Computations and structures in $sl(n)$-link homology.

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Abstract

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The thesis studies $sl(n)$ and HOMFLY-PT link homology. We begin by constructing a version of $sln)$-link homology, which assigns the $U(n)$-equivariant cohomology of $\mathbb{CP}^{n-1}$ to the unknot. This theory specializes to the Khovanonv-Rozansky $sl(n)$-homology and we are motivated by the "universal" rank two Frobenius extension studied by M. Khovanov in (20) for $sl(2)$-homology. This framework allows one to work with graded, rather than filtered, objects and should prove useful in investigating structural properties of the $sl(n)$-homology theories.

We proceed by using the diagrammatic calculus for Soergel bimodules, developed by B. Elias and M. Khovanov in (1), to prove that Rouquier complexes, and ultimately HOMFLY-PT link homology, is functorial. Upon doing so we are able to explicitly write the chain map generators of the movie moves and compute over the integers. This is joint work with B. Elias.

In suite, we take the above diagrammatic calculus and construct and integral version of HOMFLY-PT link homology, which we also extend to an integral version of $sl(n)$-homology with the aid Rasmussen's specral sequences between the two (33). We reprove invariance under the Reidemeister moves in this context and highlight the computational power of the calculus at hand.

The last part of the thesis concerns an example. We show that for a particular class of tangles the $sl(n)$-link homology is entirely "local," i.e. has no "thick" edges, and its homology depends only on the underlying Frobenius structure of the algebra assigned to the unknot.

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To my parents,

To my sister,

And to my Bellas.

Chapter 1

Introduction

In the decade following the discovery of the Jones polynomial by V. Jones (14) in 1983 a slew of new invariants in low dimensional topology came to light. Included in this extensive list is a large family, known as "quantum link invariants," which arises from quantum groups and their representations. In brief, given a tangle T and an appropriate collection of representations $V_1, \ldots, V_k$ of a quantum group G, one constructs an element $F(T)$ of the endomorphism ring of $V_1 \otimes \cdots \otimes V_k$, which is an invariant of T. "Closing off" the tangle corresponds to taking trace of this operator $F(T)$, this trace being a polynomial in $R[q, q^{-1}]$ with R generally a field or a commutative ring. The subject of quantum invariants has not only played a pivotal role relating various branches of mathematics and physics such as representation theory, operator algebras, low dimensional topology, and statistical mechanics, but has evolved to be extremely interesting and powerful in its own right.

In (17) M. Khovanov constructed a link homology theory with Euler characteristic the Jones, or $sl(2)$, polynomial. From this work a completely novel viewpoint arose with regards to quantum invariants - the viewpoint of categorification. Roughly speaking, categorification refers to lifting a given mathematical structure to that of a higher order. For example: a natural number can be regarded as the dimension of a vector space or the Euler characteristic of a (co)-homology theory, a polynomial with integral coefficients as the quantum dimension of a vector space or the Euler characteristic of a bi-graded homology theory, a group as the Grothendieck group of some category, a group action as the consequence of functorial actions on some category, etc. One approach to constructing Khovanov homology begins with Kauffman's solid-state model for the Jones polynomial. This is gotten by resolving each crossing of a link diagram in either the oriented or un-oriented resolution and assigning to each the polynomial $(q+q^{-1})^n$ where n is the number of circles resulting in a given resolution; a weighted alternating sum, with weights corresponding to that of each resolution, is taken and after some shifts one arrives at the Jones polynomial. The homology theory takes this alternating sum and lifts it to a complex of tensor products of Frobenius algebras, each of quantum dimension $q + q^{-1}$, with appropriate shifts in the quantum grading; the differential maps arise from the underlying Frobenius algebra structure, anti-commute, and are grading preserving, so that the bi-graded Euler characteristic of the homology theory is the same as that of the complex, i.e. the Jones polynomial. (We will discuss this and other related constructions in great detail in the next chapter.) This categorification of the Jones polynomial was a seminal moment in providing a completely new approach to quantum invariants, has paved the way for a number of other categorification constructions, and has proven to be powerful as well as deeply connected to other branches of mathematics. One example of the breadth of Khovanov homology was seen when, using subsequent work of E. S. Lee (25) on variants of the theory, J. Rasmussen found a purely combinatorial proof of the Milnor conjecture for the slice genus of torus knots (34) (previously proved by P. Kronheimer and T. Mrowka using gauge theory); moreover, applications to transverse links and tight contact structures have also been discovered (32), (3); and perhaps what is most surprising is the existence of a spectral sequence connecting Khovanov homology and Heegaard Floer homology (31). Many papers have been written on or relating to this subject and millions of examples have been calculated, but it is clear that more discoveries are to come.

In the last few years categorification and, in particular, that of topological invariants has flourished into a subject of its own right. This has been a study finding connections and ramifications over a vast spectrum of mathematics, including areas such as low-dimensional topology, representation theory, algebraic geometry, as well as others. Following the original work of M. Khovanov on the categorification of the Jones polynomial in, came a slew of link homology theories lifting other quantum invariants. With a construction that utilized a tool previously developed in an algebra-geometric context - matrix factorizations - M. Khovanov and L. Rozansky produced the $sl(n)$ and HOMFLY-PT link homology theories. Albeit computationally intensive, it was clear from the onset that thick interlacing structure was hidden within. The most insightful and influential work in uncovering these inner-connections was that of J. Rasmussen in (33), where he constructed a spectral sequence from the HOMFLY-PT to the $sl(n)$-link homology. This was a major step in deconstructing the pallet of how these theories come together, yet many structural questions remained and still remain unanswered, waiting for a new approach. Close to the time of the original work, M. Khovanov produced an equivalent categorification of the HOMFLY-PT polynomial in (21), but this time using Hochschild homology of Soergel bimodules and Rouquier complexes of (36). The latter proved to be more computation-friendly and was used by B. Webster to calculate many examples in (43). However, at the present moment the Khovanov and Khovanov-Rozansky approach is by far not the only one; these theories and their generalizations to other representations and Lie algebras have come about via Category O, derived categories of coherent sheaves, perverse sheaves on Grassmanians and Springer fibers, as well as other geometric constructions (see for example (8), (9), (41), (42)).

In the meantime of all this development, a new flavor of categorification came into light. With the work of A. Lauda and M. Khovanov on the categorification of quantum groups in (26), a diagrammatic calculus originating in the study of 2-categories arrived into the foreground. This graphical approach proved quite fruitful and was soon used by B. Elias and M. Khovanov in (1) to rewrite the work of Soergel , and en suite by B. Elias and the author to repackage Rouquier's complexes and to prove that they are functorial over braid-cobordisms (2) (not just projectively functorial as was known before). An immediate advantage to this construction was the inherent ease of calculation, at least comparative ease, and the fact that it worked equally well over the integers as well as over fields.

The thesis is divided into four sections, each centered around a particular result and conjectural motivation. Since most of these require different techniques and at times a rather unique approach, we present the required background information separately for each section, skipping only those details which have already been presented.

We begin with the construction of an "equivariant" version of $sl(n)$-link homology, which was motivated by the study of the following conjecture.

Conjecture 1.1. *Consider the deformation of $sl(n)$-homology that assigns*

$$H_{n,f(x)}(unknot) = \mathbb{Q}[\alpha_1, \dots, \alpha_k][x]/(f(x)) \quad where \quad f(x) = (x-\alpha_1)^{r_1} \dots (x-\alpha_k)^{r_k}.$$

Then

$$H_{n,f(x)}(L) \cong \bigoplus_{\phi:\{c(L)\}\to\{\alpha_i\}} \bigotimes_{i=1}^{k} H_{k_i}(L_i)$$

where the sum is taken over colorings ϕ of the components of L by the roots $\{\alpha_j\}$ and L_i is the sublink corresponding to components colored by α_i, with $H_{k_i}(L_i)$ its $sl(k_i)$-link homology.

The conjecture arose from the work of E. S. Lee (25) on $sl(2)$-link homology and later was supported by M. Mackaay and P. Vaz (29) where they consider the corresponding scenario for $sl(3)$-homology. Such a decomposition theorem would be interesting in its own right, but would also tell a lot about of what to expect from $sl(n)$-homology as a link invariant. The "equivariant" version allows us to consider this deformation in the context of graded rather than filtered objects and is a direct generalization of Khovanov's work on Frobenius extension and link homology (20). The main result is the following:

Theorem 1.2. *For every $n \in \mathbb{N}$ there exists a bigraded homology theory that is an invariant of links, such that*

$$H_n(unknot) = \mathbb{Q}[a_0, \dots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \dots + a_1x + a_0),$$

where setting $a_i = 0$ for $0 \leq i \leq n-2$ in the chain complex gives the Khovanov-Rozansky invariant, i.e. a bigraded homology theory of links with Euler characteristic the quantum sl_n-polynomial $P_n(L)$.

The following chapter is a work joint with B. Elias; as mentioned above, this follows from his and M. Khovanov's diagrammatic calculus for Soergel bimodules. We extend this graphical approach to Rouquier complexes and show the following:

Theorem 1.3. *There is a functor F from the category of combinatorial braid cobordisms to the category of complexes of Soergel bimodules up to homotopy, lifting Rouquier's construction (i.e. such that F sends crossings to Rouquier complexes).*

This functoriality remains true even over Z and, in addition, we are able to exhibit all movie move generators explicitly and calculate the movies with relative ease.

In the next chapter we extend the above results to produce an integral version of HOMFLY-PT homology and utilizing Rasmussen's spectral sequence from HOMFLY-PT to $sl(n)$-link homology construct an integral version of the latter. (Previous integral constructions existed only for the $sl(2)$ and $sl(3)$ variants.) Once again, the graphical calculus plays an invaluable role in the ease of calculations. We prove the following:

Theorem 1.4. *Given a link $L \subset S^3$, the groups $H(L)$ and $\overline{H}(L)$ are invariants of L and when tensored with $\mathbb{Q}$ are isomorphic to the unreduced and reduced versions, respectively, of the Khovanov-Rozansky HOMFLY-PT link homology. Moreover, these integral homology theories give rise to functors from the category of braid cobordisms to the category of complexes of graded R-bimodules.*

Theorem 1.5. *Given a link L, there exists a spectral sequence E^n where the E^1 page is isomorphic to the integral HOMFLY-PT link homology of L, E^n is an invariant of L for all n, and E^∞ categorifies the quantum $sl(n)$-link polynomial $P_L(q^n, q)$.*

The last chapter takes a step back to the original Khovanov-Rozansky construction and focuses on a particular class of tangles where the homology is of a rather unique flavor. Using ideas from (4) we show that for this class of tangles, and hence for knots and links composed of these, the Khovanov-Rozansky complex reduces to one that is quite simple, or one without any "thick" edges. In particular we will consider the tangle in figure 1.1 and show that its associated complex is homotopic to the one below, with some grading shifts and basic maps which we leave out for now. All of the necessary details will be provided within the chapter.

The inherent interest in this particular class of tangles, or what is implied by the above result, comes from the fact that the complexes for the knots and links constructed from them and their mirrors are entirely "local," i.e. to calculate the homology we only need to

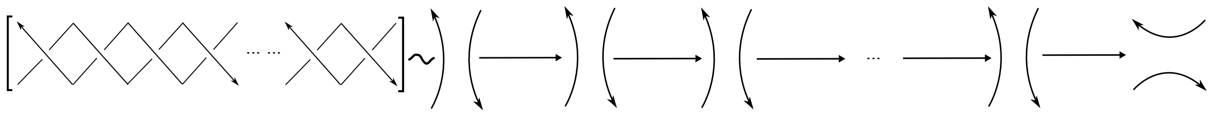


Figure 1.1: Our main tangle and its reduced complex

exploit the Frobenius structure of the underlying algebra assigned to the unknot. Hence, here the calculations and complexity is similar to that of $sl(2)$-homology. We will also discuss a general algorithm, basically the one described in (4), to compute these homology groups in a more time-efficient manner. The chapter will end with a comparison our results with similar computations in the version of $sl(3)$-homology found in (18), which we refer to as the "foam" version (foams are certain types of cobordisms described in the $sl(3)$ paper), and give an explicit isomorphism between the two versions.

Chapter 2

Equivariant $sl(n)$-link homology

2.1 Introduction

In (17), M. Khovanov introduced a bigraded homology theory of links, with Euler characterstic the Jones polynomial, now widely known as “Khovanov homology.” In short, the construction begins with the Kauffman solid-state model for the Jones polynomial and associates to it a complex where the ‘states’ are replaced by tensor powers of a certain Frobenius algebra. In the most common variant, the Frobenius algebra in question is $\mathbb{Z}[x]/(x^2)$, a graded algebra with $\deg(1) = 1$ and $\deg(x) = -1$, i.e. of quantum dimension $q^{-1} + q$, this being the value of the unreduced Jones polynomial of the unknot. This algebra defines a 2-dimensional TQFT which provides the maps for the complex. (A 2-dimensional TQFT is a tensor functor from oriented $(1+1)$-cobordisms to R-modules, with R a commutative ring, that assigns R to the empty 1-manifold, a ring A to the circle, where A is also a commutative ring with a map $\iota : R \longrightarrow A$ that is an inclusion, $A \otimes_R A$ to the disjoint union of two circles, etc.) In (19) M. Khovanov extended this to an invariant of tangles by associating to a tangle a complex of bimodules and showing that that the isomorphism class of this complex is an invariant in the homotopy category. The operation of “closing off” the tangles gave complexes isomorphic to the orginal construction for links.

Variants of this homology theory quickly followed. In (25), E.S. Lee deformed the algebra above to $\mathbb{Z}[x]/(x^2-1)$ introducing a different invariant, and constructed a spectral sequence with E^2 term Khovanov homology and E^∞ term the ‘deformed’ version. Even though this homology theory was no longer bigraded and was essentially trivial, it allowed Lee to prove structural properties of Khovanov homology for alternating links. J. Rasmussen used Lee's construction to establish results about the slice genus of a knot, and give a purely combinatorial proof of the Milnor conjecture (34). In (4), D. Bar-Natan introduced a series of such invariants repackaging the original construction in, what he called, the "world of topological pictures." It became quickly obvious that these theories were not only powerful invariants, but also interesting objects of study in their own right. M. Khovanov unified the above constructions in (20), by studying how rank two Frobenius extensions of commutative rings lead to link homology theories. We give an overview these results below.

Frobenius Extensions Let $\iota : R \longrightarrow A$ be an inclusion of commutative rings. We say that ι is a *Frobenius extension* if there exists an A-bimodule map $\Delta : A \longrightarrow A \otimes_R A$ and an R-module map $\varepsilon : A \longrightarrow R$ such that Δ is coassociative and cocommutative, and $(\varepsilon \otimes Id)\Delta = Id$. We refer to Δ and ε as the *comultiplication* and *trace* maps, respectively.

This can be defined in the non-commutative world as well, see (15), but we will work with only commutative rings. We denote by $\mathscr{F} = (R, A, \varepsilon, \Delta)$ a Frobenius extension together with a choice of Δ and ε, and call $\mathscr{F}$ a *Frobenius system*. Lets look at some examples from (20); we'll try to be consistent with the notation.

- $\mathscr{F}_1 = (R_1, A_1, \varepsilon_1, \Delta_1)$ where $R_1 = \mathbb{Z}$, $A_1 = \mathbb{Z}[x]/(x^2)$ and

$$\varepsilon_1(1) = 0, \quad \varepsilon_1(x) = 1, \quad \Delta_1(1) = 1 \otimes x + x \otimes 1, \quad \Delta_1(x) = x \otimes x.$$

  This is the Frobenius system used in the original construction of Khovanov Homology (17).

- The constuction in (17) also worked for the following system: $\mathscr{F}_2 = (R_2, A_2, \varepsilon_2, \Delta_2)$ where $R_2 = \mathbb{Z}[c], A_2 = \mathbb{Z}[x, c]/(x^2)$ and

$$\varepsilon_2(1) = -c, \quad \varepsilon_2(x) = 1, \quad \Delta_2(1) = 1 \otimes x + x \otimes 1 + cx \otimes x, \quad \Delta_2(x) = x \otimes x.$$

  Here $\deg(x) = 2, \deg(c) = -2$.

- $\mathscr{F}_3 = (R_3, A_3, \varepsilon_3, \Delta_3)$ where $R_3 = \mathbb{Z}[t], A_3 = \mathbb{Z}[x],\ \iota : t \longmapsto x^2$ and

$$\varepsilon_3(1) = 0, \quad \varepsilon_3(x) = 1, \quad \Delta_3(1) = 1 \otimes x + x \otimes 1, \quad \Delta_3(x) = x \otimes x + t1 \otimes 1.$$

  Here $\deg(x) = 2, \deg(t) = 4$ and the invariant becomes a complex of graded, free $\mathbb{Z}[t]$-modules (up to homotopy). This was Bar-Natan's modification found in (4), with t a formal variable equal to 1/8'th of his invariant of a closed genus 3 surface. The framework of the Frobenius system $\mathscr{F}_3$ gives a nice interpretation of Rasmussen's results, allowing us to work with graded rather than filtered complexes, see (20) for a more in-depth discussion.

- $\mathscr{F}_5 = (R_5, A_5, \varepsilon_5, \Delta_5)$ where $R_5 = \mathbb{Z}[h,t], A_5 = \mathbb{Z}[h,t][x]/(x^2 - hx - t)$ and

$$\varepsilon_5(1) = 0, \quad \varepsilon_5(x) = 1, \quad \Delta_5(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1, \quad \Delta_5(x) = x \otimes x + t1 \otimes 1.$$

  Here $\deg(h) = 2, \deg(t) = 4$.

Proposition 2.1. *(M.Khovanov (20)) Any rank two Frobenius system is obtained from $\mathscr{F}_5$ by a composition of base change and twist.*

[Given an invertible element $y \in A$ we can "twist" ε and Δ, defining a new comultiplication and counit by $\varepsilon'(x) = \varepsilon(yx), \quad \Delta'(x) = \Delta(y^{-1}x)$ and, hence, arriving at a new Frobenius system. For example: $\mathscr{F}_1$ and $\mathscr{F}_2$ differ by twisting with $y = 1 + cx \in A_2$.]

We can say $\mathscr{F}_5$ is "universal" in the sense of the proposition, and this sytem will be of central interest to us being the model case for the construction we embark on. For example, by sending $h \longrightarrow 0$ in $\mathscr{F}_5$ we arrive at the system $\mathscr{F}_3$. Note, if we change to a field of characteristic other than 2, h can be removed by sending $x \longrightarrow x - \dfrac{h}{2}$ and by modifying $t = -\dfrac{h^2}{4}$.

Cohomology and Frobenius extensions There is an interpretation of rank two Frobenius systems that give rise to link homology theories via equivariant cohomology. Let us recall some definitions.

Given a topological group G that acts continuously on a space X we define the *equivariant cohomology* of X with respect to G to be

$$H^*_G(X, R) = H^*(X \times_G EG, R),$$

where $H^*(-, R)$ denotes singular cohomology with coefficients in a ring R, EG is a contractible space with a free G action such that $EG/G = BG$, the classifying space of G, and $X \times_G EG = X \times EG/(gx, e) \sim (x, eg)$ for all $g \in G$. For example, if $X = \{p\}$ a point then $H^*_G(X, R) = H^*(BG, R)$. Returning to the Frobenius extension encountered we have:

- $G = \{e\}$, the trivial group. Then $R_1 = \mathbb{Z} = H^*_G(p, \mathbb{Z})$ and $A_1 = H^*_G(\mathbb{S}^2, \mathbb{Z})$.
- $G = SU(2)$. This group is isomorphic to the group of unit quaternions which, up to sign, can be thought of as rotations in 3-space, i.e. there is a surjective map from $SU(2)$ to $SO(3)$ with kernel $\{I, -I\}$. This gives an action of $SU(2)$ on $\mathbb{S}^2$.

  $R_3 = \mathbb{Z}[t] \cong H^*_{SU(2)}(p, \mathbb{Z}) = H^*(BSU(2), \mathbb{Z}) = H^*(\mathbb{HP}^\infty, \mathbb{Z})$,

  $A_3 = \mathbb{Z}[x] \cong H^*_{SU(2)}(\mathbb{S}^2, \mathbb{Z}) = H^*(\mathbb{S}^2 \times_{SU(2)} ESU(2), \mathbb{Z}) = H^*(\mathbb{CP}^\infty, \mathbb{Z}), \quad x^2 = t.$
- $G = U(2)$. This group has an action on $\mathbb{S}^2$ with the center $U(1)$ acting trivially.

  $R_5 = \mathbb{Z}[h, t] \cong H^*_{U(2)}(p, \mathbb{Z}) = H^*(BU(2), \mathbb{Z}) = H^*(Gr(2, \infty), \mathbb{Z})$,

  $A_5 = \mathbb{Z}[h, x] \cong H^*_{U(2)}(\mathbb{S}^2, \mathbb{Z}) = H^*(\mathbb{S}^2 \times_{U(2)} EU(2), \mathbb{Z}) \cong H^*(BU(1) \times BU(1), \mathbb{Z}).$

  $Gr(2, \infty)$ is the Grassmannian of complex 2-planes in $\mathbb{C}^\infty$; its cohomology ring is freely generated by h and t of degree 2 and 4, and $BU(1) \cong \mathbb{CP}^\infty$. Notice that A_5 is a polynomial ring in two generators x and $h - x$, and R_5 is the ring of symmetric functions in x and $h - x$, with h and $-t$ the elementary symmetric functions.

Other Frobenius systems and their cohomological interpretations are studied in (20), but $\mathscr{F}_5$ with its "universality" property will be our starting point and motivation.

$\mathbf{sl_n}$-link homology Following (17), M. Khovanov constructed a link homology theory with Euler characteristic the quantum sl_3-link polynomial $P_3(L)$ (the Jones polynomial is

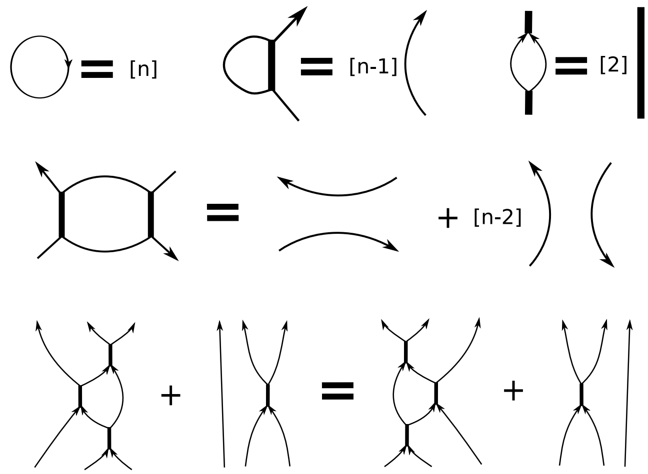


Figure 2.1: MOY graph skein relation $[i] := \dfrac{q^i - q^{-i}}{q - q^{-1}}$

the sl_2-invariant) (18). In succession, M. Khovanov and L. Rozansky introduced a family of link homology theories categorifying all of the quantum sl_n-polynomials and the HOMFLY-PT polynomial, see (23) and (22). The equivalence of the specializations of the Khovanov-Rozansky theory to the original contructions were easy to see in the case of $n = 2$ and recently proved in the case of $n = 3$, see (28).

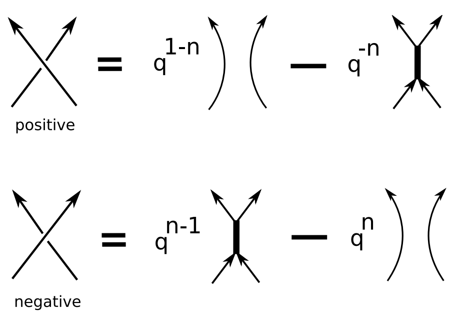


Figure 2.2: Skein formula for $P_n(L)$

The sl_n-polynomial $P_n(L)$ associated to a link L can be computed in the following two ways. We can resolve the crossings of L and using the rules in figure 2.2, with a selected value of the unknot, arrive at a recursive formula, or we could use the Murakami, Ohtsuki, and Yamada (13) calculus of planar graphs (this is the sl_n generalization of the Kauffman solid-state model for the Jones polynomial). Given a diagram D of a link L and resolution Γ

of this diagram, i.e. a trivalent graph, we assign to it a polynomial $P_n(\Gamma)$ which is uniquely determined by the graph skein relations in figure 2.1. Then we sum $P_n(\Gamma)$, weighted by powers of q, over all resolutions of D, i.e.

$$P_n(L) = P_n(D) := \sum_{resolutions} \pm q^{\alpha(\Gamma)} P_n(\Gamma),$$

where $\alpha(\Gamma)$ is determined by the rules in figure 2.2. The consistency and independence of the choice of diagram D for $P_n(\Gamma)$ are shown in (13).

