o(N;\Lambda) \otimes S(N;\Lambda)$. For each arithmetic point $P \in \mathcal{X}(\mathcal{I})$,

$$(T(p) \otimes 1)\varpi(F)(P) = (1 \otimes T(p))\varpi(F)(P).$$

Hence $\varpi(F) \in S^o(N;\Lambda) \otimes S(N;\Lambda) \cong S^o(N;\Lambda) \otimes S^o(N;\Lambda)$.

# 5 One Variable $p$-adic $L$-Functions

In this chapter, we construct $p$-adic measures attached to modular symbols and to modular forms, to define a one variable $p$-adic $L$-function for $\mathrm{Sp}_2 \times \mathrm{GL}_2$.

## 5.1 $p$-adic measures attached to modular symbols

We recall the two-variable $p$-adic $L$-function by Greenberg and Stevens in [GS]. Put $\mathcal{X}_0 = \mathrm{Spec}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$, and $\mu_\Phi = \Phi(\{0\} - \{i\infty\}) \in D$ for any modular symbol $\Phi \in \mathrm{Symb}_{\Gamma_0(N)}(D)$. It's called the special value of the $L$-function by Greenberg and Stevens.

Define the standard two-variable $p$-adic $L$-function $L_p(\Phi)$ on $\mathcal{X}_0 \times \mathcal{X}_0$ by

$$L_p(\Phi, P, \delta) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} P(x)\delta(y/x) d\mu_\Phi(x, y),$$

for $(P, \delta) \in \mathcal{X}_0 \times \mathcal{X}_0$.

Moreover, for fixed $\delta$, there's a unique $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$-morphism

$$L_p(\cdot, \delta) : \mathrm{Symb}_{\Gamma_0(N)}(D) \to \mathbb{Z}_p[[\mathbb{Z}_p^\times]], \tag{37}$$

such that $L_p(\Phi, \delta)(P) = L_p(\Phi, P, \delta)$ for all $P \in \mathcal{X}_0$. We extend the map (37) by $\mathcal{I}$-linearity:

$$L_p(\cdot, \delta) : H_c^1(\Gamma_0(N), \mathcal{D}) \to \mathcal{I}. \tag{38}$$

Then we associate to each $\Phi \in H^1_c(\Gamma_0(N), \mathcal{D})$ a two-variable $p$-adic $L$-function $L_p(\Phi)$ on $\mathcal{X}(\mathcal{I}) \times \mathcal{X}_0$ by

$$L_p(\Phi, P, \delta) = L_p(\Phi, \delta)(P),$$

for $(P, \delta) \in \mathcal{X}(\mathcal{I}) \times \mathcal{X}_0$.

Let $\mathbf{f}$ and $\Phi = \Phi_{\mathbf{f}}$ have the same meaning in Chapter 4.

**Theorem 5.1 ([GS])** *The two-variable p-adic L-function $L_p(\Phi_{\mathbf{f}})$ interpolates critical values of the $\Lambda$-adic form $\mathbf{f}$. Suppose $\delta$ has signature $(s_0, \psi\omega^{1-s_0})$, and $\psi$ is an even character. For each arithmetic point $P \in \mathcal{X}(\mathcal{I})$, there exist a p-adic period $\Omega_P \in \overline{\mathbb{Q}}_p$ and a complex period $\Omega_{\mathbf{f}_P^+} \in \mathbb{C}$, such that*

$$L_p(\Phi_{\mathbf{f}}, P, \delta) =$$

$$\Omega_P \cdot a(p, \mathbf{f}_P)^{-r}(1 - a(p, \mathbf{f}_P)\psi\omega^{1-s_0}(p)p^{s_0-1})\frac{\tau(\psi\omega^{1-s_0})(-1)^{\frac{s_0-1}{2}}}{M}\frac{A(s_0, \psi\omega^{1-s_0}, \mathbf{f}_P)}{\Omega_{\mathbf{f}_P^+}},$$