To contruct their homology theories, Khovanov and Rozansky first categorify the graph polynomial $P_n(\Gamma)$. They assign to each graph a 2-periodic complex whose cohomology is a graded $\mathbb{Q}$-vector space $H(\Gamma) = \oplus_{i\in\mathbb{Z}} H^i(\Gamma)$, supported only in one of the cohomological degrees, such that

$$P_n(\Gamma) = \sum_{i\in\mathbb{Z}} dim_{\mathbb{Q}} H^i(\Gamma) q^i.$$

These complexes are made up of *matrix factorizations*, which we will discuss in detail later. They were first seen in the study of isolated hypersurface singularities in the early and mid-eighties, see (11), but have since seen a number of applications. The graph skein relations for $P_n(\Gamma)$ are mirrored by isomorphisms of matrix factorizations assigned to the corresponding trivalent graphs in the homotopy category.

Nodes in the cube of resolutions of L are assigned the homology of the corresponding trivalent graph, and maps between resolutions, see figure 2.3, are given by maps between matrix factorizations which further induce maps on cohomology. The resulting complex is proven to be invariant under the Reidemeister moves. The homology assigned to the unknot is the Frobenius algebra $\mathbb{Q}[x]/(x^n)$, the rational cohomology ring of $\mathbb{CP}^{n-1}$.

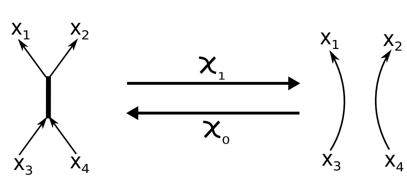


Figure 2.3: Maps between resolutions

The main goal of this chapter is to generalize the above construction by extending the Khovanov-Rozansky homology to that of $\mathbb{Q}[a_0, \ldots, a_{n-1}]$-modules, where the a_i's are coefficients, such that

$$H_n(\emptyset) = \mathbb{Q}[a_0, \ldots, a_{n-1}],$$

$$H_n(unknot) = \mathbb{Q}[a_0, \ldots, a_{n-1}][x]/(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0).$$

Our contruction is motivated by the "universal" Frobenius system $\mathscr{F}_5$ introduced in (20) and its cohomological interpretation, i.e. for every n we would like to construct a homology theory that assigns to the unknot the analogue of $\mathscr{F}_5$ for $n \geq 2$. Notice that,

$$\mathbb{Q}[a_0, \ldots, a_{n-1}] \cong H^*_{U(n)}(p, \mathbb{Q}) = H^*(BU(n), \mathbb{Q}) = H^*(Gr(n, \infty), \mathbb{Q}),$$

$$\mathbb{Q}[a_0, \ldots, a_{n-1}][x]/(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) \cong H^*_{U(n)}(\mathbb{CP}^{n-1}, \mathbb{Q}).$$

In practice, we will change basis as above for $\mathscr{F}_5$, getting rid of a_{n-1}, and work with the algebra

$$H_n(unknot) = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0).$$

Theorem 2.2. *For every $n \in \mathbb{N}$ there exists a bigraded homology theory that is an invariant of links, such that*

$$H_n(unknot) = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0),$$

where setting $a_i = 0$ for $0 \leq i \leq n-2$ in the chain complex gives the Khovanov-Rozansky invariant, i.e. a bigraded homology theory of links with Euler characteristic the quantum sl_n-polynomial $P_n(L)$.

The chapter is organized in the following way: we begin with a review of the basic definitions, work out the necessary statements for matrix factorizations over the ring

$\mathbb{Q}[a_0, \ldots, a_{n-2}]$, assign complexes to planar trivalent graphs and prove MOY-type decompositions. Then we explain how to construct our invariant of links and move on to the proofs of invariance under the Reidemeister moves. We conclude with a discussion of open questions and a possible generalization.

2.2 Matrix Factorizations

Basic definitions: Let R be a Noetherian commutative ring, and let $\omega \in R$. A *matrix factorization* with *potential* ω is a collection of two free R-modules M^0 and M^1 and R-module maps $d^0 : M^0 \to M^1$ and $d^1 : M^1 \to M^0$ such that

$$d^0 \circ d^1 = \omega Id \quad \text{and} \quad d^1 \circ d^0 = \omega Id.$$

The d^i's are referred to as 'differentials' and we often denote a matrix factorization by

$$M = \qquad M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^0$$

Note M^0 and M^1 need not have finite rank, but later when dealing with graded modules we will insist that the gradings are bounded from below, as this is necessary for the proof of proposition 2.4.

A *homomorphism* $f : M \to N$ of two factorizations is a pair of homomorphisms $f^0 : M^0 \to N^0$ and $f^1 : M^1 \to N^1$ such that the following diagram is commutative:

$$\begin{tikzcd}
M^0 \arrow[r, "d^0"] \arrow[d, "f^0"] & M^1 \arrow[r, "d^1"] \arrow[d, "f^1"] & M^0 \arrow[d, "f^0"] \\
N^0 \arrow[r, "d^0"] & N^1 \arrow[r, "d^1"] & N^0.
\end{tikzcd}$$

Let M_ω^{all} be the category with objects matrix factorizations with potential ω and morphisms homomorphisms of matrix facotrizations. This category is additive with the direct sum of two factorizations taken in the obvious way. It is also equipped with a shift functor $\langle 1 \rangle$ whose square is the identity,

$$M\langle 1 \rangle^i = M^{i+1}$$

$$d^i_{M\langle 1\rangle} = -d^{i+1}_M,\ i = 0, 1\ mod\ 2.$$

We will also find the following notation useful. Given a pair of elements $b, c \in R$ we will denote by $\{b, c\}$ the factorization

$$R \xrightarrow{b} R \xrightarrow{c} R.$$

If $\mathbf{b} = (b_1, \ldots, b_k)$ and $\mathbf{c} = (c_1, \ldots, c_k)$ are two sequences of elements in R, we will denote by $\{\mathbf{b}, \mathbf{c}\} := \otimes_i \{b_i, c_i\}$ the tensor product factorization, where the tensor product is taken over R. We will call the pair $(\mathbf{b}, \mathbf{c})$ *orthogonal* if

$$\mathbf{bc} := \sum_i b_i c_i = 0.$$

Hence, the factorization $\{\mathbf{b}, \mathbf{c}\}$ is a complex if and only if the pair $(\mathbf{b}, \mathbf{c})$ is orthogonal. If in addition the sequence $\mathbf{c}$ is R-regular the cohomology of the complex becomes easy to determine. [Recall that a sequence $(r_1, \ldots, r_n)$ of elements of R is called *R-regular* if r_i is not a zero divisor in the quotient ring $R/(r_1, \ldots, r_{i-1})$.]

Proposition 2.3. *If $(\mathbf{b}, \mathbf{c})$ is orthogonal and $\mathbf{c}$ is R-regular then*

$$H^0(\{\mathbf{b}, \mathbf{c}\}) \cong R/(c_1, \ldots, c_k) \textit{ and } H^1(\mathbf{b}, \mathbf{c}) = 0.$$

For more details we refer the reader to (23) section 2.

Homotopies of matrix factorizations: A homotopy h between maps $f, g : M \to N$ of factorizations is a pair of maps $h^i : M^i \to N^{i-1}$ such that $f - g = h \circ d_M + d_N \circ h$ where d_M and d_N are the differentials in M and N respectively.

Example: Any matrix factorization of the form

$$R \xrightarrow{r} R \xrightarrow{\omega} R\ ,$$

or of the form

$$R \xrightarrow{\omega} R \xrightarrow{r} R\ ,$$

with $r \in R$ invertible, is null-homotopic. Any factorization that is a direct sum of these is also null-homotopic.

Let HMF_ω^{all} be the category with the same objects as MF_ω^{all} but fewer morphisms:

$$Hom_{HMF}(M,N) := Hom_{MF}(M,N)/\{null - homotopic\ \ morphisms\}.$$

Consider the free R-module $Hom(M,N)$ given by

$$Hom^0(M,N) \xrightarrow{d} Hom^1(M,N) \xrightarrow{d} Hom^0(M,N)\ ,$$

where

$$Hom^0(M,N) = Hom(M^0,N^0) \oplus Hom(M^1,N^1),$$

$$Hom^1(M,N) = Hom(M^0,N^1) \oplus Hom(M^1,N^0),$$

and the differential given in the obvious way, i.e. for $f \in Hom^i(M,N)$ and $m \in M$

$$(df)(m) = d_N(f(m)) + (-1)^i f(d_M(m)).$$

It is easy to see that this is a 2-periodic complex, and following the notation of (23), we denote its cohomology by

$$Ext(M,N) = Ext^0(M,N) \oplus Ext^1(M,N).$$

Notice that

$$Ext^0(M,N) \cong Hom_{HMF}(M,N),$$

$$Ext^1(M,N) \cong Hom_{HMF}(M,N\langle 1\rangle).$$

Tensor Products: As mentioned above, given two matrix factorizations M_1 and M_2 with potentials ω_1 and ω_2, respectively, their tensor product is given as the tensor product of complexes, and a quick calculation shows that $M_1 \otimes M_2$ is a matrix factorization with potential $\omega_1 + \omega_2$. Note that if $\omega_1 + \omega_2 = 0$ then $M_1 \otimes M_2$ becomes a 2-periodic complex.

To keep track of differentials of tensor products of factorizations we introduce the labelling scheme used in (23). Given a finite set I and a collection of matrix factorizations

M_a for $a \in I$, consider the Clifford ring $Cl(I)$ of the set I. This ring has generators $a \in I$ and relations

$$a^2 = 1, \quad ab + ba = 0, \quad a \neq b.$$

As an abelian group it has rank $2^{|I|}$ and a decomposition

$$Cl(I) = \bigoplus_{J \subset I} \mathbb{Z}_J,$$

where $\mathbb{Z}_J$ has generators - all ways to order the set J and relations

$$a \dots bc \dots e + a \dots cd \dots e = 0$$

for all orderings $a \dots bc \dots e$ of J.

For each $J \subset I$ not containing an element a there is a 2-periodic sequence

$$\mathbb{Z}_J \xrightarrow{r_a} \mathbb{Z}_{J \sqcup \{a\}} \xrightarrow{r_a} \mathbb{Z}_J,$$

where r_a is right multiplication by a in $Cl(I)$ (note: $r_a^2 = 1$).

Define the tensor product of factorizations M_a as the sum over all subsets $J \subset I$, of

$$(\otimes_{a \in J} M_a^1) \otimes (\otimes_{b \in I \setminus J} M_b^0) \otimes_{\mathbb{Z}} \mathbb{Z}_J,$$

with differential

$$d = \sum_{a \in I} d_a \otimes r_a,$$

where d_a is the differential of M_a. Denote this tensor product by $\otimes_{a \in I} M_a$. If we assign a label a to a factorization M we write M as

$$M^0(\emptyset) \longrightarrow M^1(a) \longrightarrow M^0(\emptyset).$$

An easy but useful exercise shows that if M has finite rank then $Hom(M, N) \cong N \otimes_R M_-^*$, where M_-^* is the factorization

$$(M^0)^* \xrightarrow{-(d^1)^*} (M^1)^* \xrightarrow{(d^0)^*} (M^0)^*.$$

Cohomology of matrix factorizations

Suppose now that R is a local ring with maximal ideal $\mathbf{m}$ and M a factorization over R. If we impose the condition that the potential $\omega \in \mathbf{m}$ then

$$M^0/\mathbf{m}M \xrightarrow{d^0} M^1/\mathbf{m}M \xrightarrow{d^1} M^0/\mathbf{m}M \ ,$$

is a 2 periodic complex, since $d^2 = \omega \in \mathbf{m}$. Let $H(M) = H^0(M) \oplus H^1(M)$ be the cohomology of this complex.

Proposition 2.4. *Let M be a matrix factorization over a local ring R, with potential ω contained in the maximal ideal $\mathbf{m}$, and let $r \in R$. The following are equivalent:*

1)$H(M) = 0.$

2)$H^0(M) = 0.$

3)$H^1(M) = 0.$

4) *M is null-homotopic.*

5) *M is isomorphic to a, possibly infinite, direct sum of*

$$M = \qquad R \xrightarrow{r} R \xrightarrow{\omega} R \ ,$$

and

$$M = \qquad R \xrightarrow{\omega} R \xrightarrow{r} R \ .$$

Proof: The proof is the same as in (23), and we only need to notice that it extends to factorizations over any commutative, Noetherian, local ring. The idea is as follows: consider a matrix representing one of the differentials and suppose that it has an entry not in the maximal ideal, i.e. an invertible entry; then change bases and arrive at block-diagonal matrices with blocks representing one of the two types of factorizations listed above (both of which are null-homotpic). Using Zorn's lemma we can decompose M as a direct sum of $M_{es} \oplus M_c$ where M_c is made up of the null-homotopic factorizations as above, i.e. the "contractible" summand, and M_{es} the factor with corresponding submatrix containing no invertible entries, i.e. the "essential" summand. Now it is easy to see that $H(M) = 0$ if and only if M_{es} is trivial. $\square$

Proposition 2.5. *If $f : M \to N$ is a homomorphism of factorizations over a local ring R then the following are equivalent:*

1) *f is an isomorphism in HMF_ω^{all}.*
2) *f induces an isomorphism on the cohomologies of M and N.*

Proof: This is done in (23). Decompose M and N as in the proposition above and notice that the cohomology of a matrix factorization is the cohomology of its essential part. Now a map of two free R-modules $L_1 \to L_2$ that induces an isomorphism on $L_1/\mathbf{m} \cong L_2/\mathbf{m}$ is an isomorphism of R-modules. □

Corollary 2.6. *Let M be a matrix factorization over a local ring R. The decomposition $M \cong M_{es} \oplus M_c$ is unique; moreover if M has finite-dimensional cohomology then it is the direct sum of a finite rank factorization and a contractible factorization.*

Let MF_ω be the category whose objects are factorizations with finite-dimensional cohomology and let HMF_ω be corresponding homotopy category.

Matrix factorizations over a graded ring

Let $R = \mathbb{Q}[a_0, \dots, a_{n-2}][x_1, \dots, x_k]$, a graded ring of homogeneous polynomials in variables $x_1, \dots, x_k$ with coefficients in $\mathbb{Q}[a_0, \dots, a_{n-2}]$. The gradings are as follows: $\deg(x_i) = 2$ and $\deg(a_i) = 2(n-i)$ with $i = 0, \dots n-2$. Furthermore let $\mathbf{m} = \langle a_0, \dots, a_{n-2}, x_1, \dots, x_k \rangle$ the maximal homogeneous ideal, and let $\mathbf{a} = \langle a_0, \dots, a_{n-2} \rangle$ the ideal generating the ring of coefficients.

A matrix factorization M over R naturally becomes graded and we denote $\{i\}$ the grading shift up by i. Note that $\{i\}$ commutes with the shift functor $\langle 1 \rangle$. All of the categories introduced earlier have their graded counterparts which we denote with lowercase. Let hmf_ω^{all} is the homotopy category of graded matrix factorizations with the grading bounded from below.

Proposition 2.7. *Let $f : M \longrightarrow N$ be a homomorphism of matrix factorizations over $R = \mathbb{Q}[\alpha_0, \dots, \alpha_{n-2}][x_1, \dots, x_k]$ and let $\overline{f} : M/\mathbf{a}M \longrightarrow N/\mathbf{a}N$ be the induced map. Then f is an isomorphism of factorizations if and only if $\overline{f}$ is.*

Proof: One only needs to notice that modding out by the ideal $\mathbf{a}$ we arrive at factorizations over $\overline{R} = \mathbb{Q}[x_1, \ldots, x_k]$, the graded ring of homogeneous polynomials with coefficients in $\mathbb{Q}$ and maximal ideal $\mathbf{m}' = \langle x_1, \ldots, x_k \rangle$. Since $\mathbf{a} \subset \mathbf{m}$ and $\mathbf{m}' \subset \mathbf{m}$, $H(M) = H(M/\mathbf{a}M)$ for any factorization and, hence, the induced maps on cohomology are the same, i.e. $H(f) = H(\overline{f})$. Since an isomorphism on cohomology implies an isomorphism of factorizations over R and $\overline{R}$ the proposition follows. $\square$

The matrix factorizations used to define the original link invariants in (23) were defined over $\overline{R}$. With the above proposition we will be able to bypass many of the calculations nessesary for MOY-type decompositions and Reidemeister moves, citing those from the original paper. This simple observation will prove to be one of the most useful.

The category $\mathbf{hmf}_\omega$ is Krull-Schmidt: In order to prove that the homology theory we assign to links is indeed a topological invariant with Euler characteristic the quantum sl_n-polynomial, we first need to show that the algebraic objects associated to each resolution, i.e. to a trivalent planar graph, satisfy the MOY relations (13). Since the objects in question are complexes constructed from matrix factorizations, the MOY decompositions are reflected by corresponding isomorphisms of complexes in the homotopy category. Hence, in order for these relations to make sense, we need to know that if an object in our category decomposes as a direct sum then it does so uniquely. In other words we need to show that our category is Krull-Schmidt. The next subsection establishes this fact for hmf_ω, the homotopy category of graded matrix factorizations over $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, \ldots, x_k]$ with finite dimensional cohomology.

Given a homogeneous, finite rank, factorization $M \in mf_\omega$, and a degree zero idempotent $e : M \longrightarrow M$ we can decompose M uniquely as the kernel and cokernel of e, i.e. we can write $M = eM \oplus (1 - e)M$. We need to establish this fact for hmf_ω; that is, we need to know that given a degree zero idempotent $e \in Hom_{hmf_\omega}(M, M)$ we can decompose M as above, and that this decomposition is unique up to homotopy.

Proposition 2.8. *The category hmf_ω has the idempotents splitting property.*

Proof: We follow (23) . Let $I \subset Hom_{mf_\omega}(M, M)$ be the ideal consisting of maps that induce the trivial map on cohomology. Given any such map $f \in I$, we see that every entry in the matrices representing f must be contained in $\mathbf{m}$, i.e. the entries must be of non-zero degree. Since a degree zero endomorphism of graded factorizations cannot have matrix entries of arbitrarily large degree, we see that there exists an $n \in \mathbb{N}$ such that $f^n = 0$ for every $f \in I$, i.e. I is nilpotent.

Let K be the kernel of the map $Hom_{mf_\omega}(M, M) \longrightarrow Hom_{hmf_\omega}(M, M)$. Clearly $K \subseteq I$ and, hence, K is also nilpotent. Since nilpotent ideals have the idempotents lifting property, see for example (6) Thm. 1.7.3, we can lift any idempotent $e \in Hom_{hmf_\omega}(M, M)$ to $Hom_{mf_\omega}(M, M)$ and decompose $M = eM \oplus (1 - e)M$. □

Proposition 2.9. *The category hmf_ω is Krull-Schmidt.*

Proof: Proposition 8 and the fact that any object in hmf_ω is isomorphic to one of finite rank, having finite dimensional cohomology, imply that the endomorphism ring of any indecomposable object is local. Hence, hmf_ω is Krull-Schmidt. See (6) for proofs of these facts. □

Planar Graphs and Matrix Factorizations

Our graphs are embedded in a disk and have two types of edges, unoriented and oriented. Unoriented edges are called "thick" and drawn accordingly; each vertex adjoining a thick edge has either two oriented edges leaving it or two entering. In figure 5.2 left x_1, x_2 are outgoing and x_3, x_4 are incoming. Oriented edges are allowed to have marks and we also allow closed loops; points of the boundary are also referred to as marks. See for example figure 2.4. To such a graph Γ we assign a matrix factorization in the following manner:

Let

$$P(x) = \frac{1}{n+1}x^{n+1} + \frac{a_{n-2}}{n-1}x^{n-1} + \cdots + \frac{a_1}{2}x^2 + a_0 x.$$

Thick edges: To a thick edge t as in figure 5.2 left we assign a factorization C_t with potential $\omega_t = P(x_1) + P(x_2) - P(x_3) - P(x_4)$ over the ring $R_t = \mathbb{Q}[a_0, \dots, a_{n-2}][x_1, x_2, x_3, x_4]$.

Since $x^k + y^k$ lies in the ideal generated by $x + y$ and xy we can write it as a polynomial $g_k(x + y, xy)$. More explicitly,

$$g_k(s_1, s_2) = s_1^k + k \sum_{1 \le i \le \frac{k}{2}} \frac{(-1)^i}{i} \binom{k-1-i}{i-1} s_2^i s_1^{k-2i}$$

Hence, $x_1^k + x_2^k - x_3^k - x_4^k$ can be written as

$$x_1^k + x_2^k - x_3^k - x_4^k = (x_1 + x_2 - x_3 - x_4)u_k' + (x_1x_2 - x_3x_4)u_k''$$

where

$$u_k' = \frac{x_1^k + x_2^k - g_k(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4},$$

$$u_k'' = \frac{g_k(x_3 + x_4, x_1x_2) - x_3^k - x_4^k}{x_1x_2 - x_3x_4}.$$

[Notice that our u_{n+1}' and u_{n+1}'' are the same as the u_1 and u_2 in (23), respectively.]

Let

$$\mathcal{U}_1 = \frac{1}{n+1}u_{n+1}' + \frac{a_{n-2}}{n-1}u_{n-1}' + \cdots + \frac{a_1}{2}u_2' + a_0,$$

and

$$\mathcal{U}_2 = \frac{1}{n+1}u_{n+1}'' + \frac{a_{n-2}}{n-1}u_{n-1}'' + \cdots + \frac{a_1}{2}u_2''.$$

Define C_t to be the tensor product of graded factorizations

$$R_t \xrightarrow{\mathcal{U}_1} R_t\{1-n\} \xrightarrow{x_1+x_2-x_3-x_4} R_t,$$

and

$$R_t \xrightarrow{\mathcal{U}_2} R_t\{3-n\} \xrightarrow{x_1x_2-x_3x_4} R_t,$$

with the product shifted by $\{-1\}$.

Arcs: To an arc α bounded by marks oriented from j to i we assign the factorization L_j^i

$$R_\alpha \xrightarrow{\mathcal{P}_{ij}} R_\alpha \xrightarrow{x_i - x_j} R_\alpha,$$

where $R_\alpha = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_i, x_j]$ and

$$\mathcal{P}_{ij} = \frac{P(x_i) - P(x_j)}{x_i - x_j}.$$

Finally, to an oriented loop with no marks we assign the complex $0 \to \mathcal{A} \to 0 = \mathcal{A}\langle 1\rangle$ where $\mathcal{A} = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0)$. [Note: to a loop with marks we assign the tensor product of L^i_j's as above, but this turns out to be isomorphic to $A\langle 1\rangle$ in the homotopy category.]

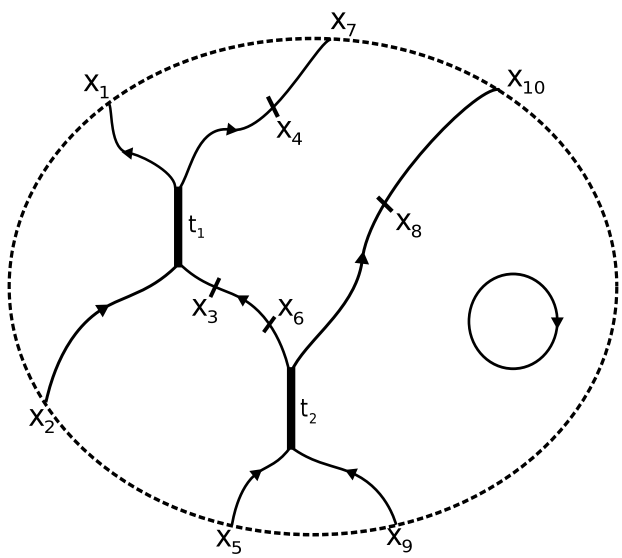


Figure 2.4: A planar graph

We define $C(\Gamma)$ to be the tensor product of C_t over all thick edges t, L^i_j over all edges α from j to i, and $A\langle 1\rangle$ over all oriented markless loops. This tensor product is taken over appropriate rings such that $C(\Gamma)$ is a free module over $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][\{x_i\}]$ where the x_i's are marks. For example, to the graph in figure 2.4 we assign $C(\Gamma) = L^7_4 \otimes C_{t_1} \otimes L^3_6 \otimes C_{t_2} \otimes L^{10}_8 \otimes A\langle 1\rangle$ tensored over $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_4]$, $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_3]$, $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_6]$, $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_8]$ and $\mathbb{Q}[a_0, \ldots, a_{n-2}]$ respectively. $C(\Gamma)$ becomes a $\mathbb{Z} \oplus \mathbb{Z}_2$-graded complex with the $\mathbb{Z}_2$-grading coming from the matrix factorization. It has potential $\omega = \sum_{i \in \partial\Gamma} \pm P(x_i)$, where $\partial\Gamma$ is the set of all boundary marks and the $+$, $-$ is determined by whether the direction of the edge corresponding to x_i is towards or away from the boundary. [Note: if Γ is a closed graph the potential is zero and we have an honest 2-complex.]