*where $M$ is the conductor of $\psi\omega^{1-s_0}$.*

Let's fix a fundamental discriminant $-D < 0$, with $-D \equiv 1 \bmod 8$, such that the complete $L$-function $A(k, f, \chi_{-D}) \neq 0$, where $\chi_{-D}$ is the Dirichlet character associated to $\mathbb{Q}(\sqrt{-D})/\mathbb{Q}$. Suppose $P \in \mathcal{X}(\mathcal{I})$, with signature $(2k, \mathrm{id})$. Put $\delta = P_{k-1,\chi_{-D}} \in \mathcal{X}_0$. Then $\delta$ is totally determined by the weight of $P$, we denote $\delta(P) = \delta$. It's the central value of the $L$-function. We then define a one-variable $p$-adic $L$-function $L_p(\mathbf{f})(P) = L_p(\Phi_{\mathbf{f}}, P, \delta(P))$. Here's the specialization to central values :

$$L_p(\Phi_{\mathbf{f}})(P) = \Omega_P \cdot a(p, \mathbf{f}_P)^{-r} \cdot (1 - \chi_{-D}(p) a(p, \mathbf{f}_P)^{-1} p^{k-1}) \cdot \frac{\tau(\chi_{-D})(-1)^{\frac{k-1}{2}}}{D} \cdot \frac{A(k, \chi_{-D}, \mathbf{f}_P)}{\Omega^{+}_{\mathbf{f}_P}}. \tag{39}$$

Here, $r$ is a positive integer such that $\chi_{-D}$ is of conductor $p^r$ on $\mathbb{Z}_p^\times$, and $\tau(\chi_{-D})$ is the Gauss sum w.r.t $\chi_{-D}$.

## 5.2 Linear form

Denote $\mathcal{L}_K$ as the fractional field of $\Lambda_K$. Let $\mathcal{M}$ be a finite extension of $\mathcal{L}_K$ defined over $K$, and $\mathcal{J}$ be the integral closure of $\Lambda_K$ in $\mathcal{M}$.

Suppose $\lambda' : h^o(N; \mathcal{O}_K) \to \mathcal{J}$ is a homomorphism of $\Lambda_K$-algebras. Let $\mathbf{g}$ be the $\Lambda$-adic cusp form corresponding to $\lambda'$.

There's an isomorphism $h^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{M} \cong \mathcal{M} \oplus \mathcal{B}$. Let $\mathbf{1}_{\mathbf{g}}$ be the idempotent corresponding to the first factor of the above decomposition.

Put $h^o_{\mathcal{J}} = h^o(N; \mathcal{O}_K) \otimes_{\Lambda_K} \mathcal{J}$, and denote its image in $\mathcal{B}$ as $h(\mathcal{B})$. Then the congruence module is defined as $\mathcal{C} = (\mathcal{J} \oplus h(\mathcal{B}))/h^o_{\mathcal{J}}$ by Hida. It's a $\mathcal{J}$-torsion module. Fix a generator $H_{\mathbf{g}}$ of the annihilator of the congruence module $\mathcal{C}$.

Let $\mathbf{g}_i \in S^o(N; \Lambda_K)$ $(i = 0, 1, 2, \cdots, m)$ be a basis, and suppose $\mathbf{g} = \mathbf{g}_0$. Define $\Lambda_K$-adic linear forms $l^i : S^o(N; \Lambda_K) \to \Lambda_K$ by

$$l^{(i)}(\mathbf{g}_j) = \delta_{i,j}.$$

Suppose $\mathbf{SK}(\mathbf{f}, \omega^a)$ is a $\Lambda$-adic Saito-Kurokawa lifting of $\mathbf{f}$, where $\omega$ is the Teichmuller character and $a$ is an integer. $\varpi(\mathbf{SK}(\mathbf{f}, \omega^a))$ is its pull back on $\mathcal{H}\times\mathcal{H}$. We define a linear form

$$\langle \varpi(\mathbf{SK}(\mathbf{f}), \omega^a), \mathbf{g}\times\mathbf{g}\rangle := l^{(0)} \times l^{(0)}(\varpi(\mathbf{SK}(\mathbf{f}), \omega^a)|H_{\mathbf{g}}\cdot 1_{\mathbf{g}_Q} \times H_{\mathbf{g}}\cdot 1_{\mathbf{g}_Q}) \in \mathcal{I}\hat{\otimes}\mathcal{J}.$$