Example: Let us look at the factorization assigned to an oriented loop with two marks x and y. We start out with the factorization L_x^y assigned to an arc and then "close it off," which corresponds to moding out by the ideal generated by the relation $x = y$, see figure 2.5. We arrive at

$$R \xrightarrow{x^n+a_{n-2}x^{n-2}+\cdots+a_1x+a_0} R \xrightarrow{0} R,$$

where $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]$.

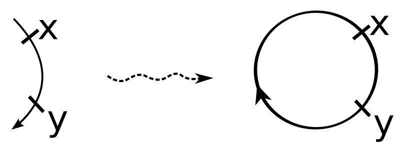


Figure 2.5: "Closing off" an arc

The homology of this complex is supported in degree 1, with

$$H^1(L_x^y/\langle x = y\rangle) = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0).$$

This is the algebra $\mathcal{A}$ we associated to an oriented loop with no marks. As we set out to define a homology theory that assigns to the unkot the $U(n)$-equivariant cohomology of $\mathbb{CP}^{n-1}$, this example illustrates the choice of potential $P(x)$. Notice that $\mathcal{A}$ has a natural Frobenius algebra structure with trace map ε and unit map ι.

$$\varepsilon : \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0) \longrightarrow \mathbb{Q}[a_0, \ldots, a_{n-2}],$$

given by

$$\varepsilon(x^{n-1}) = 1; \quad \varepsilon(x^i) = 0, \; i \leq n-2,$$

and

$$\iota : \mathbb{Q}[a_0, \ldots, a_{n-2}] \longrightarrow \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0),$$

$$\iota(1) = 1.$$

Notice that $\varepsilon(x^i)$ is not equal to zero for $i \geq n$ but a homogeneous polynomial in the a_i's. Many of the calculations in (23) necessary for the proofs of invariance would fail due to this fact; proposition 4 will be key in getting around this difference. Of course, setting $a_i = 0$, for all i, gives us the same Frobenius algebra, unit and trace maps as in (23).

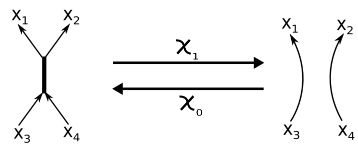


Figure 2.6: Maps χ_0 and χ_1

The maps χ_0 and χ_1: We now define maps between matrix factorizations associated to a thick edge and two disjoint arcs as in figure 5.2. Let Γ^0 correspond to the two disjoint arcs and Γ^1 to the thick edge.

$C(\Gamma^0)$ is the tensor product of L_4^1 and L_3^2. If we assign labels a, b to L_4^1, L_3^2 respectively, the tensor product can be written as

$$\begin{pmatrix} R(\varnothing) \\ R(ab)\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R(a)\{1-n\} \\ R(b)\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R(\varnothing) \\ R(ab)\{2-2n\} \end{pmatrix},$$

where

$$P_0 = \begin{pmatrix} \mathcal{P}_{14} & x_2 - x_3 \\ \mathcal{P}_{23} & x_4 - x_1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ \mathcal{P}_{23} & -\mathcal{P}_{14} \end{pmatrix},$$

and $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, x_2, x_3, x_4]$.

Assigning labels a' and b' to the two factorizations in $C(\Gamma^1)$, we have that $C(\Gamma^1)$ is given by

$$\begin{pmatrix} R(\varnothing)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix} \xrightarrow{Q_0} \begin{pmatrix} R(a')\{-n\} \\ R(b')\{2-n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R(\varnothing)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix},$$

where

$$Q_0 = \begin{pmatrix} \mathcal{U}_1 & x_1x_2 - x_3x_4 \\ \mathcal{U}_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ \mathcal{U}_2 & -\mathcal{U}_1 \end{pmatrix}.$$

A map between $C(\Gamma^0)$ and $C(\Gamma^1)$ can be given by a pair of 2×2 matrices. Define $\chi_0 : C(\Gamma^0) \to C(\Gamma^1)$ by

$$U_0 = \begin{pmatrix} x_4 - x_2 + \mu(x_1 + x_2 - x_3 - x_4) & 0 \\ k_1 & 1 \end{pmatrix}, U_1 = \begin{pmatrix} x_4 + \mu(x_1 - x_4) & \mu(x_2 - x_3) - x_2 \\ -1 & 1 \end{pmatrix},$$

where

$$k_1 = (\mu - 1)\mathcal{U}_2 + \frac{\mathcal{U}_1 + x_1\mathcal{U}_2 - \mathcal{P}_{23}}{x_1 - x_4}, \ for\ \mu \in \mathbb{Z}$$

and $\chi_1 : C(\Gamma^1) \to C(\Gamma^0)$ by

$$V_0 = \begin{pmatrix} 1 & 0 \\ k_2 & k_3 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & x_3 + \lambda(x_2 - x_3) \\ 1 & x_1 + \lambda(x_4 - x_1) \end{pmatrix}.$$

where

$$k_2 = \lambda\mathcal{U}_2 + \frac{\mathcal{U}_1 + x_1\mathcal{U}_2 - \mathcal{P}_{23}}{x_4 - x_1}, \ k_3 = \lambda(x_3 + x_4 - x_1 - x_2) + x_1 - x_3, \ for\ \lambda \in \mathbb{Z}.$$

It is easy to see that different choices of μ and λ give homotopic maps. These maps are degree 1. We encourage the reader to compare the above factorizations and maps to that of (23), and notice the difference stemming from the fact that here we are working with new potentials.

Just like in (23) we specialize to $\lambda = 0$ and $\mu = 1$, and compute to see that the composition $\chi_1\chi_0 = (x_1 - x_3)I$, where I is the identity matrix, i.e. $\chi_1\chi_0$ is multiplication by $x_1 - x_3$, which is homotopic to multiplication by $x_4 - x_2$ as an endomorphism of $C(\Gamma^0)$. Similarly $\chi_0\chi_1 = (x_1 - x_3)I$, which is also homotopic to multiplication by $x_4 - x_2$ as an endomorphism of $C(\Gamma^1)$.

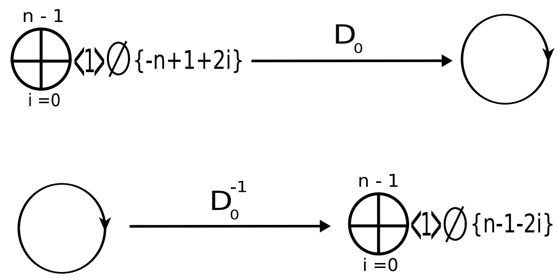


Figure 2.7: Direct Sum Decomposition 0

Direct Sum Decomposition 0

where $D_0 = \sum_{i=0}^{n-1} x^i \iota$ and $D_0^{-1} = \sum_{i=0}^{n-1} \varepsilon x^{n-1-i}$.

By the pictures above, we really mean the complexes assigned to them, i.e. $\emptyset\langle 1\rangle$ is the complex with $\mathbb{Q}[a_0, \ldots, a_{n-2}]$ sitting in homological grading 1 and the unknot is the complex $\mathcal{A}\langle 1\rangle$ as before. The map εx^i is a composition of maps

$$\mathcal{A}\langle 1\rangle \xrightarrow{x^i} \mathcal{A}\langle 1\rangle \xrightarrow{\varepsilon} \emptyset\langle 1\rangle,$$

where x^i is multiplication and ε is the trace map.

The map $x^i\iota$ is analogous. It is easy to check that the above maps are grading preserving and their composition is an isomorphism in the homotopy category. $\square$

Direct Sum Decomposition I We follow (23) closely. Recall that here matrix factorizations are over the ring $R = \mathbb{Q}[a_0, \ldots, a_{n-2}]$.

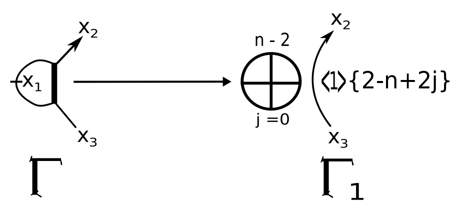


Figure 2.8: Direct Sum Decomposition I

Proposition 2.10. *The following two factorizations are isomorphic in hmf_ω.*

$$C(\Gamma) \cong \sum_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{2-n+2i\}.$$

Proof: Define grading preserving maps α_i and β_i for $0 \le i \le n-2$, as in (23),

$$\alpha_i \ : \ C(\Gamma_1)\langle 1\rangle \longrightarrow C(\Gamma)\{n-2-2i\},$$

$$\alpha_i \ = \ \sum_{j=0}^{i} x_1^j x_2^{i-j}\alpha,$$

where $\alpha = \chi_0 \circ \iota'$ is defined to be the composition in figure 5.3. [$\iota' = \iota \otimes Id$ where Id corresponds to the inclusion of the arc Γ_1 into the disjoint union of the arc and circle, and ι is the unit map.]

$$\beta_i \ : \ C(\Gamma)\{n-2-2i\} \longrightarrow C(\Gamma_1)\langle 1\rangle,$$

$$\beta_i \ = \ \beta x_1^{n-i-2},$$

where $\beta = \varepsilon' \circ \chi_1$, see figure 5.4. [Similarly, $\varepsilon' = \varepsilon \otimes Id$.]

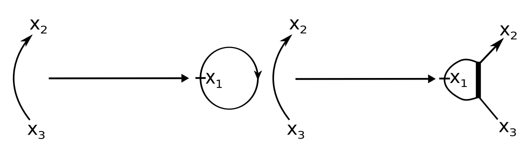


Figure 2.9: The map α

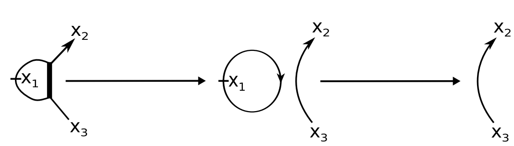


Figure 2.10: The map β

Define maps:

$$\alpha' = \sum_{i=0}^{n-2} \alpha_i \ : \ \sum_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{2-n+2i\} \longrightarrow C(\Gamma),$$

and

$$\beta' = \sum_{i=0}^{n-2} \beta_i \ : \ C(\Gamma) \longrightarrow C(\Gamma_1)\langle 1\rangle\{2-n+2i\}.$$

In (23) it was shown that these maps are isomorphisms of factorizations over the ring $\overline{R} = \mathbb{Q}[x_1, x_2, x_3, x_4]$. By Proposition 2.7 we are done.□

Direct Sum Decomposition II

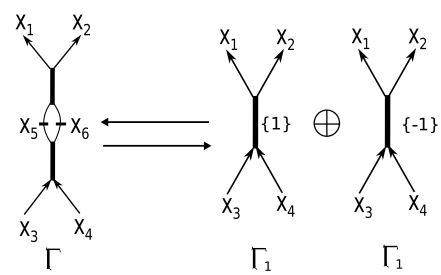


Figure 2.11: Direct Sum Decomposition II

Proposition 2.11. *There is an isomorphism of factorizations in* hmf_ω

$$C(\Gamma) \cong C(\Gamma_1)\{1\} \oplus C(\Gamma_1)\{-1\}.$$

Proof: See (23).□

Direct Sum Decomposition III

Proposition 2.12. *There is an isomorphism of factorizations in* hmf_ω

$$C(\Gamma) \cong C(\Gamma_2) \oplus \left(\oplus_{i=0}^{n-3} C(\Gamma_1)\langle 1\rangle\{3-n+2i\}\right).$$

Proof: Define grading preserving maps α_i, β_i for $0 \le i \le n-3$

$$\alpha_i \ : \ C(\Gamma_1)\langle 1\rangle\{3-n+2i\} \longrightarrow C(\Gamma)$$

$$\alpha_i \ = \ x_5 \alpha,$$

where $\alpha = \chi_0' \circ \iota'$, $\iota' = Id \otimes \iota \otimes Id$ with identity maps on the two arcs, and χ_0' the composition of two χ_0's corresponding to merging the two arcs into the circle, see figure 2.13.

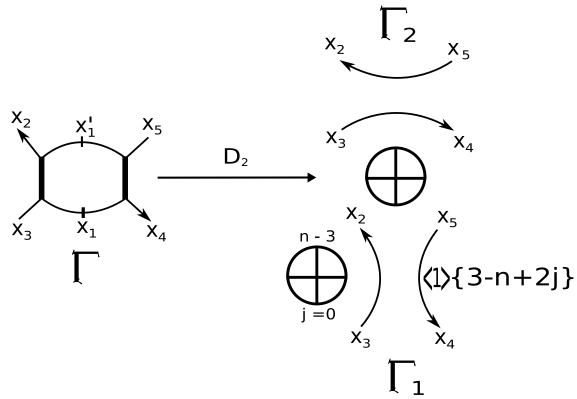


Figure 2.12: Direct Sum Decomposition III

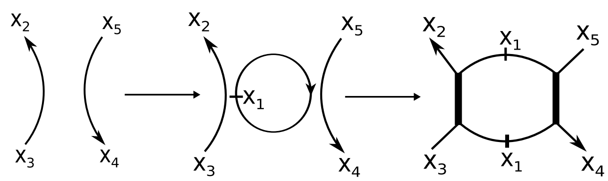


Figure 2.13: The map α

$$\beta_i \ : \ C(\Gamma) \longrightarrow C(\Gamma_1)\langle 1\rangle\{3-n+2i\}$$

$$\beta_i \ = \ \sum_{i=0}^{n-3} \beta \sum_{a+b+c=n-3-i} x_2^a x_4^b x_1^c,$$

where β is defined as in figure 5.5.

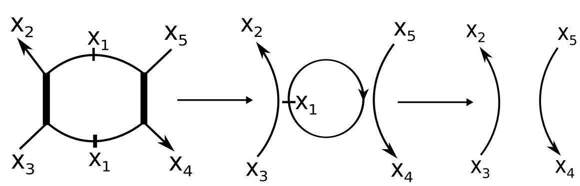


Figure 2.14: The map β

$$S : C(\Gamma) \longrightarrow C(\Gamma_2).$$

In addition, let S be the map gotten by "merging" the thick edges together to form two disjoint horizontal arcs, as in the top righ-hand corner above; an exact description of

S won't really matter so we will not go into details and refer the interested reader to (23).

Let $\alpha' = \sum_{i=0}^{n-3} \alpha_i$ and $\beta' = \sum_{i=0}^{n-3} \beta_i$. In (23) it shown that $S \oplus \beta'$ is an isomorphism in hmf_ω, with inverse $S^{-1} \oplus \alpha'$, so by Proposition 2.7 we are done. $\square$.

[Note: we abuse notation throughout by using a direct sum of maps to indicate a map to or from a direct summand.]

Direct Sum Decomposition IV

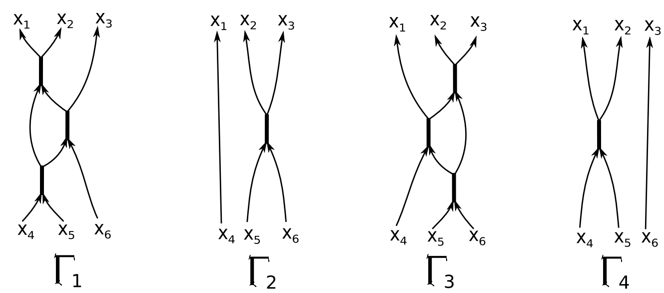


Figure 2.15: The factorizations in Direct Sum Decomposition IV

Proposition 2.13. *There is an isomorphism in hmf_ω*

$$C(\Gamma_1) \oplus C(\Gamma_2) \cong C(\Gamma_3) \oplus C(\Gamma_4).$$

Proof: Notice that $C(\Gamma_1)$ turns into $C(\Gamma_3)$ if we permute x_1 with x_3, and $C(\Gamma_2)$ turns into $C(\Gamma_4)$ if we permute x_2 and x_4. The proposition is proved by introducing a new factorization Υ that is invariant under these permutations and showing that $C(\Gamma_1) \cong \Upsilon \oplus C(\Gamma_4)$, and $C(\Gamma_3) \cong \Upsilon \oplus C(\Gamma_2)$. Since these decompositions hold for matrix factorizations over the ring $\overline{R} = \mathbb{Q}[x_1, \ldots, x_6]$, they hold here as well. We refer the reader to (23) for details. $\square$

2.3 Tangles and complexes

By a tangle T we mean an oriented, closed one manifold embedded in the unit ball $\mathbb{B}^3$, with boundary points of T lying on the equator of the bounding sphere $\mathbb{S}^2$. An isotopy of tangles preserves the boundary points. A diagram D for T is a generic projection of T onto the plane of the equator.

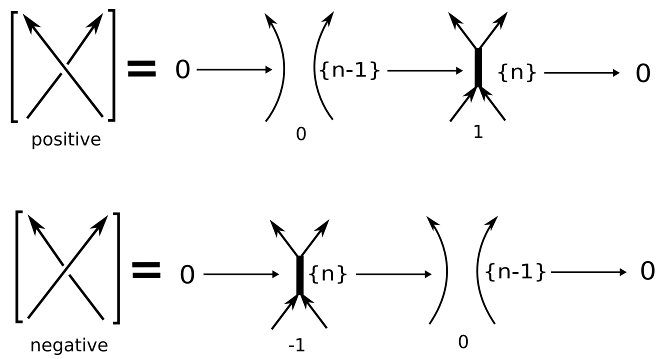


Figure 2.16: Complexes associated to pos/neg crossings; the numbers below the diagrams are cohomological degrees.

Given such a diagram D and a crossing p of D we resolve it in two ways, depending on whether the crossing is positive or negative, and assign to p the corresponding complex C^p, see figure 5.6 . We define $C(D)$ to be the comples of matrix factorizations which is the tensor product of C^p, over all crossings p, of L_j^i over arcs $j \to i$, and of $\mathcal{A}\langle 1\rangle$ over all crossingless markless circles in D. The tensor product is taken over appropriate polynomial rings, so that $C(D)$ is free and of finite rank as an R-module, where $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, \ldots, x_k]$, and the x_i's are on the boundary of D. This complex is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ graded.

For example, the complex associated to the tangle in figure 2.17 is gotten by first tensoring C^{p_1} with C^{p_2} over the ring $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_3, x_4]$, then tensoring $C^{p_1} \otimes C^{p_2}$ with L_1^2 over $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_2]$, and finally tensoring $C^{p_1} \otimes C^{p_2} \otimes L_1^2$ with $\mathcal{A}\langle 1\rangle$ over $\mathbb{Q}[a_0, \ldots, a_{n-2}]$.

Theorem 2.14. *If D and D' are two diagrams representing the same tangle T, then $C(D)$ and $C(D')$ are isomorphic modulo homotopy in the homotopy category hmf_ω, i.e. the isomorphism class of $C(D)$ is an invariant of T.*

The proof of this statement involves checking the invariance under the Reidemeister

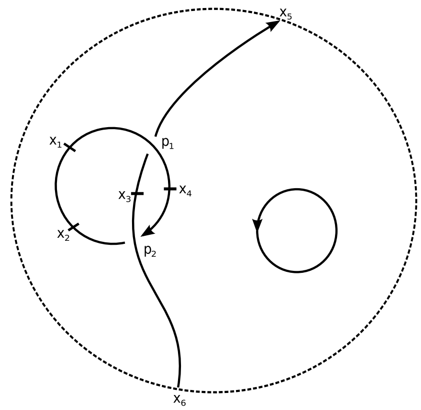


Figure 2.17: Diagram of a tangle

moves to which the next section is devoted.

Link Homology When the tangle in question is a link L, i.e. there are no boundary points and $R = \mathbb{Q}[a_0, \ldots, a_{n-2}]$, complexes of matrix factorizations associated to each resolution have non-trivial cohomology only in one degree (in the cyclic degree which is the number of components of L modulo 2). The grading of the cohomology of $C(L)$ reduces to $\mathbb{Z} \oplus \mathbb{Z}$. We denote the resulting cohomology groups of the complex $C(L)$ by

$$H_n(L) = \oplus_{i,j \in \mathbb{Z}} H_n^{i,j}(L),$$

and the Euler characteristic by

$$P_n(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j dim_R H_n^{i,j}(L).$$

It is clear from the construction that

Corollary 2.15. *Setting the a_i's to zero in the chain complex we arrive at the Khovanov-Rozansky homology, with Euler characteristic the quantum sl_n-polynomial of L.*

2.4 Invariance under the Reidemeister moves

R1: To the tangle in figure 2.18 left we associate the following complex

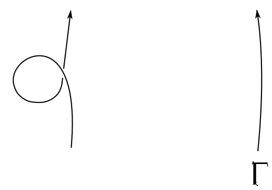


Figure 2.18: Reidemeister I

$$0 \longrightarrow C(\Gamma_1)\{1-n\} \xrightarrow{\chi_0} C(\Gamma_2)\{-n\} \longrightarrow 0.$$

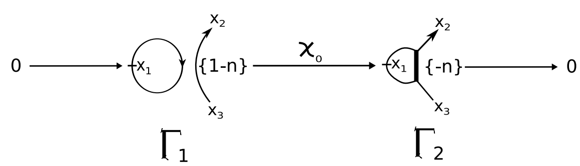


Figure 2.19: Reidemeister 1 complex

Using direct decompositions 0 and I, and for a moment forgoing the overall grading shifts, we see that this complex is isomorphic to

$$0 \longrightarrow \bigoplus_{i=0}^{n-1} C(\Gamma)\{1-n+2i\} \xrightarrow{\Phi} \bigoplus_{j=0}^{n-2} C(\Gamma)\{1+n-2j\} \longrightarrow 0,$$

where

$$\begin{aligned}
\Phi &= \beta' \circ \chi_0 \circ \sum_{i=0}^{n-1} x_1^i \iota' \\
&= \big(\sum_{j=0}^{n-2} \varepsilon' \circ \chi_1 x_1^{n-j-2}\big) \circ \chi_0 \circ \sum_{i=0}^{n-1} x_1^i \iota' \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-2} \varepsilon' \circ \chi_1 \circ \chi_0 x_1^{n-j+i-2} \circ \iota' \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-2} \varepsilon'(x_1 - x_2) x_1^{n-j+i-2} \circ \iota' \\
&= \varepsilon'\big(\sum_{i=0}^{n-1}\sum_{j=0}^{n-2}(x_1^{n-j+i-1} - x_2 x_1^{n-j+i-2})\big)\iota'.
\end{aligned}$$

Hence, Φ is an upper triangular matrix with 1's on the diagonal, which implies that up to homotopy the above complexes are isomorphic to

$$0 \longrightarrow C(\Gamma)\{n-1\} \longrightarrow 0.$$

Recalling that we left out the overall grading shift of $\{-n+1\}$ we arrive at the desired conclusion:

$$0 \longrightarrow C(\Gamma_1)\{1-n\} \xrightarrow{\chi_0} C(\Gamma_2)\{-n\} \longrightarrow 0$$

is homotopic to

$$0 \longrightarrow C(\Gamma)\{n-1\} \longrightarrow 0.$$

The other Reidemeister 1 move is proved analogously. □

R2: The complex associated to the tangle in figure 2.20 left is

Figure 2.20: Reidemeister 2a

$$0 \longrightarrow C(\Gamma_{00})\{1\} \xrightarrow{(f_1,f_3)^t} \begin{matrix} C(\Gamma_{01}) \\ \oplus \\ C(\Gamma_{10}) \end{matrix} \xrightarrow{(f_2,-f_4)} C(\Gamma_{11})\{-1\} \longrightarrow 0.$$

Using direct decomposition II we know that

$$C(\Gamma_{10}) \cong C(\Gamma_1)\{1\} \oplus C(\Gamma_1)\{-1\}.$$

Hence, the above complex is isomorphic to

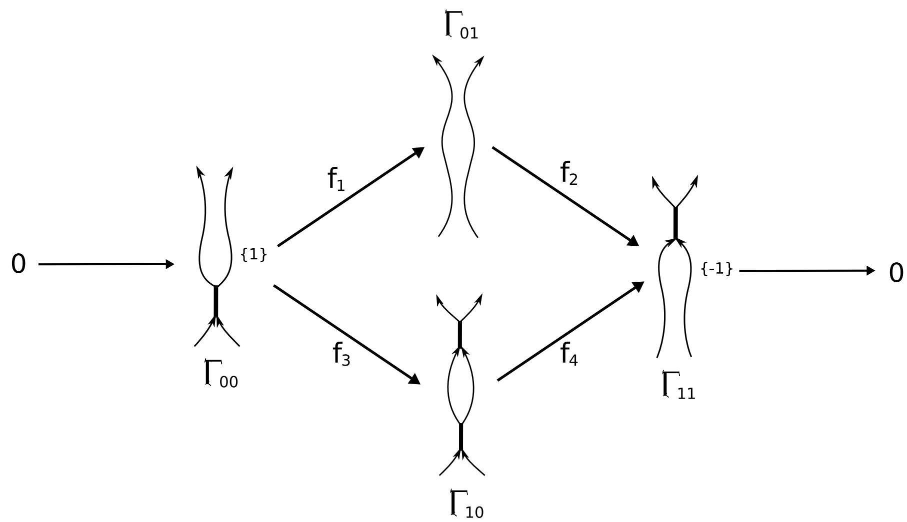


Figure 2.21: Reidemeister 2a complex

$$0 \longrightarrow C(\Gamma_{00})\{1\} \xrightarrow{(f_1,f_{03},f_{13})^t} \begin{matrix} C(\Gamma_{01}) \\ \oplus \\ C(\Gamma_1)\{1\} \\ \oplus \\ C(\Gamma_1)\{-1\} \end{matrix} \xrightarrow{(f_2,-f_{04},-f_{14})} C(\Gamma_{11})\{-1\} \longrightarrow 0,$$

where f_{03}, f_{13}, f_{04}, f_{14} are the degreee 0 maps that give the isomorphism of decomposition II. If we know that both f_{14} and f_{03} are isomorphisms then the subcomplex containing $C(\Gamma_{00}), C(\Gamma_{10})$, and $C(\Gamma_{11})$ is acyclic; moding out produces a complex homotopic to

$$0 \longrightarrow C(\Gamma_0) \longrightarrow 0.$$

The next two lemmas establish the fact that f_{14} and f_{03} are indeed isomorphisms.