For any arithmetic points $P \in \mathcal{X}(\mathcal{I})$ and $Q \in \mathcal{X}(\mathcal{J})$ with matching weights $k(P) + 2 = 2k(Q)$, $\epsilon_P = \epsilon_Q = \mathrm{id}$, suppose that $a \equiv k(Q) \mod p-1$, then we have the specialization expression:

$$\begin{aligned}
&\langle \varpi(\mathbf{SK}(\mathbf{f}), \omega^a), \mathbf{g}\times\mathbf{g}\rangle(P, Q) \\
&= H(Q)^2 \cdot \frac{\Omega_P}{\Omega^-_{\mathbf{f}_P}} \cdot \frac{\langle \varpi(\mathrm{SK}(\mathbf{f}_P)) - \varpi(\mathrm{SK}(\mathbf{f}_P))|\delta_p, h_Q\times h_Q\rangle}{\langle \mathbf{g}_Q\times\mathbf{g}_Q, h_Q\times h_Q\rangle} \\
&= H(Q)^2 \cdot \frac{\Omega_P}{\Omega^-_{\mathbf{f}_P}} \cdot (1 - p^{-2}a(p, \mathbf{g}_Q)^2) \cdot \frac{\langle \varpi(\mathrm{SK}(\mathbf{f}_P)), \mathbf{g}_Q\times\mathbf{g}_Q\rangle}{\langle \mathbf{g}_Q, \mathbf{g}_Q\rangle^2}
\end{aligned} \tag{40}$$

where $h_Q = \mathbf{g}^\rho_Q|\tau_{Np}$, and $\tau_{Np} = \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}$.

## 5.3 One variable $p$-adic $L$-functions for $\mathbf{Sp}_2 \times \mathbf{GL}_2$

Suppose $\Theta(\Phi_{\mathbf{f}})$ is the $\Lambda$-adic Shintani lifting of $\mathbf{f}$. Let $\alpha_D$ be its $D$-th Fourier coefficient. For an arithmetic point $P \in \mathcal{X}(\mathcal{I})$, suppose $h$ is the algebraic Shintani lifting of $\mathbf{f}_P$. Assume $P^1 \in \mathcal{X}(\mathcal{I}_\delta)$ is an arithmetic point lying over $P$, by Theorem 4.8, we have

$$\alpha_D(P^1) = \frac{\Omega_P}{\Omega^-_{\mathbf{f}_P}} \cdot c_h(D).$$

Fix a $\Lambda$-adic form $\mathbf{f}$, we define a $p$-adic function $\alpha_D(\mathbf{f}, P) = \alpha_D(P^1)$ on $\mathcal{X}(\mathcal{I})$.

Put $\tau_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. For a normalized eigenform $f \in S_{2k}(\Gamma_0(N))$, it's well-known that $a(n, f|\tau_N)$ is the complex conjugate of $a(n, f)$ for any $n \in \mathbb{N}$. Here $a(n, f)$ and $a(n, f|\tau_N)$ are eigenvalues of the Hecke operator $T(n)$, such that $T(n)f = a(n, f)f$ and $T(n)f|\tau_N = a(n, f|\tau_N)$.

Suppose $h'$ is the Shintani lifting of $f|\tau_N$. It's not clear that $c_{h'}(n)$ is the complex conjugate of $c_h(n)$ for any $n \in \mathbb{N}$.

Combining the $p$-adic $L$-function of a $\Lambda$-adic form $\mathbf{f}$ by Greenberg and Stevens, and the linear form in Section 5.2, we now define a one-variable $p$-adic $L$-function $L_p(\mathrm{Sym}^2(\mathbf{g}) \otimes \mathbf{f})$ for $\mathrm{Sp}_2 \times \mathrm{GL}_2$ on $\mathrm{Spec}(\mathcal{J}) \times \mathrm{Spec}(\mathcal{I})$ by

$$\begin{aligned} L_p(\mathrm{Sym}^2(\mathbf{g}) \otimes \mathbf{f})(Q, P) &:= \alpha_D(\mathbf{f}, P)^{-1} \cdot \alpha_D(\mathbf{f}|\tau_N, P)^{-1} \cdot L_p(\mathbf{f})(P) \\ &\qquad \langle \mathbf{SK}(\mathbf{f}), \mathbf{g} \times \mathbf{g} \rangle (P, Q) \cdot \langle \mathbf{SK}(\mathbf{f}|\tau_N), \mathbf{g}|\tau_N \times \mathbf{g}|\tau_N \rangle (P, Q). \end{aligned}$$