Lemma 2.16. *The space of degree* 0 *endomorphisms of* $C(\Gamma_1)$ *is isomorphic to* $\mathbb{Q}$. *The space of degree* 2 *endomorphism is* 3*-dimensional spanned by* x_1, x_2, x_3, x_4 *with only relation being* $x_1 + x_2 - x_3 - x_4 = 0$ *for* $n > 2$, *and* 2*-dimensional with the relations* $x_1 + x_2 = 0$ *and* $x_3 + x_4 = 0$ *for* $n = 2$.

Proof: The complex $Hom(C(\Gamma_1), C(\Gamma_1))$ is isomorphic to the factorization of the pair

$(\mathbf{b},\mathbf{c})$ where

$$\mathbf{b}=(x_1+x_2+x_3+x_4, x_1x_2-x_3x_4, -\mathcal{U}_1, -\mathcal{U}_2),\ \mathbf{c}=(\mathcal{U}_1,\mathcal{U}_2, x_1+x_2+x_3+x_4, x_1x_2-x_3x_4).$$

The pair $(\mathbf{b},\mathbf{c})$ is orthogonal, since this is a complex, and it is easy to see that the sequence $\mathbf{c}$ is regular ($\mathbf{c}$ is certainly regular when we set the a_i's equal to zero) and hence the cohomology of this 2-complex is

$$\mathbb{Q}[a_0,\dots,a_{n-2}][x_1,x_2,x_3,x_4]/(x_1+x_2+x_3+x_4, x_1x_2-x_3x_4,\mathcal{U}_1,\mathcal{U}_2).$$

For $n>2$ the last three terms of the above sequence are at least quadratic and, hence, have degree at least 4 (recall that $\deg a_i \geq 4$ for all i). For $n=2$, $\mathcal{U}_2=u_2''$ which is linear and we get the relations $x_1+x_2=0$, $x_3+x_4=0$. □

Lemma 2.17. *$f_{14}\neq 0$ and $f_{03}\neq 0$.*

Proof: With the above lemma the proof follows the lines of (23). □

Hence, f_{14} and f_{03} are indeed isomorphisms and we arrive at the desired conclusion.□

Figure 2.22:

R3: The complex assigned to the tangle on the left-hand side of figure 2.22 is

$$0 \longrightarrow C(\Gamma_{111}) \xrightarrow{d^{-3}} \begin{array}{c} C(\Gamma_{011})\{-1\} \\ \oplus \\ C(\Gamma_{101})\{-1\} \\ \oplus \\ C(\Gamma_{110})\{-1\} \end{array} \xrightarrow{d^{-2}} \begin{array}{c} C(\Gamma_{100})\{-2\} \\ \oplus \\ C(\Gamma_{010})\{-2\} \\ \oplus \\ C(\Gamma_{001})\{-2\} \end{array} \xrightarrow{d^{-1}} C(\Gamma_{000})\{-3\} \longrightarrow 0.$$

Direct sum decompositions II and III show that

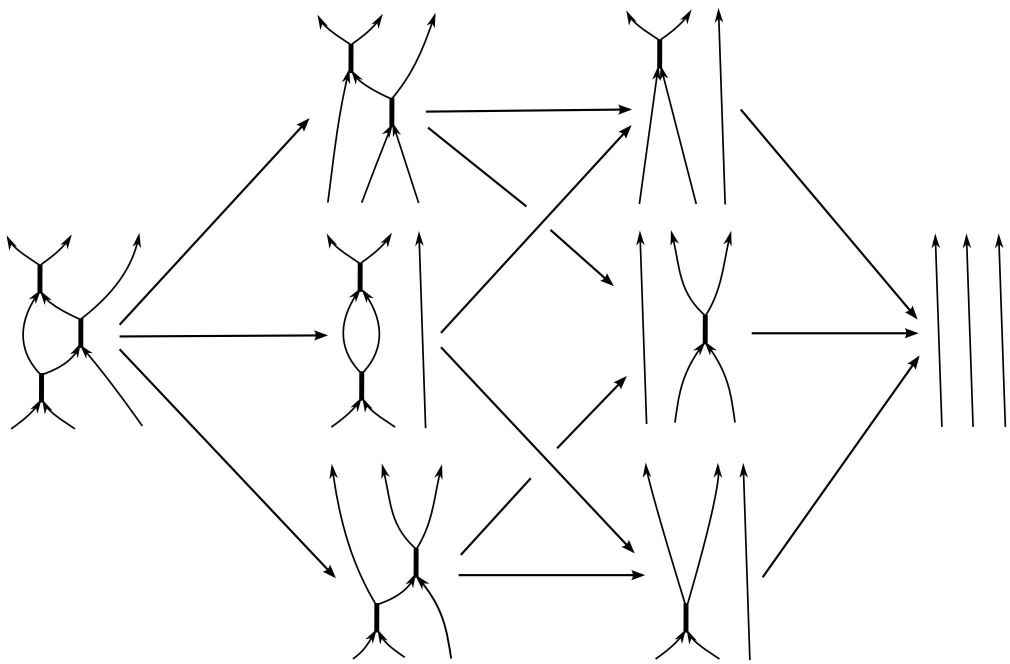


Figure 2.23: Reidemeister 3 complex

$$C(\Gamma_{101}) \cong C(\Gamma_{100})\{1\} \oplus C(\Gamma_{100})\{-1\},$$

and

$$C(\Gamma_{111}) \cong C(\Gamma_{100}) \oplus \Upsilon.$$

Inserting these and using arguments analogous to those used in the decomposition proofs we reduce the original complex to

$$0 \longrightarrow \Upsilon \xrightarrow{d^{-3}} \begin{array}{c} C(\Gamma_{011})\{-1\} \\ \oplus \\ C(\Gamma_{110})\{-1\} \end{array} \xrightarrow{d^{-2}} \begin{array}{c} C(\Gamma_{010})\{-2\} \\ \oplus \\ C(\Gamma_{100})\{-2\} \end{array} \xrightarrow{d^{-1}} C(\Gamma_{000})\{-3\} \longrightarrow 0.$$

Proposition 2.18. *Assume $n > 2$, then for every arrow in 4.12 from object A to B the space of grading-preserving morphisms*

$$Hom_{hmf}(C(A), C(B)\{-1\})$$

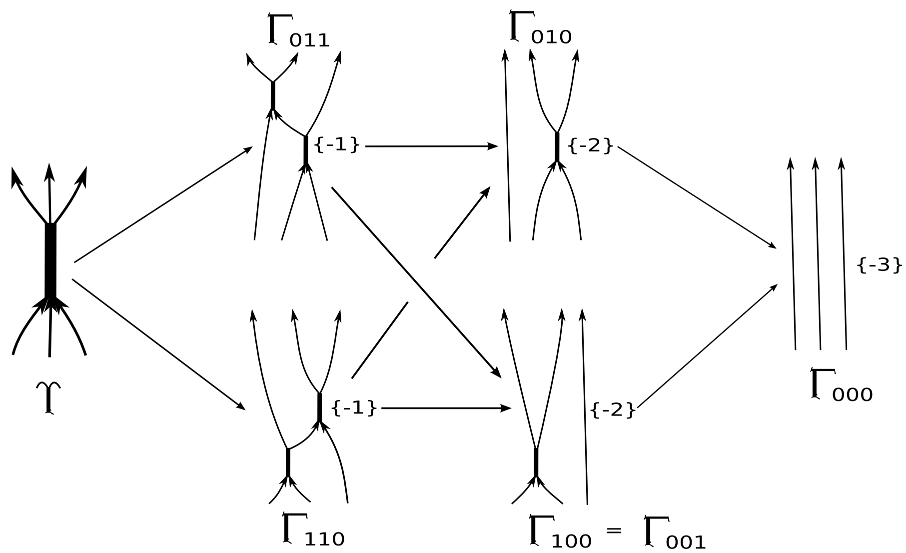


Figure 2.24: Reidemeister 3 complex reduced

is one dimensional. Moreover, the composition of any two arrows $C(A) \longrightarrow C(B)\{-1\} \longrightarrow C(C)\{-2\}$ *is nonzero.*

Proof: Once again the maps in question are all of degree ≤ 2, and noticing that these remain nonzero when we work over the ring $\mathbb{Q}[a_0, \ldots, a_{n-2}]$, we can revert to the calculations in (23).□

Hence, this complex is invariant under the "flip" which takes x_1 to x_3 and x_4 to x_6. This flip takes the complex associated to the braid on the left-hand side of figure 2.22 to the one on the right-hand side.□

2.5 Remarks

Given a diagram D of a link L let $C_n(D)$ be the equivariant sl_n chain complex constructed above. The homotopy class of $C_n(D)$ is an invariant of L and consists of free $\mathbb{Q}[a_0, \ldots, a_{n-2}]$-modules where the a_i's are coefficients with $\deg(a_i) = 2(n-i)$. The cohomology of this complex $H_n(D)$ is a graded $\mathbb{Q}[a_0, \ldots, a_{n-2}]$-module. For a moment, let us consider the case where all the $a_i = 0$ for $1 \leq i \leq n-2$, and denote by $C_{n,a}(D)$ and $H_{n,a}(D)$ the corresponding complex and cohomology groups with $a = a_0$. Here the cohomology $H_{n,a}(D)$ is a finitely generated $\mathbb{Q}[a]$-module and we can decompose it as direct sum of torsion modules $\mathbb{Q}[a]/(a^k)$ for various k and free modules $\mathbb{Q}[a]$. Let $H'_{n,a}(D) = H_{n,a}(D)/Tor_{n,a}(D)$, where $Tor_{n,a}(D)$ is the torsion submodule. Just like in the sl_2 case in (20) we have:

Proposition 2.19. *$H'_{n,a}(D)$ is a free $\mathbb{Q}[a]$-module of rank n^m, where m is the number of components of L.*

Proof: If we quotient $C_{n,a}(D)$ by the subcomplex $(a-1)C_{n,a}(D)$ we arrive at the complex studied by Gornik in (12), where he showed that its rank is n^m. The ranks of our complex and his are the same. □

In some sense this specialization is isomorphic to n copies of the trivial link homology which assigns to each link a copy of $\mathbb{Q}$ for each component, modulo grading shifts. In (29), M. Mackaay and P. Vaz studied similar variants of the sl_3-theory working over the Frobenius algebra $\mathbb{C}[x]/(x^3 + ax^2 + bx + c)$ with $a, b, c \in \mathbb{C}$ and arrived at three isomorphism classes of homological complexes depending on the number of distinct roots of the polynomial $x^3 + ax^2 + bx + c$. They showed that multiplicity three corresponds to the sl_3-homology of (18), one root of multiplicity two is a modified version of the original sl_2 or Khovanov homology, and distinct roots correspond to the “Lee-type” deformation. We expect an interpretation of their results in the equivariant version. Moreover, it would be interesting to understand these specialization for higher n and we foresee similar decompositions, i.e. we expect the homology theories to break up into isomorphism classes corresponding to the number of distinct “roots” in the decomposition of the polynomial $x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$.

The sl_2-homology and sl_3-homology for links, as well as their deformations, are defined over $\mathbb{Z}$; in chapter 4 we will present such an integral construction for all n.

Chapter 3

Functoriality of Rouquier complexes

3.1 Soergel bimodules in representation theory and link homology

For some time, the category of Soergel bimodules, here called $\mathcal{SC}$, has played a significant role in the study of representation theory, while more recently strong connections between $\mathcal{SC}$ and knot theory have come to light. Originally introduced by Soergel in (39), $\mathcal{SC}$ is an equivalent but more combinatorial description of a certain category of Harish-Chandra modules over a semisimple lie algebra $\mathfrak{g}$. The added simplicity of this formulation comes from the fact that $\mathcal{SC}$ is just a full monoidal subcategory of graded R-bimodules, where R is a polynomial ring equipped with an action of the Weyl group of $\mathfrak{g}$. Among other things, Soergel gave an isomorphism between the Grothendieck ring of $\mathcal{SC}$ and the Hecke algebra $\mathcal{H}$ associated to $\mathfrak{g}$, where the Kazhdan-Lusztig generators b_i of $\mathcal{H}$ lift to bimodules B_i which are easily described. The full subcategory generated monoidally by these bimodules B_i is here called $\mathcal{SC}_1$, and the category including all grading shifts and direct sums of objects in $\mathcal{SC}_1$ is called $\mathcal{SC}_2$. It then turns out that $\mathcal{SC}$ is actually the idempotent closure of $\mathcal{SC}_2$, which reduces the study of $\mathcal{SC}$ to the study of these elementary bimodules B_i and their tensors. For more on Soergel bimodules and their applications to representation theory see

(37; 38; 40).

An important application of Soergel bimodules was discovered by Rouquier in (36), where he observes that one can construct complexes in $\mathcal{SC}_2$ which satisfy the braid relations modulo homotopy. To the i overcrossing (resp. undercrossing) in the braid group Rouquier associates a complex, which has R in homological degree 0 and B_i in homological degree -1 (resp. 1). Giving the homotopy equivalence classes of invertible complexes in $\mathcal{SC}_2$ the obvious group structure under tensor product, this assignment extends to a homomorphism from the braid group. Using this, one can define an action of the braid group on the homotopy category of $\mathcal{SC}_2$, where the endofunctor associated to a crossing is precisely taking the tensor product with its associated complex.

Following his work with L. Rozansky on matrix factorizations and link homology in (**?**), Khovanov produced an equivalent categorification (21) of the HOMFLY-PT polynomial utilizing Rouquier's work. To a braid one associates its Rouquier complex, which naturally has two gradings: the homological grading, and the internal grading of Soergel bimodules. Then, taking the Hochschild homology of each term in the complex, one gets a complex which is triply graded (the third grading is the Hochschild homological grading). Khovanov showed that, up to degree shifts, this construction yields an equivalent triply-graded complex to the one produced by the reduced version of the Khovanov-Rozansky HOMFLY-PT link homology for the closure of the braid (see (21) and (22) for more details).

Many computations of HOMFLY-PT link homology were done by B. Webster (43), and by J. Rasmussen in (33) and (35). In (33), Rasmussen showed that given a braid presentation of a link, for every $n \in \mathbb{N}$ there exists a spectral sequence with E^1-term its HOMFLY-PT homology and the E^∞-term its $sl(n)$ homology. This was a spectacular development in understanding the structural properties of these theories, and has also proven very useful in computation (see for example (27)).

One key aspect of the original Khovanov-Rozansky theory is that it gives rise to a projective functor. The braid group can actually be realized as the isomorphism classes of objects in the category of braid cobordisms. This category, while having a topological definition, is equivalent to a combinatorially defined category, whose objects are braid diagrams, and whose morphisms are called *movies* (see Carter-Saito, (7)). For instance, performing a Reidemeister 3 move on a braid diagram would give an equivalent element of the braid group, but gives a distinct object in the braid cobordism category; however, the R3 move itself is a movie which gives the isomorphism between those two objects. It was shown in (22) that for each movie between braids one can associate a chain map between their triply-graded complexes. This assignment was known to be *projectively functorial*, meaning that the relations satisfied amongst movies in the braid cobordism category are also satisfied by their associated chain maps, up to multiplication by a scalar. Scalars take their value in $\mathbb{Q}$, the ring over which Khovanov-Rozansky theory is defined. However, these chain maps are not explicitly described even in the setting of Khovanov-Rozansky theory, and the maps they correspond to in the Soergel bimodule context are even more obscure. A more general discussion of braid group actions, including this categorification via Rouquier complexes, and their extensions to projective actions on the category of braid cobordisms can be found in (24).

Recently, in (1), B. Elias and Mikhail Khovanov gave a presentation of the category $\mathcal{SC}_1$ in terms of generators and relations. Moreover, it was shown that the entire category can be drawn graphically, thanks to the biadjointness and cyclicity properties that the category possesses. Each B_i is assigned a color, and a tensor product is assigned a sequence of colors. Morphisms between tensor products can be drawn as certain colored graphs in the plane, whose boundaries on bottom and top are the sequence of colors associated to the source and target. Composition and tensor product of morphisms correspond to vertical and horizontal concatenation, respectively. Morphisms are invariant under isotopy of the graph embedding, and satisfy a number of other relations, as described herein. In addition to providing a presentation, this graphical description is useful because one can use pictures to encapsulate a large amount of information; complicated calculations involving compositions of morphisms can be visualized intuitively and written down suffering only minor headaches.

Because of the simplicity of the diagrammatic calculus, we were able to calculate explicitly the chain maps which correspond to each generating cobordism in the braid cobordism category, and check that these chain maps satisfy the same relations that braid cobordisms do. The general proofs are straightforward and computationally explicit, performable by any reader with patience, time, and colored chalk. While we use some slightly more sophisticated machinery to avoid certain incredibly lengthy computations, the machinery is completely unnecessary. This makes the results of Rouquier and Khovanov that much more concrete, and implies the following new result.

Theorem 3.1. *There is a functor F from the category of combinatorial braid cobordisms to the category of complexes in $\mathcal{SC}_2$ up to homotopy, lifting Rouquier's construction (i.e. such that F sends crossings to Rouquier complexes).*

Soergel bimodules are generally defined over certain fields $\Bbbk$ in the literature, because one is usually interested in Soergel bimodules as a categorification of the Hecke algebra, and in relating indecomposable bimodules to the Kazhdan-Lusztig canonical basis. However, we invite the reader to notice that the diagrammatic construction in (1) can be made over any ring, and in particular over $\mathbb{Z}$. In fact, all our proofs of functoriality still work over $\mathbb{Z}$. We discuss this in detail in section 3.5.2. In the subsequent paper, we plan to use the work done here to define HOMFLY-PT and $sl(n)$-link homology theories over $\mathbb{Z}$, a construction which is long overdue. We also plan to investigate the Rasmussen spectral sequence in this context.

At the given moment there does not exist a diagrammatic calculus for the higher Hochschild homology of Soergel bimodules. Some insights have already been obtained, although a full understanding had yet to emerge. We plan to develop the complete picture, which should hopefully give an explicit and easily computable description of functoriality in the link homology theories discussed above.

The organization of this chapter is as follows. In Section 3.2 we go over all the previous constructions that are relevant to this paper. This includes the Hecke algebra, Soergel's categorification $\mathcal{SC}$, the graphical presentation of $\mathcal{SC}$, the combinatorial braid cobordism category, and Rouquier's complexes which link $\mathcal{SC}$ to braids. In Section 3.2.6 we describe the conventions we will use in the remainder of the paper to draw Rouquier complexes for movies. In Section 3.3 we define the functor from the combinatorial braid cobordism category to the homotopy category of $\mathcal{SC}$, and in Section 3.4 we check the movie move relations to verify that our functor is well-defined. These checks are presented in numerical order, not in logical order, but a discussion of the logical dependency of the proofs, and of the simplifications that are used, can be found in Section 3.4.1. Section 3.5 contains some useful statements for the interested reader, but is not strictly necessary. Some additional light is shed on the generators and relations of $\mathcal{SC}$ in Section 3.5.1, where it is demonstrated how the relations arise naturally from movie moves. In Section 3.5.2 we briefly describe how one might construct the theory over $\mathbb{Z}$, so that future papers may use this result to define link homology theories over arbitrary rings.

3.2 Constructions

3.2.1 The Hecke Algebra

The Hecke algebra $\mathcal{H}$ of type A_∞ has a presentation as an algebra over $\mathbb{Z}\left[t,t^{-1}\right]$ with generators b_i, $i \in \mathbb{Z}$ and the *Hecke relations*

$$\begin{aligned} b_i^2 &= (t+t^{-1})b_i && (3.1)\\ b_ib_j &= b_jb_i \text{ for } |i-j| \geq 2 && (3.2)\\ b_ib_{i+1}b_i + b_{i+1} &= b_{i+1}b_ib_{i+1} + b_i. && (3.3)\end{aligned}$$

For any subset $I \subset \mathbb{Z}$, we can consider the subalgebra $\mathcal{H}(I) \subset \mathcal{H}$ generated by b_i, $i \in I$, which happens to have the same presentation as above. Usually only finite I are considered.

We write the monomial $b_{i_1}b_{i_2}\cdots b_{i_d}$ as $b_{\underline{i}}$ where $\underline{\boldsymbol{i}} = i_1 \dots i_d$ is a finite sequence of indices; by abuse of notation, we sometimes refer to this monomial simply as $\underline{\boldsymbol{i}}$. If $\underline{\boldsymbol{i}}$ is as above, we say the monomial has *length* d. We call a monomial *non-repeating* if $i_k \neq i_l$ for $k \neq l$. The empty set is a sequence of length 0, and $b_\emptyset = 1$.

Let ω be the t-antilinear anti-involution which fixes b_i, i.e. $\omega(t^a b_{\underline{i}}) = t^{-a} b_{\sigma(\underline{i})}$ where σ reverses the order of a sequence. Let $\epsilon\colon \mathcal{H} \to \mathbb{Z}\left[t,t^{-1}\right]$ be the $\mathbb{Z}\left[t,t^{-1}\right]$-linear map which is uniquely specified by $\epsilon(xy) = \epsilon(yx)$ for all $x,y \in \mathcal{H}$ and $\epsilon(b_{\underline{i}}) = t^d$, whenever $\underline{\boldsymbol{i}}$ is a non-repeating sequence of length d. Let $(,)\colon \mathcal{H} \times \mathcal{H} \to \mathbb{Z}\left[t,t^{-1}\right]$ be the map which sends $(x,y) \mapsto \epsilon(\omega(x)y)$. Via the inclusion maps, these structures all descend to each $\mathcal{H}(I)$ as well.

We say $i,j \in \mathbb{Z}$ are *adjacent* if $|i-j| = 1$, and are *distant* if $|i-j| \geq 2$.

For more details on the Hecke algebra in this context, see (1).

3.2.2 The Soergel Categorification

In (39), Soergel introduced a monoidal category categorifying the Hecke algebra for a finite Weyl group W of type A. We will denote this category by $\mathcal{SC}(I)$, or by $\mathcal{SC}$ when I is irrelevant. Letting V be the geometric representation of W over a field $\Bbbk$ of characteristic $\neq 2$, and R its coordinate ring, the category $\mathcal{SC}$ is given as a full additive monoidal subcategory of graded R-bimodules (whose objects are now commonly referred to as Soergel bimodules). This category is not abelian, for it lacks images, kernels, and the like, but it is idempotent closed. In fact, $\mathcal{SC}$ is given as the idempotent closure of another full additive monoidal subcategory $\mathcal{SC}_1$, whose objects are called Bott-Samuelson modules. The category $\mathcal{SC}_1$ is generated monoidally over R by objects B_i, $i \in I$, which satisfy

$$B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\} \tag{3.4}$$

$$B_i \otimes B_j \cong B_j \otimes B_i \text{ for distant } i, j \tag{3.5}$$

$$B_i \otimes B_j \otimes B_i \oplus B_j \cong B_j \otimes B_i \otimes B_j \oplus B_i \text{ for adjacent } i, j. \tag{3.6}$$

The Grothendieck group of $\mathcal{SC}(I)$ is isomorphic to $\mathcal{H}(I)$, with the class of B_i being sent to b_i, and the class of $R\{1\}$ being sent to t.

One useful feature of this categorification is that it is easy to calculate the dimension of Hom spaces in each degree. Let $\mathrm{HOM}(M,N) \stackrel{\text{def}}{=} \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(M, N\{m\})$ be the graded vector space (actually an R-bimodule) generated by homogeneous morphisms of all degrees. Let $B_{\underline{i}} \stackrel{\text{def}}{=} B_{i_1} \otimes \cdots \otimes B_{i_d}$. Then $\mathrm{HOM}(B_{\underline{i}}, B_{\underline{j}})$ is a free left R-module, and its graded rank over R is given by $(b_{\underline{i}}, b_{\underline{j}})$.

For two subsets $I \subset I' \subset \mathbb{Z}$, the categories $\mathcal{SC}(I)$ and $\mathcal{SC}(I')$ are embedded in bimodule categories over different rings $R(I)$ and $R(I')$, but there is nonetheless a faithful inclusion of categories $\mathcal{SC}(I) \to \mathcal{SC}(I')$. This functor is not full: the size of R itself will grow, and $\mathrm{HOM}(B_\emptyset, B_\emptyset) = R$. However, the graded rank over R does not change, since the value of ϵ and hence $(,)$ does not change over various inclusions. Effectively, the only difference in Hom spaces under this inclusion functor is base change on the left, from $R(I)$ to $R(I')$.

As a result of this, most calculations involving morphisms between Soergel bimodules will not depend on which I we work over. When I is infinite, the ring R is no longer

Noetherian, and we do not wish to deal with such cases. However, the categories $\mathcal{SC}(I)$ over arbitrary finite I will all work essentially the same way. A slightly more rigorous graphical statement of this property is forthcoming. In particular, the calculations we do for the Braid group on m strands will also work for the braid group on $m+1$ strands, and so forth.

3.2.3 Soergel Diagrammatics

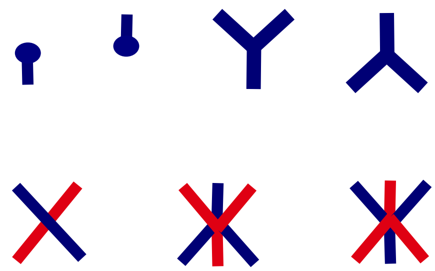
In (1), the category $\mathcal{SC}_1$ was given a diagrammatic presentation by generators and relations, allowing morphisms to be viewed as isotopy classes of certain graphs. We review this presentation here, referring the reader to (1) for more details. We will first deal with the case where $W = S_{n+1}$, or where $I = \{1, 2, \ldots, n\}$, and then discuss what the inclusions of categories from the previous section imply for the general setting.

Remark 3.2. Technically, (1) gave the presentation for a slightly different category, which we temporarily call $\mathcal{SC}'_1$. The category presented here is a quotient of $\mathcal{SC}'_1$ by the central morphism corresponding to e_1, the first symmetric polynomial. This is discussed briefly in Section 4.5 of (1). Moreover, $\mathcal{SC}'_1$ is also a faithful extension of $\mathcal{SC}_1$, so that the main results of this paper apply to the extension as well. We use $\mathcal{SC}_1$ instead because it is the "minimal" category required for our results (no extensions are necessary), and because it streamlines the presentation. We leave it as an exercise to see that the definition of $\mathcal{SC}_1$ below agrees with the e_1 quotient of the category defined in (1).

The first subtlety to be addressed is that $\mathcal{SC}_1$ is only equivalent to the e_1 quotient of $\mathcal{SC}'_1$ when one is working over a base ring $\Bbbk$ where $n+1$ is invertible. Otherwise, the quotient of $\mathcal{SC}'_1$ is still a non-trivial faithful extension.

For a discussion of the advantages to using $\mathcal{SC}'_1$, see Section 3.5.2.

An object in $\mathcal{SC}_1$ is given by a sequence of indices $\underline{\boldsymbol{i}}$, which is visualized as d points on the real line $\mathbb{R}$, labelled or "colored" by the indices in order from left to right. Sometimes these objects are also called $B_{\underline{i}}$. Morphisms are given by pictures embedded in the strip $\mathbb{R} \times [0, 1]$ (modulo certain relations), constructed by gluing the following generators horizontally and vertically:



For instance, if "blue" corresponds to the index i and "red" to j, then the lower right generator is a morphism from jij to iji. The generating pictures above may exist in various colors, although there are some restrictions based on of adjacency conditions.

We can view a morphism as an embedding of a planar graph, satisfying the following properties:

1. Edges of the graph are colored by indices from 1 to n.

2. Edges may run into the boundary $\mathbb{R} \times \{0, 1\}$, yielding two sequences of colored points on $\mathbb{R}$, the top boundary $\underline{\boldsymbol{i}}$ and the bottom boundary $\underline{\boldsymbol{j}}$. In this case, the graph is viewed as a morphism from $\underline{\boldsymbol{j}}$ to $\underline{\boldsymbol{i}}$.

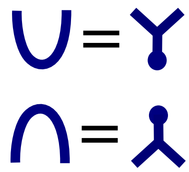
3. Only four types of vertices exist in this graph: univalent vertices or "dots", trivalent vertices with all three adjoining edges of the same color, 4-valent vertices whose adjoining edges alternate in colors between i and j distant, and 6-valent vertices whose adjoining edges alternate between i and j adjacent.

The degree of a graph is +1 for each dot and -1 for each trivalent vertex. 4-valent and 6-valent vertices are of degree 0. The term *graph* henceforth refers to such a graph embedding.

By convention, we color the edges with different colors, but do not specify which colors match up with which $i \in I$. This is legitimate, as only the various adjacency relations between colors are relevant for any relations or calculations. We will specify adjacency for all pictures, although one can generally deduce it from the fact that 6-valent vertices only join adjacent colors, and 4-valent vertices join only distant colors.

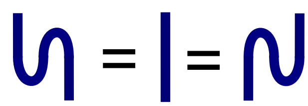
As usual in a diagrammatic category, composition of morphisms is given by vertical concatenation, and the monoidal structure is given by horizontal concatenation.

In writing the relations, it will be useful to introduce a pictures for the "cup" and "cap":

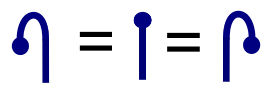


(3.7)

We then allow $\Bbbk$-linear sums of graphs, and apply the relations below to obtain our category $\mathcal{SC}_1$. Some of these relations are redundant. For a more detailed discussion of the remarks in the remainder of this section, see (1).

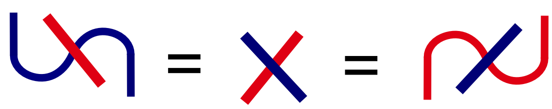


(3.8)

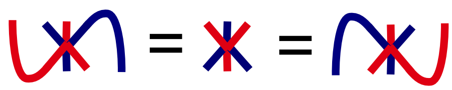


(3.9)

(3.10)



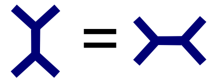
(3.11)



(3.12)

Remark 3.3. The relations (3.8) through (3.12) together imply that the morphism specified by a particular graph embedding is independent of the isotopy class of the embedding. We could have described the category more simply by defining a morphism to be an isotopy class of a certain kind of planar graph. However, it is useful to understand that these "isotopy relations" exist, because they will appear naturally in the study of movie moves (see Section 3.5.1).

Other relations are written in a format which already assumes that isotopy invariance is given. Some of these relations contain horizontal lines, which cannot be constructed using the generating pictures given; nonetheless, such a graph is isotopic to a number of different pictures which are indeed constructible, and it is irrelevant which version you choose, so the relation is unambiguous.

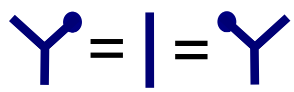


(3.13)

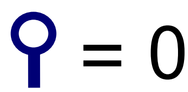
Remark 3.4. Relation (3.13) effectively states that a certain morphism is invariant under 90 degree rotation. To simplify drawings later on, we often draw this morphism as follows:

Note that morphisms will still be isotopy invariant with this convention.

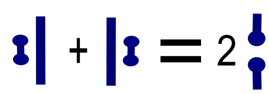
Here are the remainder of the one color relations.



(3.14)

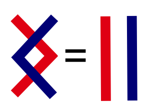


(3.15)

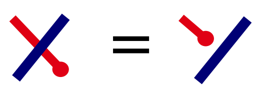


(3.16)

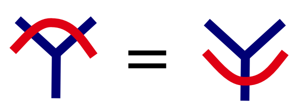
In the following relations, the two colors are distant.



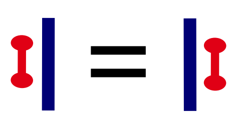
(3.17)



(3.18)

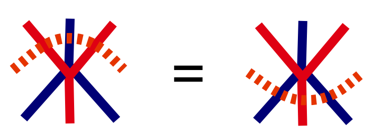


(3.19)



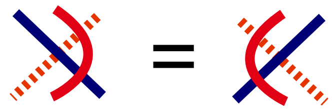
(3.20)

In this relation, two colors are adjacent, and both distant to the third color.



(3.21)

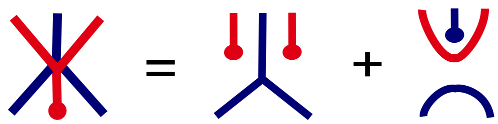
In this relation, all three colors are mutually distant.



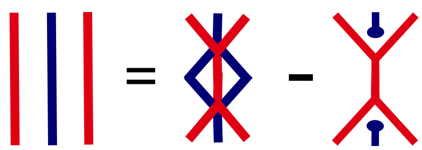
(3.22)

Remark 3.5. Relations (3.17) thru (3.22) indicate that any part of the graph colored i and any part of the graph colored j "do not interact" for i and j distant. That is, one may visualize sliding the j-colored part past the i-colored part, and it will not change the morphism. We call this the *distant sliding property*.

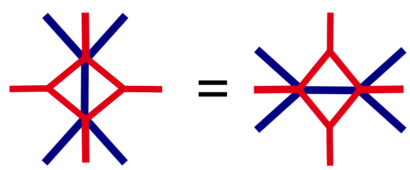
In the following relations, the two colors are adjacent.



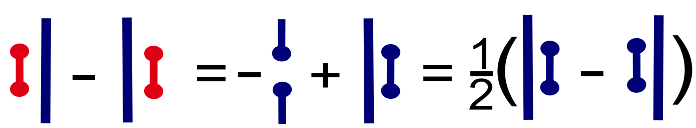
(3.23)



(3.24)

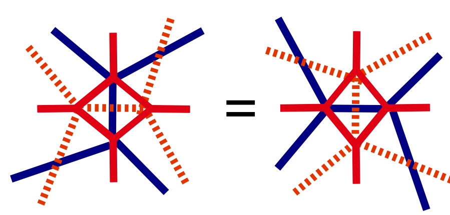


(3.25)



(3.26)

The last equality in (3.26) is implied by (3.16), so it is not necessary to include as a relation. In this final relation, the colors have the same adjacency as $\{1, 2, 3\}$.



(3.27)

Remark 3.6. Because of isotopy invariance, the object B_i in $\mathcal{SC}_1$ is self-biadjoint. In particular, instead of viewing the graph in $\mathbb{R} \times [0, 1]$ as a morphism from $\underline{\boldsymbol{i}}$ to $\underline{\boldsymbol{j}}$, we could twist it around and view it in the lower half plane (with no bottom boundary) as a morphism from $\emptyset$ to $\underline{\boldsymbol{i}}\sigma(\underline{\boldsymbol{j}})$. Thus, we need only investigate morphisms from $\emptyset$ to $\underline{\boldsymbol{i}}$, to determine all Hom spaces.

Remark 3.7. There is a functor from this category into the category of R-bimodules, sending a line colored i to B_i and each generator to an appropriate bimodule map. The functor gives an equivalence of categories between this graphically defined category and the subcategory $\mathcal{SC}_1$ of R-bimodules mentioned in the previous section, so the use of the same name is legitimate.

We refer to any connected component of a graph which is a dot connected directly to the boundary as a *boundary dot*, and to any component equal to two dots connected by an edge as a *double dot*.

Remark 3.8. Relations (3.16), (3.20), and (3.26) are collectively called *dot slides*. They indicate how one might attempt to move a double dot from one region of the graph to another.

The following theorem and corollary are the most important results from (1), and the crucial fact which allows all other proofs to work.

Theorem 3.9. *Consider a morphism $\phi\colon \underline{\boldsymbol{i}} \to \emptyset$, and suppose that the index i appears in $\underline{\boldsymbol{i}}$ zero times (respectively, once). Then ϕ can be rewritten as a linear combination of graphs, for which each graph has the following property: the only edges of the graph colored i are included in double dots (respectively, as well as a single boundary dot connecting to $\underline{\boldsymbol{i}}$), and moreover, all these double dots are in the leftmost region of the graph. This result may be obtained simultaneously for multiple indices i. We could also have chosen the rightmost region for the slide.*

The space $\mathrm{HOM}_{\mathcal{SC}_1}(\emptyset, \emptyset)$ is the free commutative polynomial ring generated by f_i, the double dot colored i, for various $i \in I$. This is a graded ring, with the degree of f_i is 2.

Remark 3.10. Another corollary of the more general results in (1) is that, when a color only appears twice in the boundary one can (under certain conditions on other colors present) reduce the graph to a form where that color only appears in a line connecting the two boundary appearances (and double dots as usual). In particular, if no color appears more than twice on the boundary, then under certain conditions one can reduce all graphs to a form that has no trivalent vertices, and hence all morphisms have nonnegative degree. We will use this fact to help check movie moves 8 and 9, in whose contexts the appropriate conditions do hold.

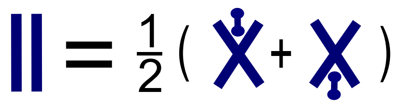
The proof of this theorem involves using the relations to reduce a single color at a time within a graph (while doing arbitrary things to the other colors). Once a color is reduced to the above form, the remainder of the graph no longer interacts with that color. Then we repeat the argument with another color on the rest of the graph, and so on and so forth.

Remark 3.11. There is a natural identification of the polynomial ring of double dots and the coordinate ring R of the geometric representation. Because of this, a combination of double dots is occasionally referred to as a polynomial. Placing double dots in the lefthand or righthand region of a diagram will correspond to the left and right action of R on Hom spaces.

Remark 3.12. Now we are in a position to see how the inclusion $\mathcal{SC}_1(I) \subset \mathcal{SC}_1(I')$ behaves. Let $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$ be objects in $\mathcal{SC}_1(I)$, and k an index in $I' \setminus I$. Applying Theorem 3.9 to the color k, we can assume that in $\mathcal{SC}_1(I')$ all morphisms from $\underline{\boldsymbol{i}}$ to $\underline{\boldsymbol{j}}$ will be (linear combinations of) graphs where k only appears in double dots on the left. Doing this to each color in $I' \setminus I$, we will have a collection of double dots next to a morphism which only uses colors in I. Therefore the map $\mathrm{HOM}_{\mathcal{SC}(I)}(\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}) \otimes \Bbbk[f_k,\ k \in I' \setminus I] \to \mathrm{HOM}_{\mathcal{SC}(I')}(\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}})$ is surjective. In fact, it is an isomorphism. We say that the inclusion functor is *fully faithful up to base change*. Of course, this result does not make it any easier to take a graph, which may have an arbitrarily complicated k-colored part, and reduce it to the simple form where k only appears in double dots on the left.

If we wished to define $\mathcal{SC}_1(I)$ for some $I \subset \{1, \ldots, n\}$, the correct definition would be to consider graphs which are only colored by indices in I. With this definition, inclusion functors are still fully faithful up to base change.

Now we see where the isomorphisms (3.4) through (3.6) come from. To begin, we have the following implication of (3.16):

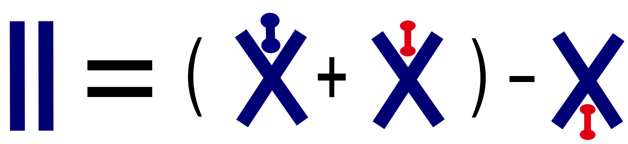


(3.28)

We let $\mathcal{SC}_2$ be the category formally containing all direct sums and grading shifts of objects in $\mathcal{SC}_1$, but whose morphisms are forced to be degree 0. Then (3.28) expresses the direct sum decomposition

$$B_i \otimes B_i = B_i\{1\} \oplus B_i\{-1\}$$

since it decomposes the identity id_{ii} as a sum of two orthogonal idempotents, each of which is the composition of a projection and an inclusion map of the appropriate degree. If one does not wish to use non-integral coefficients, and an adjacent color is present, then the following implication of (3.26) can be used instead; this is again a decomposition of id_{ii} into orthogonal idempotents.



(3.29)

Relation (3.17) expresses the isomorphism

$$B_i \otimes B_j = B_j \otimes B_i$$

for i and j distant.

The category $\mathcal{SC}$ is the Karoubi envelope, or idempotent completion, of the category $\mathcal{SC}_2$. Recall that the Karoubi envelope of a category $\mathcal{C}$ has as objects pairs (B, e) where B is an object in $\mathcal{C}$ and e an idempotent endomorphism of B. This object acts as though it were the "image" of this projection e, and in an additive category behaves like a direct summand. For more information on Karoubi envelopes, see Wikipedia.

The two color variants of relation (3.24) together express the direct sum decompositions

$$B_i \otimes B_{i+1} \otimes B_i = C_i \oplus B_i \tag{3.30}$$

$$B_{i+1} \otimes B_i \otimes B_{i+1} = C_i \oplus B_{i+1}. \tag{3.31}$$

Again, the identity $\mathrm{id}_{i(i+1)i}$ is decomposed into orthogonal idempotents, where the first idempotent corresponds to a new object C_i in the idempotent completion, appearing as a summand in both $i(i+1)i$ and $(i+1)i(i+1)$. Technically, we get two new objects, corresponding to the idempotent in $B_{i(i+1)i}$ and the idempotent in $B_{(i+1)i(i+1)}$, but these two objects are isomorphic, so by abuse of notation we call them both C_i.

We will primarily work within the category $\mathcal{SC}_2$. However, since this includes fully faithfully into $\mathcal{SC}$, all calculations work there as well.

3.2.4 Braids and Movies

In this paper we always use the combinatorial braid cobordism category as a replacement for the topological braid cobordism category, since they are equivalent but the former is more convenient for our purposes. See Carter and Saito (7) for more details.

The category of $(n+1)$-stranded braid cobordisms can be defined as follows. The objects are arbitrary sequences of braid group generators O_i, $1 \leq i \leq n$, and their inverses $U_i = O_i^{-1}$. These sequences can be drawn using braid diagrams on the plane, where O_i is an overcrossing (the $i+1^{\text{st}}$ strand crosses over the i strand) and U_i is an undercrossing. Objects have a monoidal structure given by concatenation of sequences. A *movie* is a finite sequence of transformations of two types:

I. Reidemeister type moves, such as

$$\tau_1 O_i U_i \tau_2 \leftrightarrow \tau_1 \tau_2,$$

$$\tau_1 O_i O_j \tau_2 \leftrightarrow \tau_1 O_j O_i \tau_2 \text{ for distant } i, j$$

$$\tau_1 O_i O_{i+1} O_i \tau_2 \leftrightarrow \tau_1 O_{i+1} O_i O_{i+1} \tau_2.$$

where τ_1 and τ_2 are arbitrary braid words.

II. Addition or removal of a single O_i or U_i from a braid word

$$\tau_1 \tau_2 \leftrightarrow \tau_1 O_i^{\pm 1} \tau_2.$$

These transformations are known as *movie generators*. Morphisms in this category will consist of movies modulo *locality moves*, which ensure that the category is a monoidal category, and certain relations known as *movie moves* (it is common also to refer to locality moves as movie moves). The movie moves can be found in figures 3.1 and 3.2. Movie moves $1-10$ are composed of type I transformations and $11-14$ each contains a unique type II move. We denote the location of the addition or removal of a crossing in these last 4 movies by little black triangles. There are many variants of each of these movies: one can change the relative height of strands, can reflect the movie horizontally or vertically, or can run the movie in reverse. We refer the reader to Carter and Saito (7), section 3.

Recall that the combinatorial cobordism category is monoidal. Locality moves merely state that if two transformations are performed on a diagram in locations that do not interact (they do not share any of the same crossings) then one may change the order in which the transformations are performed. Any potential functor from the combinatorial cobordism category to a monoidal category $\mathcal{C}$ which preserves the monoidal structure will automatically satisfy the locality moves. Because of this, we need not mention the locality moves again.

Given a braid diagram P (or an object in the cobordism category), the diagram $\overline{P}$ is given by reversing the sequence defining P, and replacing all overcrossings with undercrossings and vice versa.

Note that $\overline{P}$ is the inverse of P in the group generated freely by crossings, and hence in the braid group as well.

Again, we refer the reader to (7) for more details on the combinatorial braid cobordism category.