**Conjecture 2** *For any arithmetic point $P \in \mathcal{X}(\mathcal{I})$ and $Q \in \mathcal{X}(\mathcal{J})$ with matching weights $k(P) + 2 = 2k(Q)$, $\epsilon_P = \epsilon_Q = id$, suppose that $a \equiv k(Q) \bmod p - 1$, then we have the specialization of the $p$-adic $L$-function to central values:*

$$L_p(Sym^2(\mathbf{g}) \otimes \mathbf{f})(Q, P) = \mathbf{t} \cdot H(Q)^4 \Omega_P \frac{A(2k, Sym^2(\mathbf{g}_Q) \otimes \mathbf{f}_P)}{\langle \mathbf{g}_Q, \mathbf{g}_Q \rangle^2 \Omega^+_{\mathbf{f}_P}},$$

*where*

$$\mathbf{t} = \frac{\tau(\chi_{-D})(-1)^{\frac{k-1}{2}}}{2^{2k}\sqrt{D}\xi_N} \cdot W(\mathbf{g}_Q)^2 a(p, \mathbf{f}_P)^{-r}(1 - \chi_{-D}(p) a(p, \mathbf{f}_P)^{-1} p^{k-1})(1 - a(p, \mathbf{g}_Q)^2 p^{-2})^2,$$

*and $W(\mathbf{g}_Q)$ is the root number of $\mathbf{g}_Q$.*

The last piece for the proof of this conjecture is to show that the Fourier coefficients of $h'$, the Shintani lifting of $f|\tau_N$, are complex conjugates of the Fourier coefficients of $h$, the Shintani lifting of $f$.

# References

[An1] Andrianov,A.N.: *Euler products corresponding to Siegel modular forms of genius 2*. Russ. Math. Surv. **29** (1974), pp.45-116

[An2] Andrianov,A.N.: *Modular descent and the Saito-Kurokawa conjecture*. Invent. Math. **53** (1979), pp.267-280

[An3] Andrianov,A.N.: *On functional equations satisfied by spinor Euler products for Siegel modular forms of genus 2 with characters*. Abh. Math. Sem. Univ. Hamburg **71** (2001), pp.123-142

[AS] Ash,A., Stevens,G.: *Modular forms in characteristic $\ell$ and special values of their L-functions*. Duke Math. J. **53** (1986), no.3, pp.849-868

[BFH] Bump,D., Friedberg,S., Hoffstein,J.: *Nonvanishing theorems for L-functions of modular forms and their derivatives*. Invent. Math. **102** (1990), pp.543-618

[Bo1] Bocherer,S.: *Uber die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen. II*. Math. Z. **189** (1985), pp.81-110

[Bo2] Bocherer,S.: *Uber die Funktionalgleichung automorpher L-funktionen zur Siegelschen Modulgruppe*. J. Reine Angew. Math. **362** (1985), pp.146-168

[EZ] Eichler,M., Zagier,D.: *The Theory of Jacobi Forms.* Progress in Math. 55 Boston-Birkhauser (1985)

[Ga1] Garrett,P.B.: *Pullbacks of Eisenstein series: applications.* Automorphic forms of several variables. Prog. in Math. 46 Boston-Birkhauser (1984), pp.114-137

[Ga2] Garrett,P.B.: *Decomposition of Eisenstein series: Rankin triple products.* Ann. Math. **125** (1987), pp.209-235

[GP1] Gross,B.H., Prasad,D.: *On the decomposition of a representation of $SO_n$ when restricted to $SO_{n-1}$.* Can. J. Math. **44** (1992), pp.974-1002

[GP2] Gross,B.H., Prasad,D.: *On irreducible representations of $SO_{2n+1} \times SO_{2m}$.* Can. J. Math. **46** (1994), pp.930-950

[GS] Greenberg,R., Stevens,G.: *p-adic L-functions and p-adic periods of modular forms.* Invent. Math. **111** (1993), pp.407-447