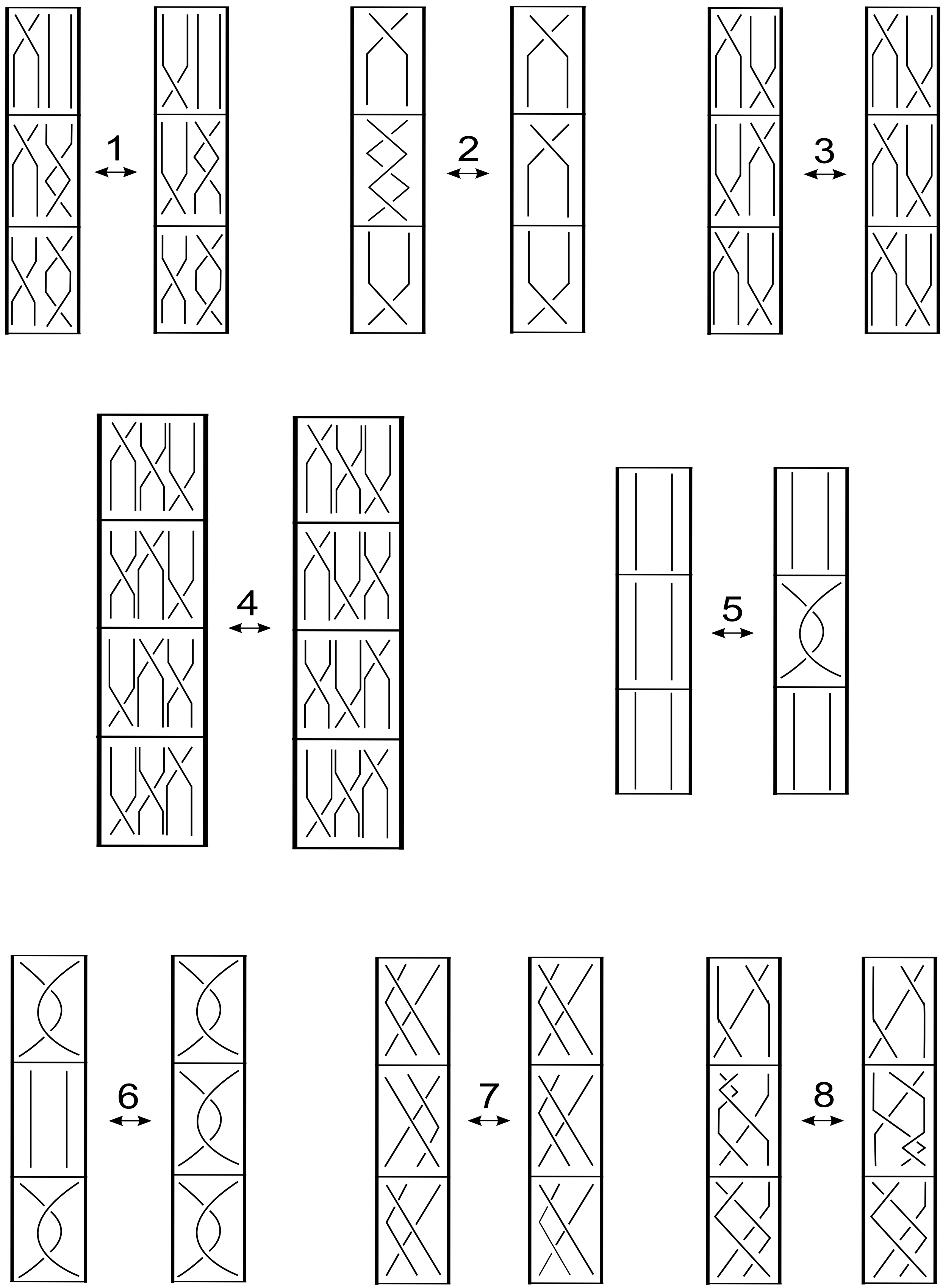


Figure 3.1: Braid movie moves $1 - 8$

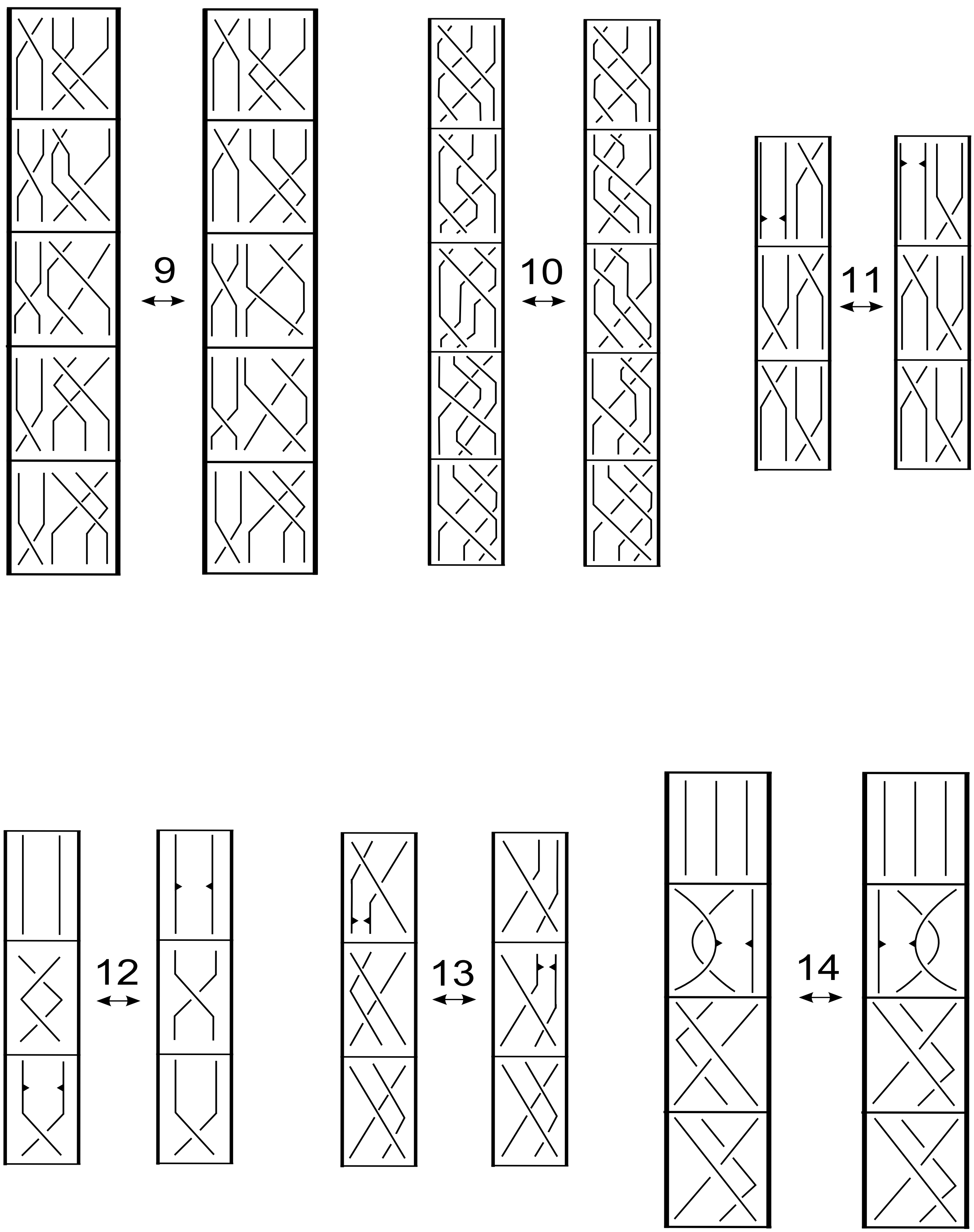


Figure 3.2: Braid movie moves $9 - 14$

3.2.5 Rouquier Complexes

Rouquier defined a braid group action on the homotopy category of complexes in $\mathcal{SC}_2$ (see (36)). To the i overcrossing, he associated a complex $B_i\{1\} \longrightarrow B_\emptyset$, and to the undercrossing, $B_\emptyset \longrightarrow B_i\{-1\}$. In each case, $B_\emptyset$ is in homological degree 0. Drawn graphically, these complexes look like:

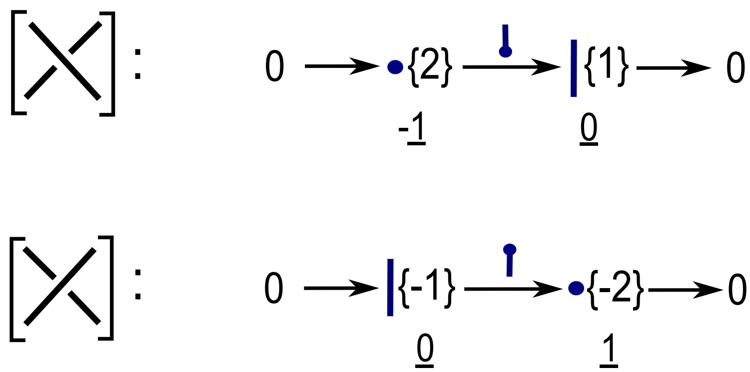


Figure 3.3: Rouquier complex for right and left crossings

We are using a (blue) dot here as a place holder for empty space.

To a braid one associates the tensor product of the complexes for each crossing. He showed in (36) that the braid relations hold amongst these complexes.

In (21), Khovanov showed that taking Hochschild cohomology of these complexes yields an invariant of the link which closes off the braid in question, and that this link homology theory is in fact identical to one already constructed by Khovanov and Rozansky in (**?**). It was shown in (24) that Rouquier's association of complexes to a braid is actually *projectively functorial*. In other words, to each movie between braids, there is a map of complexes, and these maps satisfy the movie move relations (modulo homotopy) up to a potential sign. This was not done by explicitly constructing chain maps, but instead used the formal consequences of the previously-defined link homology theory. It was known that in many cases the composed map would be an isomorphism, and that this categorification could be done over $\mathbb{Z}$ (see (24)), where the only isomorphisms are ± 1, hence the proof of projective functoriality.

The discussion of the previous sections shows that it is irrelevant which braid group we work in, because adding extra strands just corresponds to an inclusion functor which is "fully faithful after base change". In particular, when computing the space of chain maps modulo homotopy between two complexes, we need not worry about the number of strands available, except to keep track of our base ring. Hence calculations are effectively local.

3.2.6 Conventions

These are the conventions we use to draw Rouquier complexes henceforth.

We use a colored circle to indicate the empty graph, but maintain the color for reasons of sanity. It is immediately clear that in the complex associated to a tensor product of d Rouquier complexes, each summand will be a sequence of k lines where $0 \leq k \leq d$ (interspersed with colored circles, but these represent the empty graph so could be ignored). Each differential from one summand to another will be a "dot" map, with an appropriate sign.

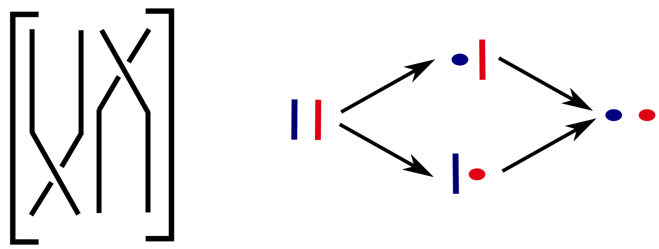
1. The dot would be a map of degree 1 if B_i had not been shifted accordingly. In $\mathcal{SC}_2$, all maps must be homogeneous, so we could have deduced the degree shift in B_i from the degree of the differential. Because of this, it is not useful to keep track of various degree shifts of objects in a complex. We will draw all the objects without degree shifts, and all differentials will therefore be maps of graded degree 1 (as well as homological degree 1). It follows from this that homotopies will have degree -1, in order to be degree 0 when the shifts are put back in. One could put in the degree shifts later, noting that $B_\emptyset$ always occurs as a summand in a tensor product exactly once, with degree shift 0.

2. Similarly, one need not keep track of the homological dimension. $B_\emptyset$ will always occur in homological dimension 0.

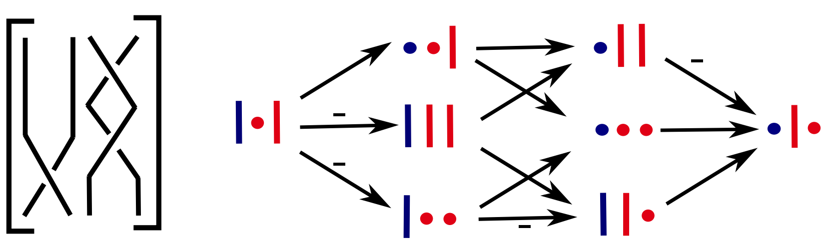
3. We will use blue for the index associated to the leftmost crossing in the braid, then red and dotted orange for other crossings, from left to right. The adjacency of these various colors is determined from the braid.

4. We read tensor products in a braid diagram from bottom to top. That is, in the following diagram, we take the complex for the blue crossing, and tensor by the

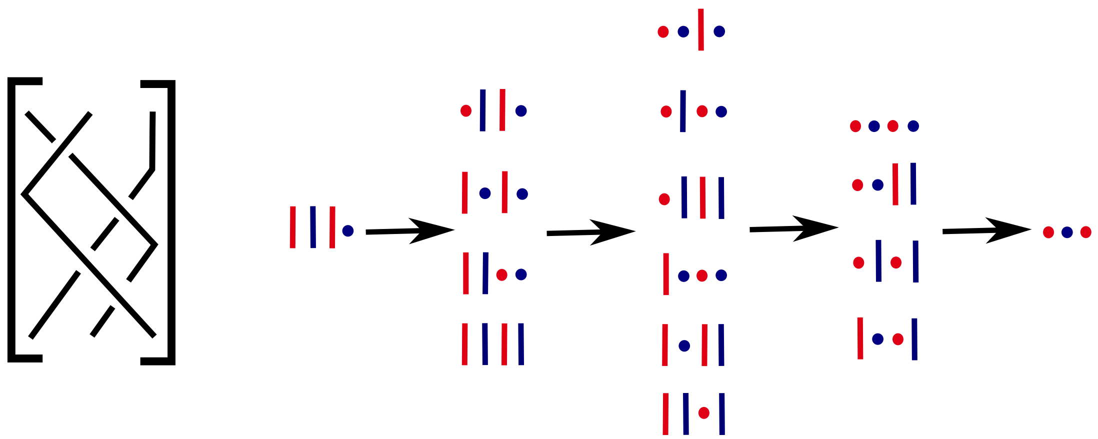
complex for the red crossing. Then we translate this into pictures by saying that tensors go from left to right. In other words, in the complex associated to this braid, blue always appears to the left of red.



5. One can deduce the sign of a differential between two summands using the Liebnitz rule, $d(ab) = d(a)b + (-1)^{|a|}ad(b)$. In particular, since a line always occurs in the basic complex in homological dimension ± 1, the sign on a particular differential is exactly given by the parity of lines appearing to the left of the map. For example,



6. When putting an order on the summands in the tensored complex, we use the following standardized order. Draw the picture for the object of smallest homological degree, which we draw with lines and circles. In the next homological degree, the first summand has the first color switched (from line to circle, or circle to line), the second has the second color switched, and so forth. In the next homological degree, two colors will be switched, and we use the lexicographic order: 1st and 2nd, then 1st and 3rd, then 1st and 4th... then 2nd and 3rd, etc. This pattern continues.



3.3 Definition of the Functor

We extend Rouquier's complexes to a functor F from the combinatorial braid cobordism category to the category of chain complexes in $\mathcal{SC}_2$ modulo homotopy. Rouquier already defined how the functor acts on objects, so it only remains to define chain maps for each of the movie generators, and check the movie move relations.

There are four basic types of movie generators: birth/death of a crossing, slide, Reidemeister 2 and Reidemeister 3.

- **Birth and Death generators**

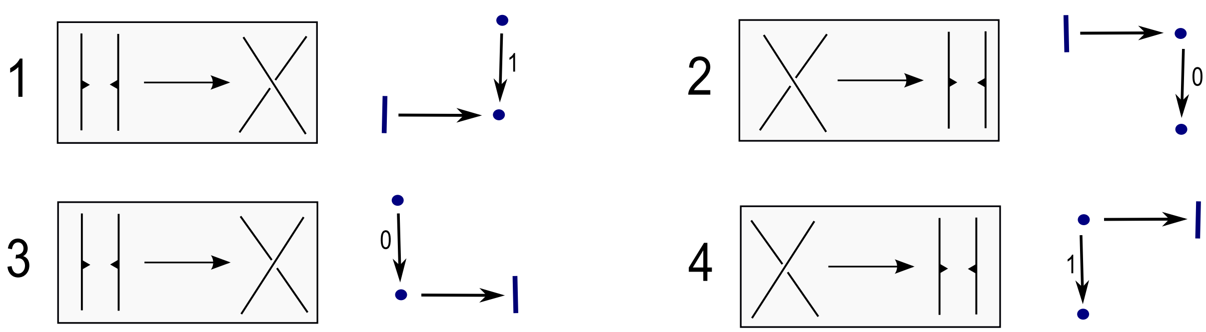


Figure 3.4: Birth and Death of a crossing generators

- **Reidemeister 2 generators**

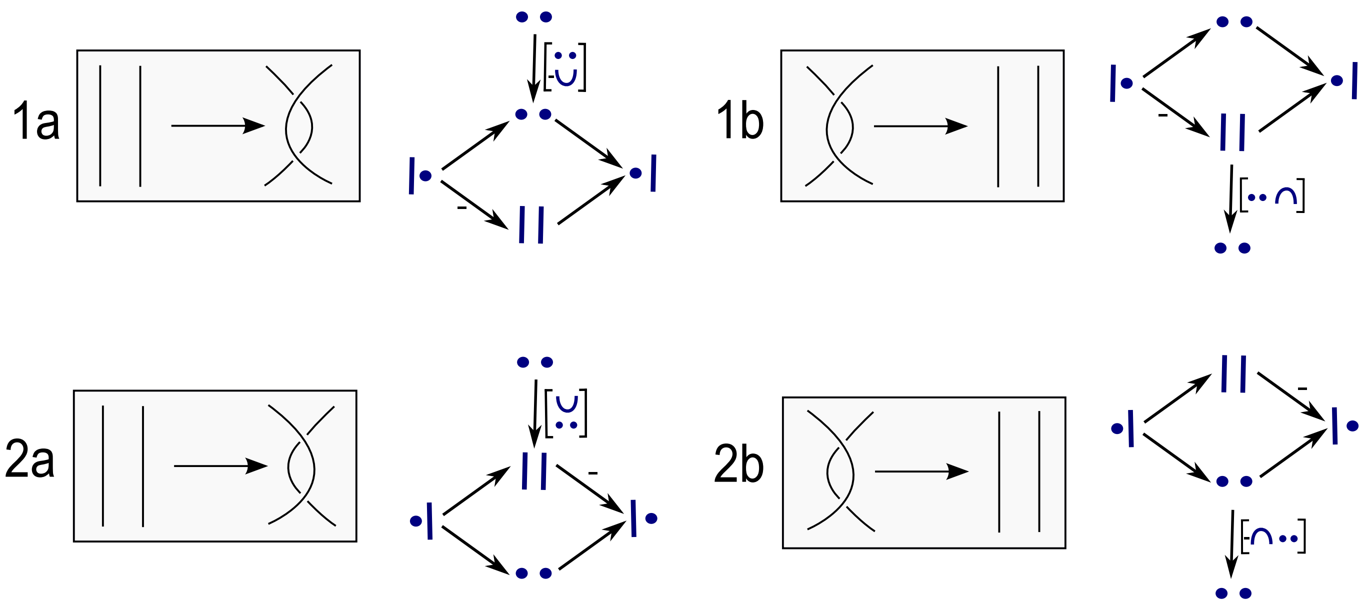


Figure 3.5: Reidemeister 2 type movie move generators

- **Slide generators**

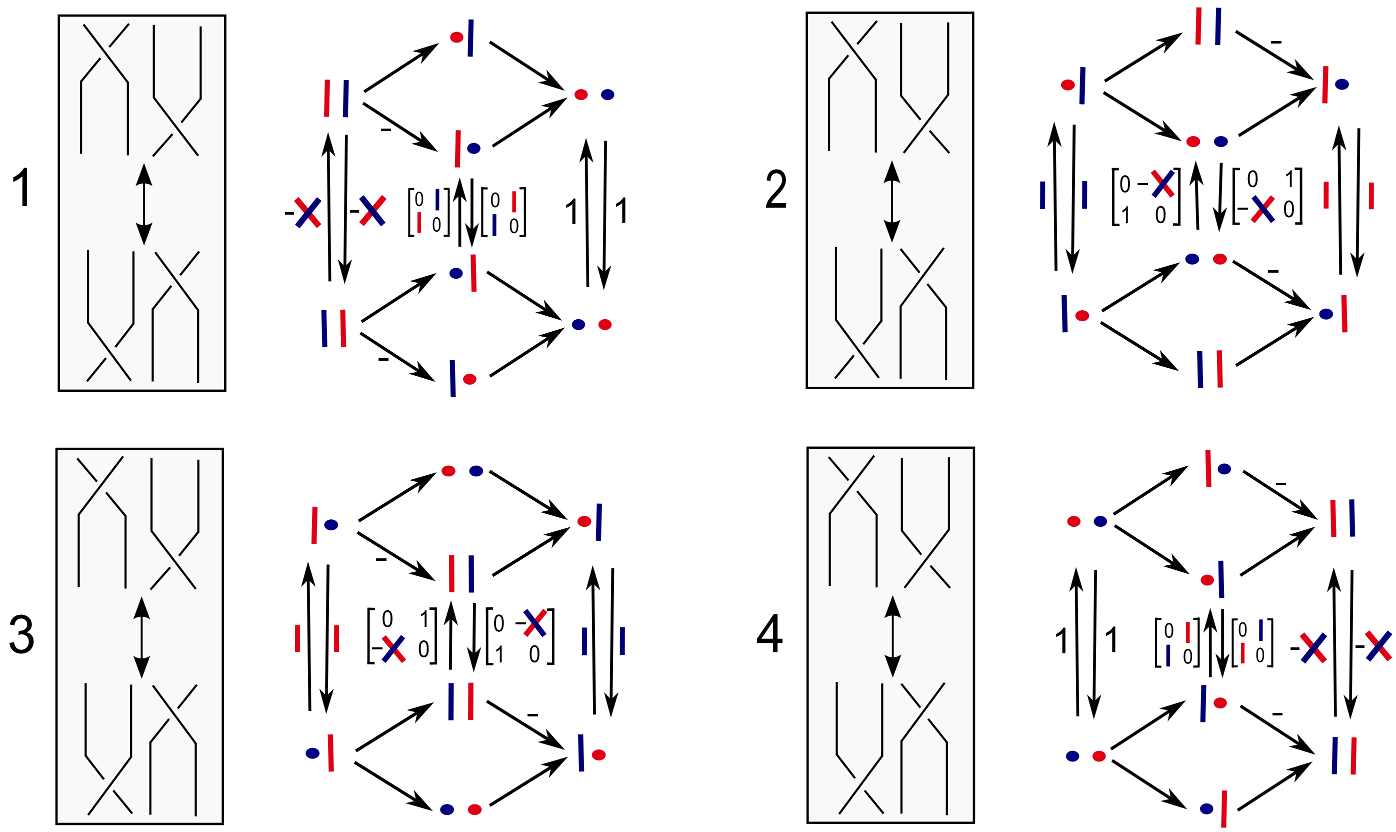


Figure 3.6: Slide generators

- **Reidemeister 3 generators** There are 12 generators in all: 6 possibilities for the height orders of the 3 strands (denoted by a number 1 through 6), and two directions for the movie (denoted "a" or "b"). Thankfully, the color-switching symmetries of the Soergel calculus allow us explicitly list only 6. The left-hand column lists the generators, and the chain complexes they correspond to; switching colors in the complexes yields the corresponding generator listed on the right. Each of these variants has a free parameter x, and the parameter used for each variant is actually independent from the other variants.

Remark 3.13. Using sequences of R2-type generators and various movie moves we could have abstained from ever defining certain R3-type variants or proving the movie moves that use them. We never use this fact, and list all here for completeness.

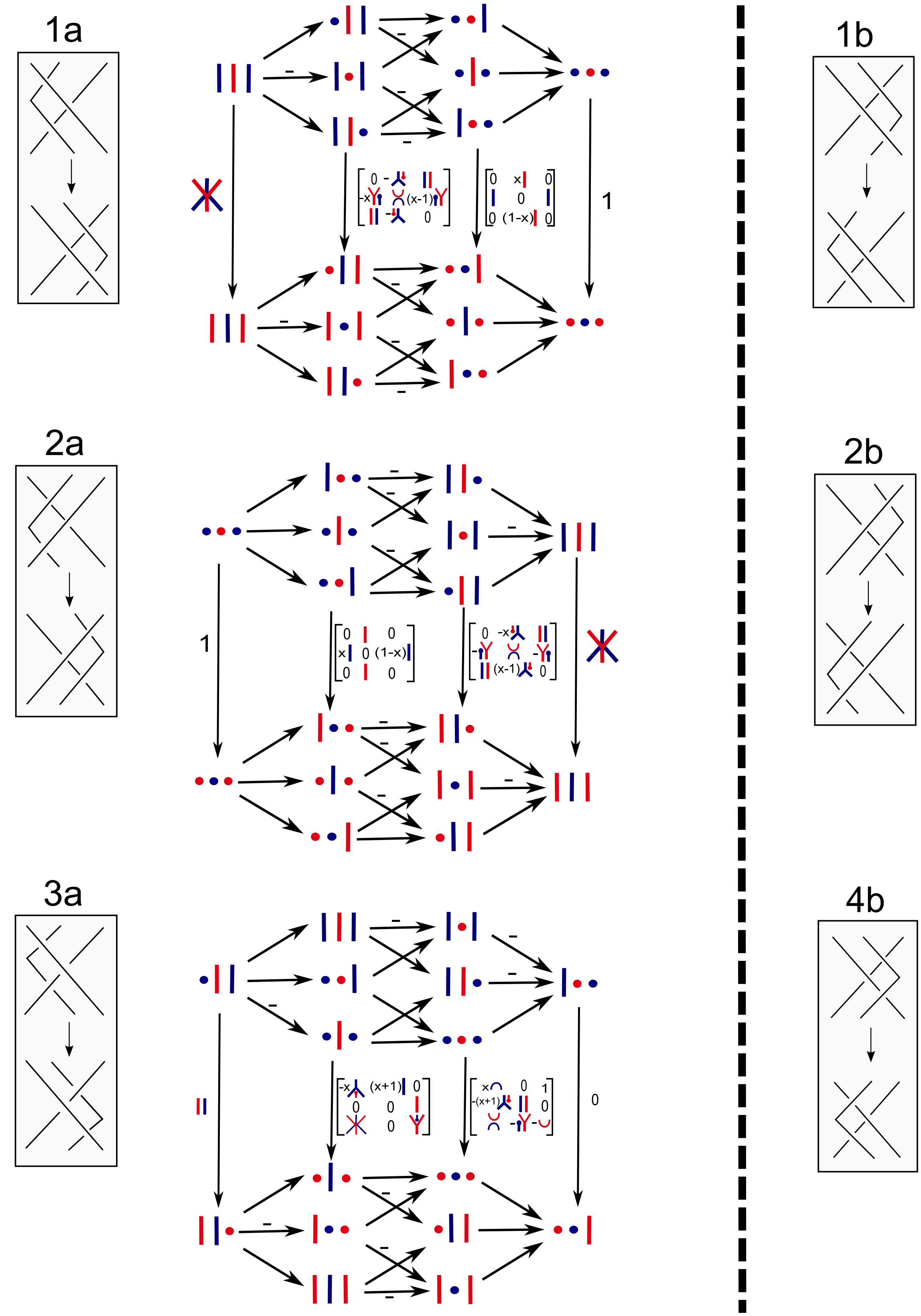


Figure 3.7: Reidemeister 3 type movie move generators

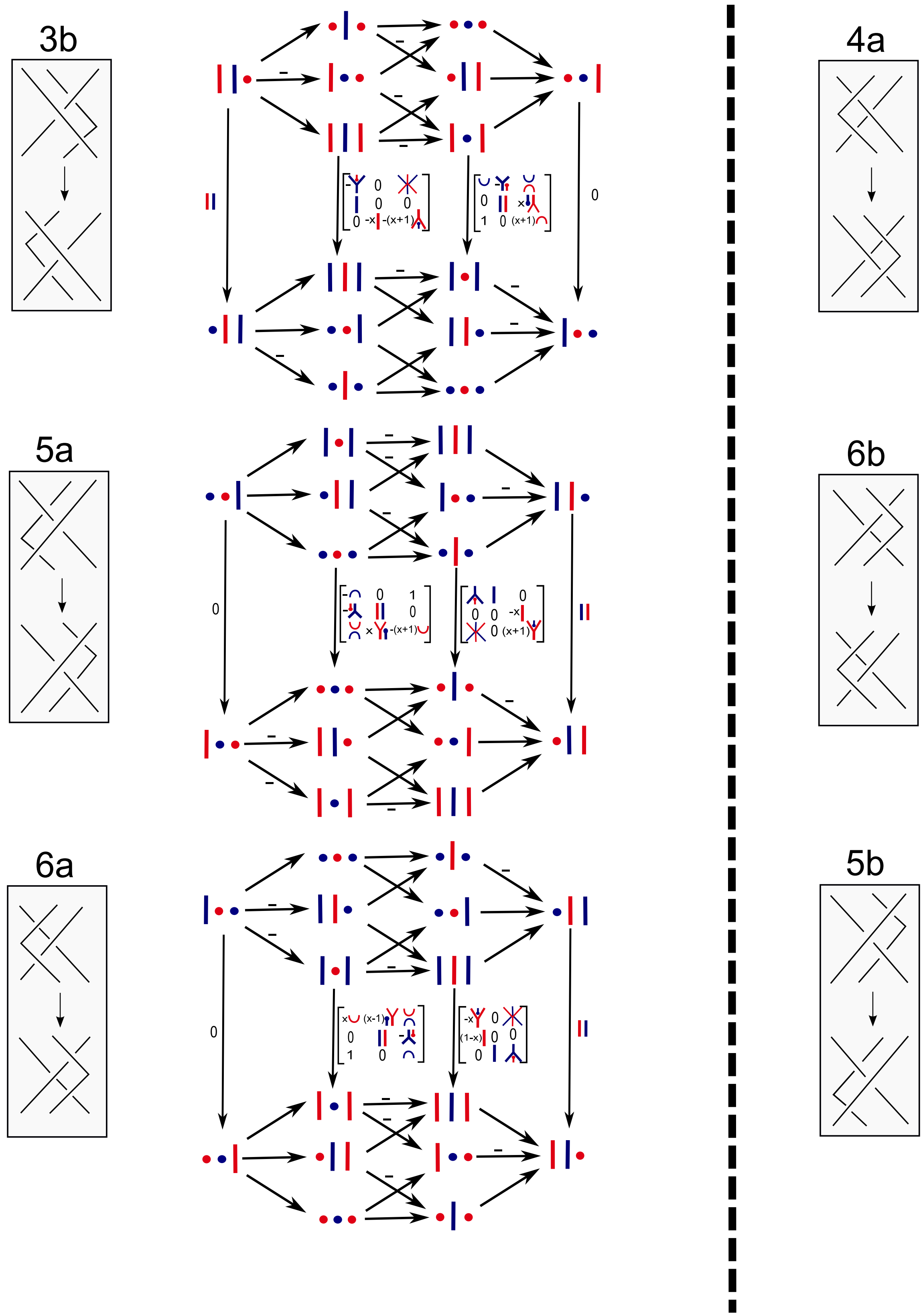


Figure 3.8: Reidemeister 3 type movie move generators

Claim 3.14. *Up to homotopy, each of the maps above is independent of x.*

Proof. We prove the claim for generator 1a above; all the others follow from essentially the same computation. One can easily observe that there are very few summands of the source complex which admit degree -1 maps to summands of the target complex. In fact, the unique (up to scalar) non-zero map of homological degree -1 and graded degree -1 is a red trivalent vertex: a red fork which sends the single red line in the second row of the source complex to the double red line in the second row of the target complex. Given two chain maps, one with free variable x and one with say x', the homotopy is given by the above fork map, with coefficient $(x-x')$. The homotopies for the other variants are exactly the same, save for the position, color, and direction of the fork (there is always a unique map of homological and graded degree -1).

Remark 3.15. For all movie generators, there is a summand of both the source and the target which is $B_\emptyset$. We have clearly used the convention that for Type I movie generators, the induced map from the $B_\emptyset$ summand in the source to the $B_\emptyset$ summand in the target is the identity map. It is true that, with this convention, the chain maps above are the unique chain maps which would satisfy the movie move relations, where the only allowable freedom is given by the choice of various parameters x (exercise). There is no choice up to homotopy, so this is a unique solution.

Remark 3.16. Ignoring this convention, each of the above maps may be multiplied by an invertible scalar. Some relations must be imposed between these scalars, which the reader can determine easily by looking at the movie moves (each side must be multiplied by the same scalar). Movie move 11 forces all slide generators to have scalar 1. Movie move 13 forces all R3 generators to have scalar 1. Movie move 14 and 2 combined force the scalar for any R2 generator to be ± 1, and then movie moves 2 and 5 force this sign to be the same for all 4 variants. Movie move 12 shows that the scalar for the birth of an overcrossing and the death of an undercrossing are related by the sign for the R2 generator. So the remaining freedom in the definition of the functor is precisely a choice of one sign and one invertible scalar.

3.4 Checking the Movie Moves

3.4.1 Simplifications

Given that the functor F has been defined explicitly, checking that the movie moves hold up to homotopy can be done explicitly. One can write down the chain maps for both complexes, and either check that they agree, or explicitly find the homotopy which gives the difference. This is not difficult, and many computations of this form were done as sanity checks. However, there are so many variants of each movie move that writing down every one would take far too long.

Thanks to Morrison, Walker, and Clark (10), a significant amount of work can be bypassed using a clever argument. The remainder of this section merely repeats results from that paper.

Let P, Q, T designate braid diagrams. $\mathrm{Hom}(P, Q)$ will designate the hom space between $F(P), F(Q)$ in the homotopy category of complexes in $\mathcal{SC}_2$. We write HOM for the graded vector space of all morphisms of complexes (not necessarily in degree 0). $\mathrm{Hom}(B_{\underline{i}}, B_{\underline{j}})$ will still designate the morphisms in $\mathcal{SC}_1$. Let 1 designate the crossingless braid diagram.

Lemma 3.17. *(see (10)) Suppose that Movie Move 2 holds. Then there is an adjunction isomorphism* $\mathrm{Hom}(PO_i, Q) \to \mathrm{Hom}(P, QU_i)$*, or more generally* $\mathrm{Hom}(PT, Q) \to \mathrm{Hom}(P, Q\overline{T})$*. Similarly for other variations:* $\mathrm{Hom}(O_iP, Q) \to \mathrm{Hom}(P, U_iQ)$*,* $\mathrm{Hom}(P, QO_i) \to \mathrm{Hom}(PU_i, Q)$*, etc.*

Proof. Given a map $f \in \mathrm{Hom}(PO_i, Q)$, we get a map in $\mathrm{Hom}(P, QU_i)$ as follows: take the R2 movie from P to PO_iU_i, then apply $f \otimes \mathrm{id}_{U_i}$ to QU_i. The reverse adjunction map is similar, and the proof that these compose to the identity is exactly Movie Move 2. □

For any braid P, $\mathrm{Hom}(P, P) \cong \mathrm{Hom}(1, P\overline{P})$.

Note that in the braid group, $P\overline{P} = 1$.

Lemma 3.18. *Suppose that Movie Moves 3, 5, 6, and 7 hold. Then if P and Q are two braid diagrams which are equal in the braid group, then* $\mathrm{Hom}(P, T) \cong \mathrm{Hom}(Q, T)$*.*

Proof. If two braid diagrams are equal in the braid group, one may be obtained from the other by a sequence of R2, R3, and distant crossing switching moves. Put together, these movie moves imply that all of the above yield isomorphisms of complexes. Thus P and Q have isomorphic complexes. $\square$

Remark 3.19. Technically, we don't even need these movie moves, only the resulting isomorphisms, which were already shown by Rouquier. However, since these movie moves are easy to prove and we desired the proofs in this paper to be self-contained, we show the movie moves directly.

Now the complex associated to 1 is just $B_\emptyset$ in homological degree 0. So $\mathrm{HOM}(1,1) = \mathrm{HOM}(B_\emptyset, B_\emptyset)$, which we have already calculated is the free polynomial ring generated by double dots. In particular, the degree 0 morphisms are just multiples of the identity. Remember, this is a non-trivial fact in the graphical context! We will say more about this in Section 3.5.1.

Putting it all together, we have

Suppose that Movie Moves 2,3,5,6,7 all hold. If P and Q are braid diagrams which are equal in the braid group, then $\mathrm{Hom}(P,Q) \cong \Bbbk$, a one-dimensional vector space.

The practical use of finding one-dimensional Hom spaces is to apply the following method.

(See (10)) Consider two complexes A and B in an additive $\Bbbk$-linear category. We say that a summand of a term in A is *homotopically isolated* with respect to B if, for every possible homotopy h from A to B, the map $dh + hd \colon A \to B$ is zero when restricted to that summand.

Lemma 3.20. *Let ϕ and ψ be two chain maps from A to B, such that $\phi \equiv c\psi$ up to homotopy, for some scalar $c \in \Bbbk$. Let X be a homotopically isolated summand of A. Then the scalar c is determined on X, that is, $\phi = c\psi$ on X.*

The proof is trivial, see (10). The final result of this argument is the following corollary.

Suppose that Movie Moves 2,3,5,6,7 all hold. If P and Q are braid diagrams which are equal in the braid group, and ϕ and ψ are two chain maps in $\mathrm{Hom}(P,Q)$ which agree on a homotopically isolated summand of P, then ϕ and ψ are homotopic.

Proof. Because the Hom space modulo homotopy is one-dimensional, we know there exists a constant c such that $\phi \equiv c\psi$. The agreement on the isolated summand implies that $c = 1$. $\square$

Most of the movie generators are isomorphisms of complexes; only birth and death are not. Hence, Movie Moves 1 through 10 all consist of morphisms P to Q, for P and Q equal in the braid group. Finding a homotopically isolated summand and checking the map on that summand alone will greatly reduce any work that needs to be done. Of course, one must show Movie Moves 2,3,5,6,7 independently before this method can be used.

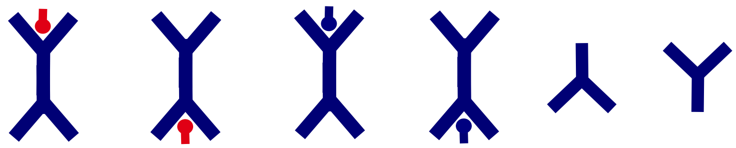
One final simplification, also found in Morrison, Walker and Clark, is that modulo Movie Move 8 all variants of Movie Move 10 are equivalent. Hence we can prove Movie Move 10 by investigating solely the overcrossing-only variant.

These simplifications apply to any functorial theory of braid cobordisms, so long as $\mathrm{Hom}(1, 1)$ is one-dimensional. Now we look at what we can say specifically about homotopically isolated summands for Rouquier complexes in $\mathcal{SC}_2$.

Any homotopy must be a map of degree -1 (if we ignore degree shifts on objects, as in our conventions). There are very few maps of negative degree in $\mathcal{SC}_2$, a fact which immediately forces most homotopies to be zero. For instance, there are no negative degree maps from $B_\emptyset$ to B_i, for any i. In an overcrossing-only braid, where $B_\emptyset$ occurs in the maximal homological grading and various B_i show up in the penultimate homological grading, the $B_\emptyset$ summand is homotopically isolated! Thus the overcrossing-only variant of Movie Move 10 will be easy. In fact, because of the convention we use that all isomorphism movie generators will restrict to multiplication by 1 from the $B_\emptyset$ summand to the $B_\emptyset$ summand, checking Movie Move 10 is immediate.

The only generators of negative degree are trivalent vertices. If each color appears no more than once in a complex, then there can be no trivalent vertices, so no homotopies are possible. This will apply to every variant of Movie Move 4, for instance.

Deducing possible homotopies is easy, as there are very few possibilities. For instance, the only nonzero maps which occur in homotopies outside of Movie Move 10 are:



We will not use these simplifications to their maximal effect, since some checks are easy enough to do without. For a discussion of other implications of checking the movie moves by hand, see Section 3.5.1.

3.4.2 Movie Moves

NOTE:

(Logical sequence in the proofs of the movie moves.) We list the movie moves in numerical order, as opposed to logical order of interdependence. To use the technical lemma about homotopically isolated summands we first need to check movie moves 2,3,5,6,7. The reader will see that we prove these through direct computation, relying on none of the other moves.

- **MM1** There are eight variants of this movie (sixteen if you count the horizontal flip, which is just a color symmetry), of which we present two explicitly here. The key fact is that every slide generator behaves the same way: chain maps on summands have either a color crossing with a minus sign, the identity map with a plus sign, or zero; these maps occur precisely between the only summands where they make sense and, hence, have the same signs on both sides of the movie. Reversing direction uniformly changes the sign on the cups or caps in the R2 move. The only interesting part of the check uses a twist of relation (3.11). We describe in detail the movie associated to the first generator in figure 3.9, and give the composition associated to generator 3 in figure 3.10. Note that this check is trivial anyway since every summand is homotopically isolated.

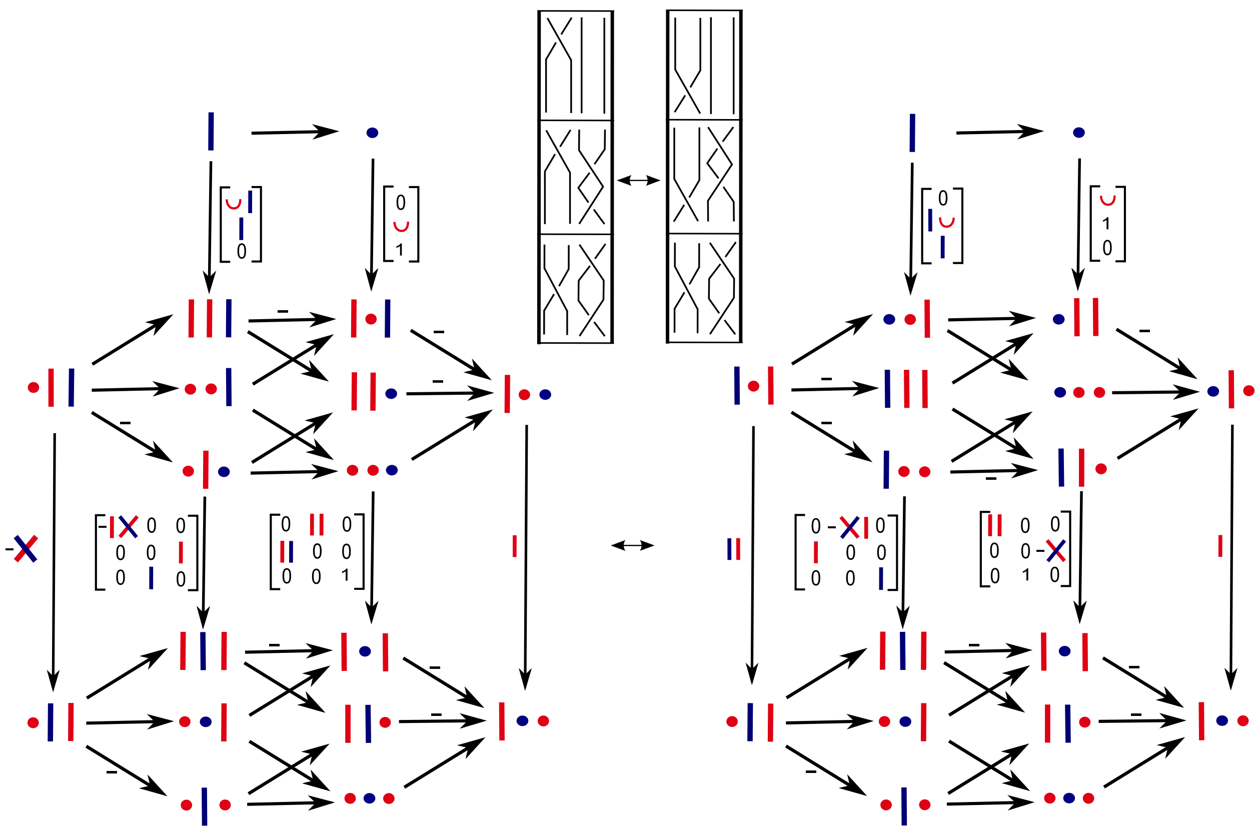


Figure 3.9: Movie Move 1 associated to slide generator 1

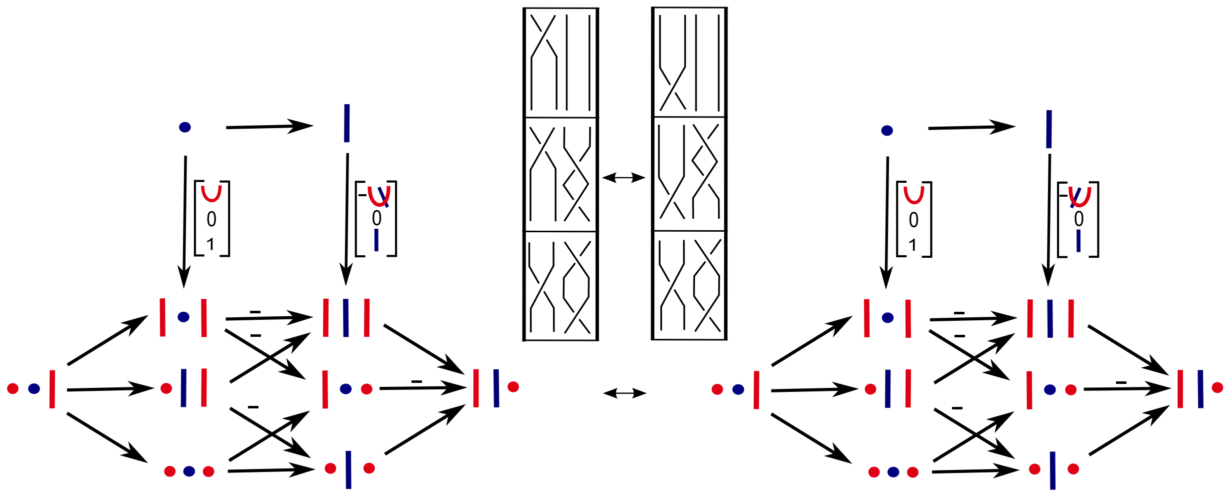


Figure 3.10: Movie Move 1 associated to slide generator 3

- **MM2** There are 4 variants to deal with here; we describe only one, and similar reasoning to that of MM1 will convince the reader that the other 3 are readily verified. The composition has the following form:

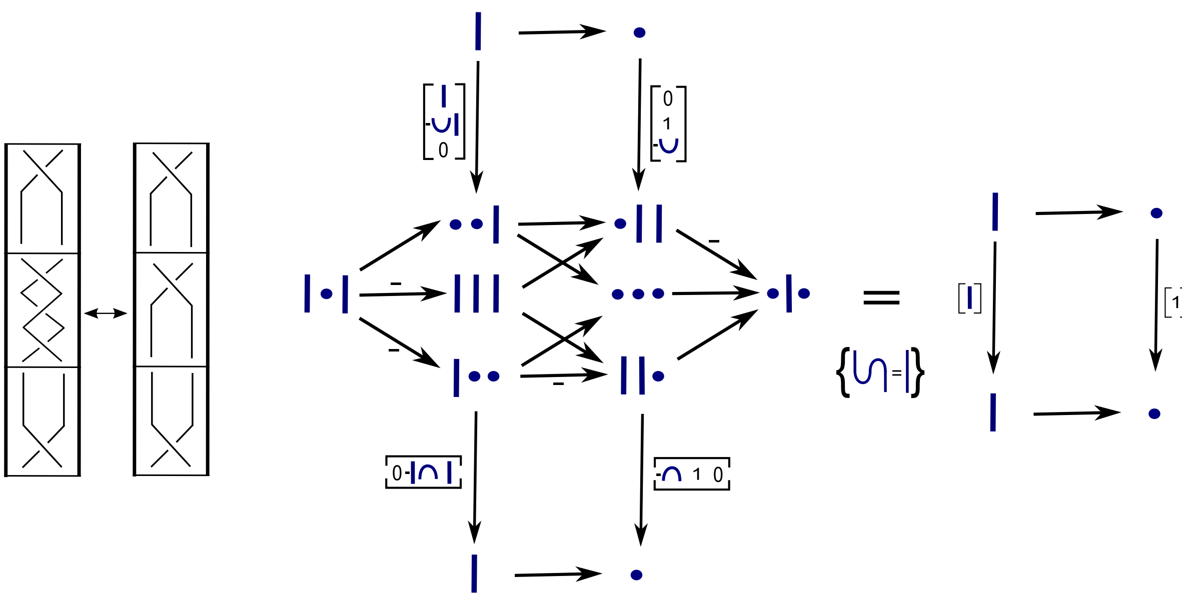


Figure 3.11: Movie Move 2

- **MM3** All 8 movie move 3 variants are essentially immediate after glancing at the slide generators, but we list one for posterity:

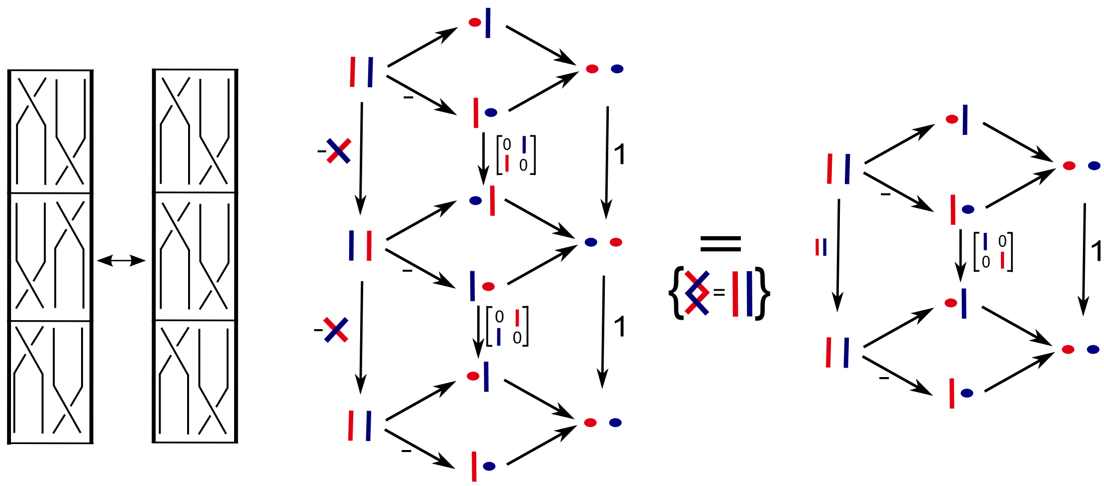


Figure 3.12: Movie Move 3

- **MM4** At this point the conscientious reader will find all 16 variants of movie move 4 quite easy, for the regularity of the slide chain maps allows one to write the compositions for the left and right-hand side at once. The maps only differ at the triple-color crossings, so we have to make use of relation (3.22).