[Hei] Heim,B.E.: *Pullbacks of Eisenstein series, Hecke-Jacobi theory and automorphic L-functions.* Automorphic forms, automorphic representations, and arithmetic. Proc. Symp. Pure Math. 66, part 2, Amer. Math. Soc. (1999), pp.201-238

[Hi1] Hida,H.: *A p-adic measure attached to the zeta functions associated with two elliptic modular forms, I.* Invent. Math. **79** (1985), pp.159-195

[Hi2] Hida,H.: *A p-adic measure attached to the zeta functions associated with two elliptic modular forms, II.* Ann. Inst. Fourier. **38** (1988), pp.1-83

[Hi3] Hida,H.: *Iwasawa modules attached to congruences of cusp forms.* Ann. Sci. École. Norm. Sup. (4) **19** (1986), pp.231-273

[Hi4] Hida,H.: *On p-adic Hecke algebras for $GL_2$ over totally real fields.* Ann. of Math. (2) **128** (1988), pp.323-382

[Ich] Ichino,A.: *Pullbacks of Saito-Kurokawa lifts.* Invent. Math. **162** (2005), pp.551-647

[Kit] Kitagawa,K.: *On standard p-adic L-functions of families of elliptic cusp forms.* Contemp. Math. vol.**165** (1994), pp.81-110

[Ko1] Kohnen,W.: *Modular forms of half-integral weight on* $\Gamma_0(4)$. Math. Ann. **248** (1980), pp.249-266

[Ko2] Kohnen,W.: *New forms of half-integral weight.* J. Reine Angew. Math. **333** (1982), pp.32-72

[Ko3] Kohnen,W.: *Fourier coefficients of modular forms of half-integral weight.* Math. Ann. **271** (1985), pp.237-268

[KZ] Kohnen,W., Zagier,D.: *Values of L-series of modular forms at the center of the critical strip.* Invent. Math. **64** (1981), pp.175-198

[Li] Li,Z.: *Pullbacks of Saito-Kurokawa lifts for higher levels.* In preparation.

[Maz] Mazur,B.: *Two variable p-adic L-functions.* unpublished lecture note, 1985

[MR1] Manickam,M.; Ramakrishnan,B.: *On Shimura, Shintani and Eichler-Zagier correspondences.* Trans. Amer. Math. Soc. **352** (2000), no.6, pp.2601-2617

[MR2] Manickam,M.; Ramakrishnan,B.: *On Saito-Kurokawa correspondence of degree two for arbitrary level.* J. Ram. Math. Soc. **17** (2002), no.3, pp.149-160

[MRV] Manickam,M.; Ramakrishnan,B.; Vasudevan,T.C.: *On Saito-Kurokawa descent for conguence subgroups.* Manuscripta Math. **81** (1993), pp.161-182

[Pan] Panchishkin,A.A.: *On the Siegel-Eisenstein measure and its applications.* Isreal J. Math. **120** (2000), pp.467-509

[Ran] Rankin,R.A.: *Contribution to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions. I. II.* Proc. of the Cambridge Phil. Soc. **35** (1939), pp.351-372

[Sh1] Shimura,G.: *Introduction to the arithmetic theory of automorphic functions.* Princeton Univ. Press (1971)

[Sh2] Shimura,G.: *On Modular Forms of Half-Integral Weight.* Ann. of Math., **97**, (1973), pp.440-481

[Sh3] Shimura,G.: *On the periods of modular forms.* Math. Ann. **229** (1977), pp.211-221

[Shi] Shintani,T.: *On construction of holomorphic cusp forms of half integral weight.* Nagoya Math. J. **58** (1975), pp.83-126

[Ste] Stevens,G.: $\Lambda$*-adic modular forms of half-integral weight and a* $\Lambda$*-adic Shintani lifting.* Comtemp. Math. **174** (1994), pp.129-151

[SU] Skinner,C., Urban,E.: *Sur les deformations* $p$*-adiques de certaines representations automorphes.* J. Inst. Math. Jussieu **5**(4) (2006), pp.629-698

[Wal] Waldspurger,J.-L.: *Correspondances de Shimura et quaternions.* Forum Math. **3** (1991), pp.219-307