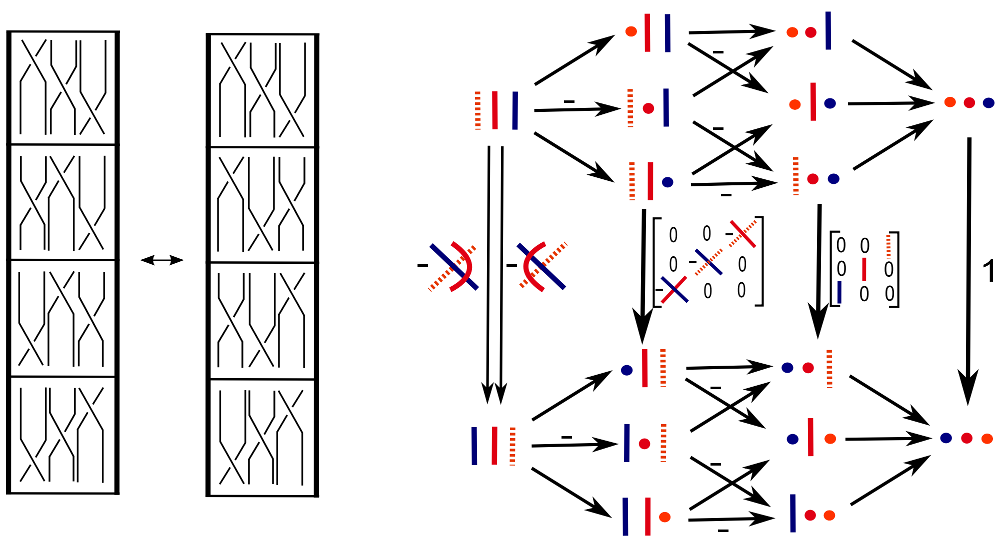


Figure 3.13: Movie Move 4

- **MM5** There are two variants of this movie, with the calculation for both almost identical. We consider the move associated to the first generator. The compostion has the following form:

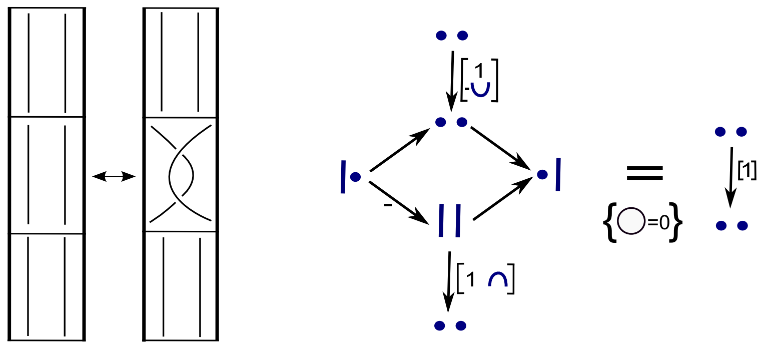


Figure 3.14: Movie Move 5

- **MM6** Again there are two variants and the calculation is almost as easy as the one for MM5; the only difference is that here we actually have to produce a homotopy. We check the variant associated to generator 1; left arrows are the identity, right the composition, and dashed the homotopy. Checking that the homotopy works requires playing with relation (3.16).

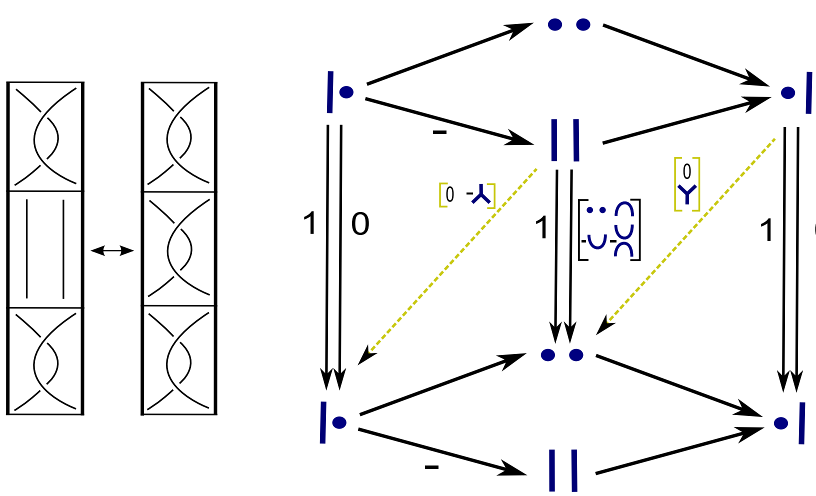
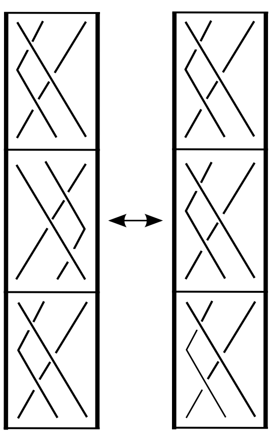


Figure 3.15: Movie Move 6

- **MM7** There are 12 variants of MM7, one for each R3 generator, and color symmetry will immediately reduce the number of different checks to 6; nevertheless, this is still a bit a drudge as each one requires a homotopy and a minor exercise in the relations. We display the movie associated to generator 1a and leave it to the very determined reader to repeat a very similar computation the remaining 5 times. The chain maps for the left-hand side of the movie are the following:



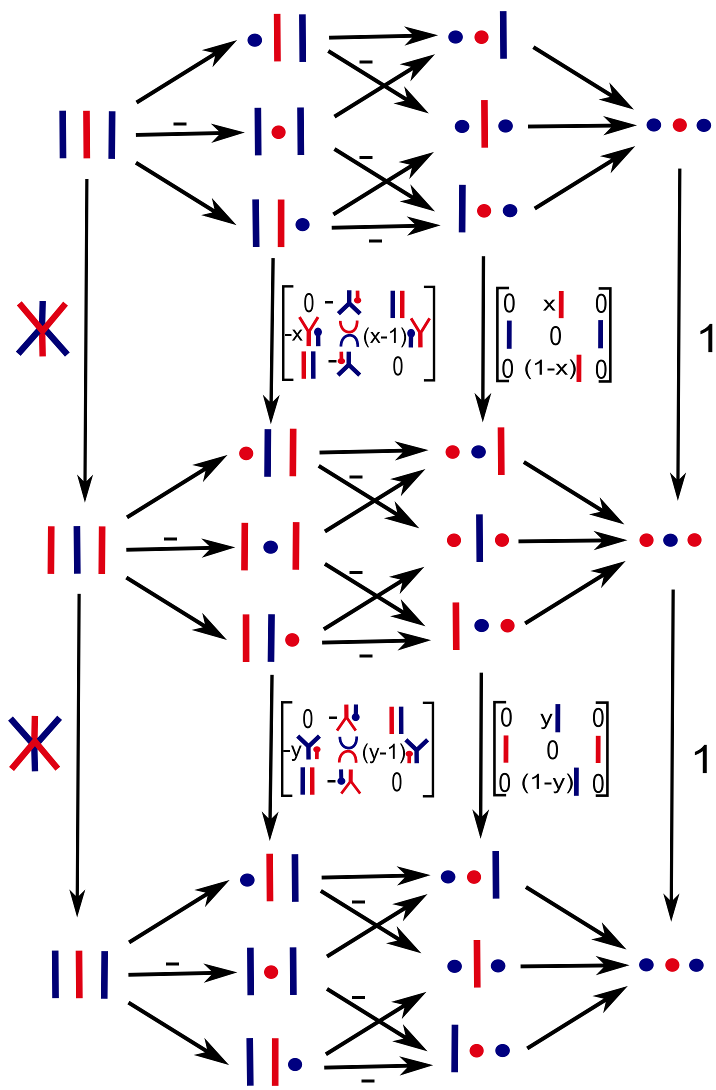


Figure 3.16: Movie Move 7

The composition and homotopy is:

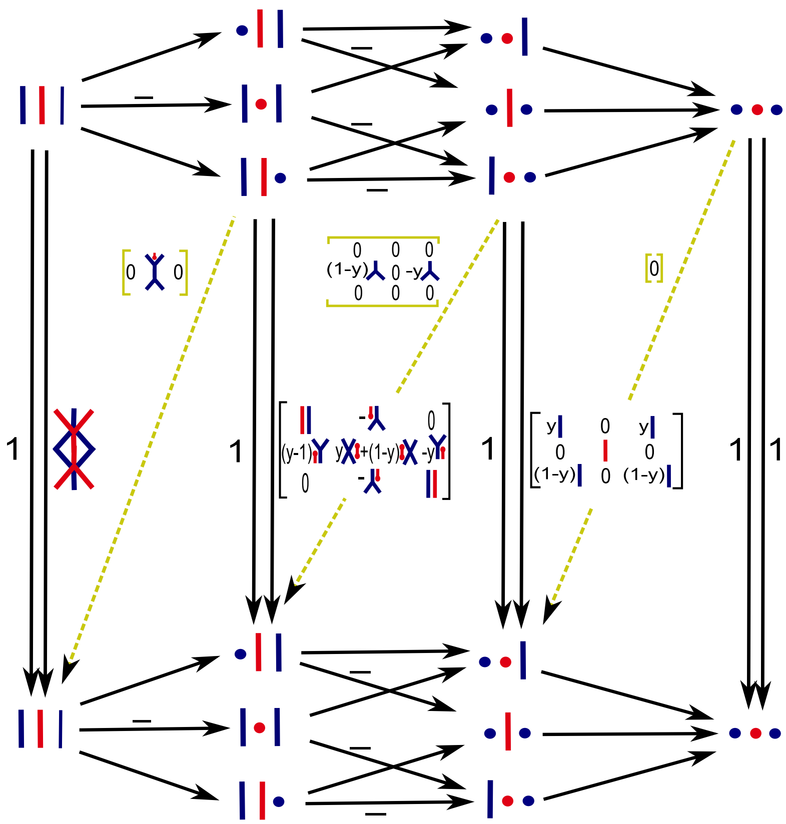
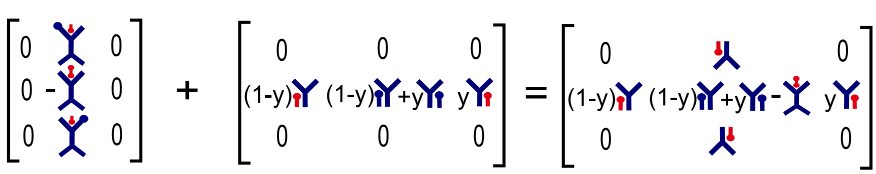
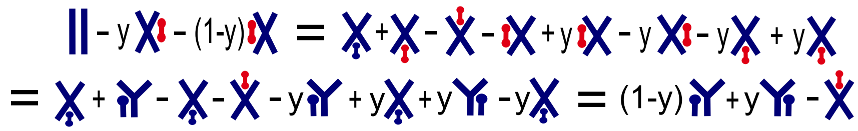


Figure 3.17: Homotopy for Movie Move 7

To check that the prescribed maps actually give a homotopy between and composition and the identity still requires some manipulation. The verification for the left-most map is simply relation (3.24). The verification for the right-most map is immediate, and for the third map is simple. This leaves us with the second map. Here $dH + Hd =$



which save for the central entry is precisely the identity minus the composition. Equality of the central entry follows from this computation:



NOTE: This computation was done using relation (3.26) numerous times.

- **MM8** There are twelve variants of MM8: 3! possibilities for height order, and two directions the movie can run. All twelve are dealt with by the same argument, using a homotopically isolated summand. There are no degree -1 maps from $B_\emptyset$ to any summand in the target, since there are at most two lines of a given color in the target, so we can assume there are no trivalent vertices. Hence the $B_\emptyset$ summand of the source is homotopically isolated, so we need only keep track of the homological degree 0 part, which significantly simplifies the calculation. We present one variant in diagram 3.18.

  Composing the chain maps for the two sides of MM8 we see that they agree on $B_\emptyset$.

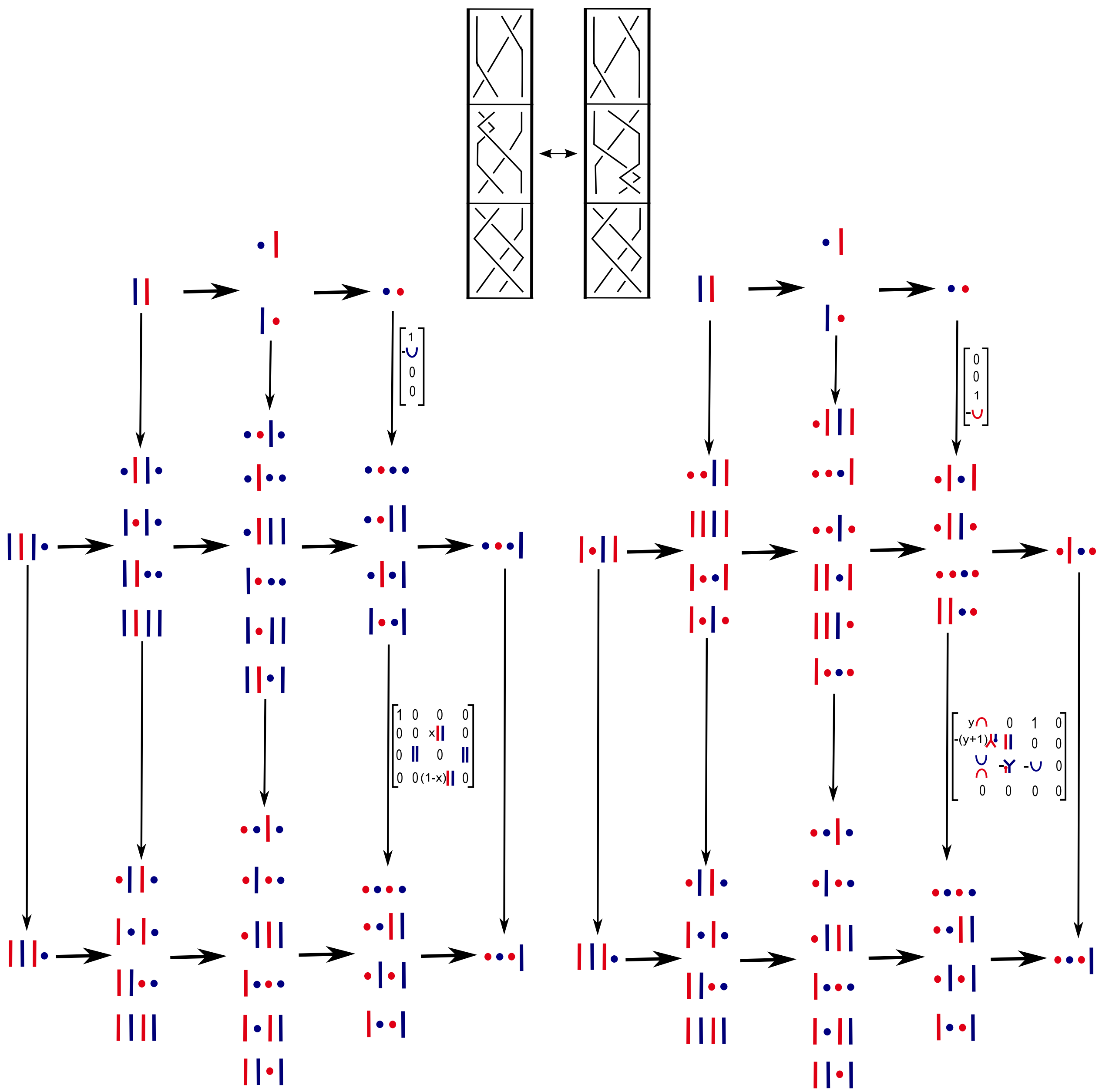


Figure 3.18: Movie Move 8

- **MM9** There are a frightful 96 versions of MM9, coming from all the different R3 moves that can be done (12 in all), the type of crossing that appears in the slide, and horizontal and vertical flips. Once again, homotopically isolated summands come to the rescue. Again, in each variant there are no more than two crossings of a given color, so all maps from $B_\emptyset$ to each summand in the target have non-negative degree. Thus the $B_\emptyset$ summand of the source is homotopically isolated. Three colors are involved, the distant color and two adjacent colors. In the $B_\emptyset$ summand, the distant-colored line does not appear, and no application of a distant slide or R3 move can make it appear. When the distant-colored line does not appear, the distant slide move acts by the identity. Thus both the right and left sides of the movie act the same way on the $B_\emptyset$ summand, namely, they perform the R3 operation to it (sending it to the appropriate summands of the target).

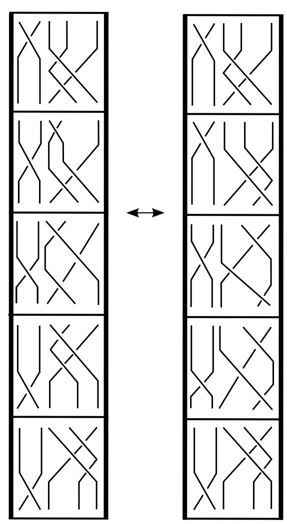


Figure 3.19: Movie Move 9

- **MM10** The sheer burden of writing down the complexes and calculating the chain maps for even one version of MM10 is best avoided at all costs. Despite at first seeming the more complicated of the movie moves, it is in the end the easiest to verify. We begin noting that, once one has shown MM8, all of the versions of MM10 are equivalent (see section 3.2.2 in (10)). So let us consider the variant with all left crossings. We see immediately that the $B_\emptyset$ summand is homotopically isolated, that it is the unique summand in homological degree 0 in every intermediate complex, and that the chain maps all act by the identity in homological degree 0. Hence both sides agree on a homotopically isolated summand.

- **MM11** There are 32 variants of MM11: 2 choices of crossing, a vertical and a horizontal flip, and the direction of the movie. Half of these have chain maps that compose to zero on both sides, since the birth of a right crossing or the death of a left crossing is the zero chain map. The rest are straightforward. We give an example below in figure 3.20.

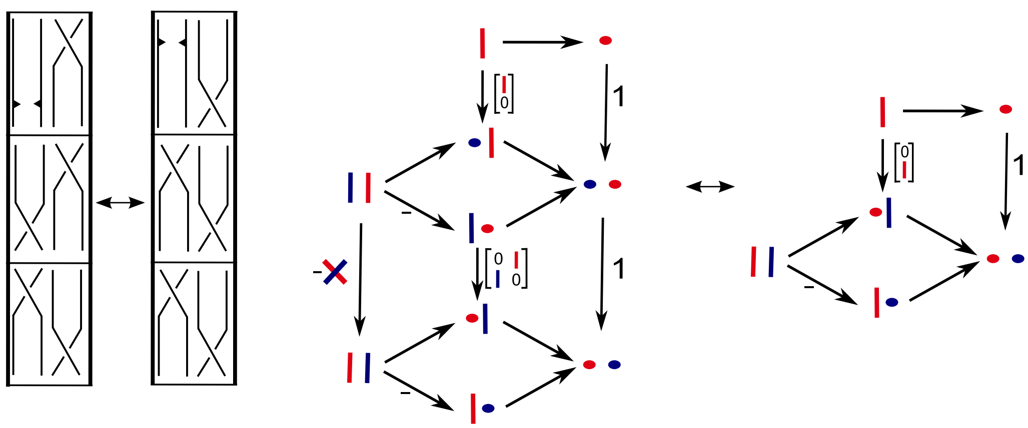


Figure 3.20: Movie Move 11

- **MM12** There are 8 variants: a choice of R2 move, a vertical flip, and the direction of the movie. Again, half of these are zero all around. Here are two variants; the other two are extremely similar.

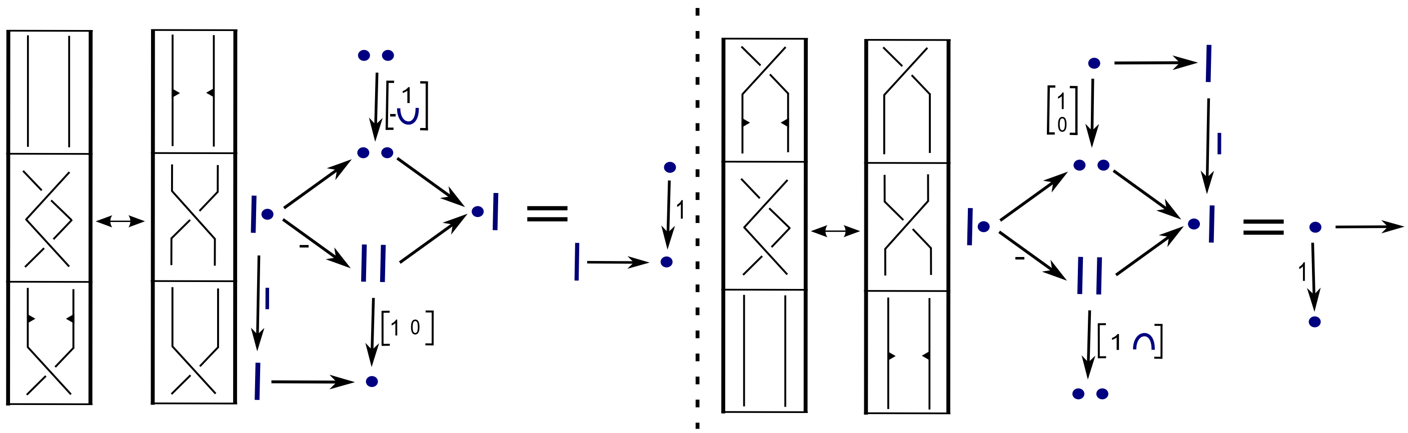


Figure 3.21: Movie Move 12

- **MM13** There are 24 variants: 12 R3 generators and two directions. Half are zero, and color symmetry for R3 generators reduces the number to check by half again. For the 6 remaining variants, the check requires little more than just writing down the composition, since the required homotopy in each instance is quite easy to guess. In figure 3.22 we describe the variant associated to to the first R3 generator.

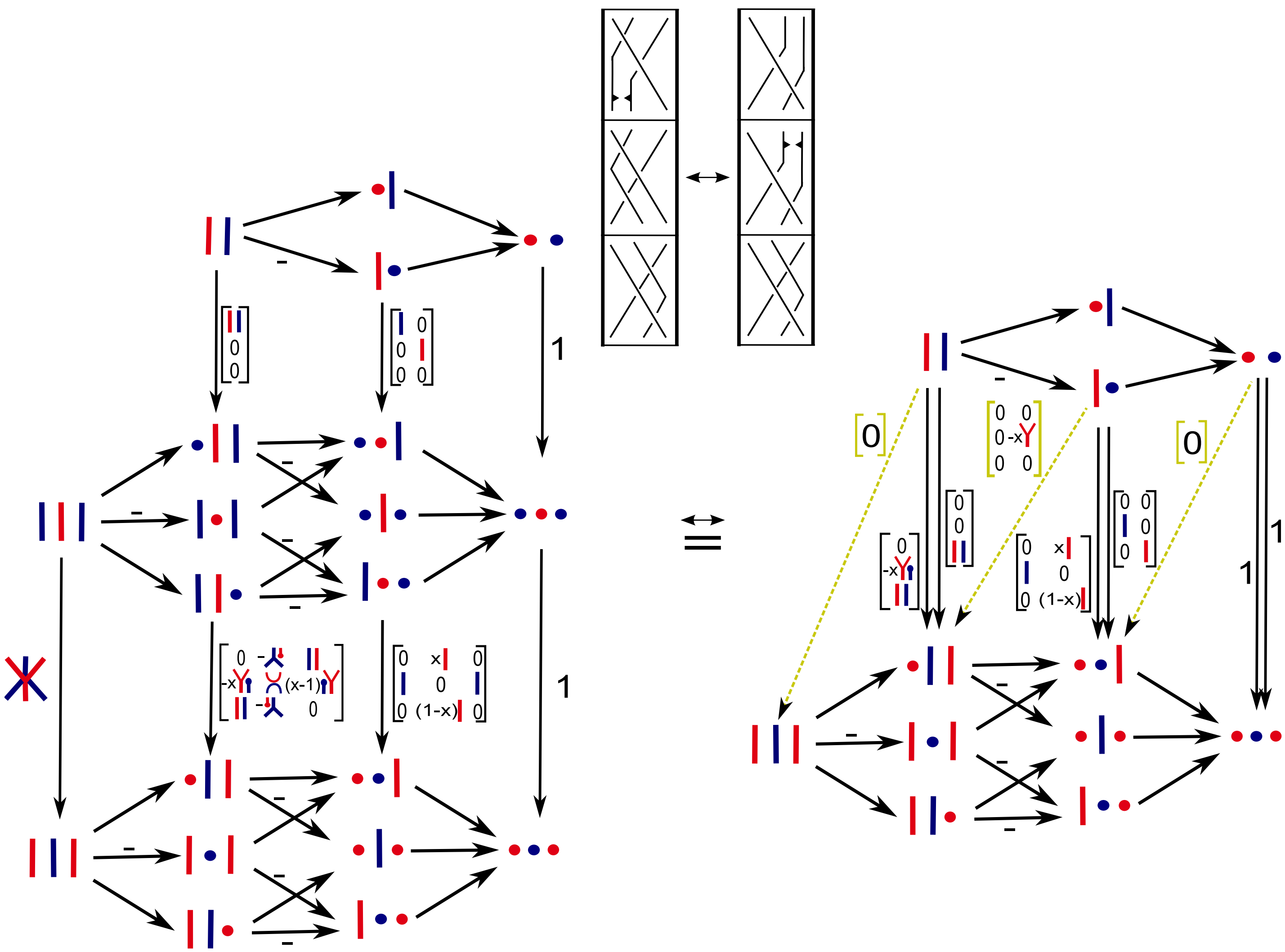


Figure 3.22: Movie Move 13

- **MM14** Since none of the R3 generators of type 1 or 2 is compatible with MM14, we are left with 16 variants: 8 R3 generators and 2 directions. As usual, there are only 4 to check. In addition to this, the initial frame of the movie corresponds to a complex supported in homological degree 0 only, so we only need write down what happens there. In figure 3.23 we describe the variant associated to the R3 generator 3a.

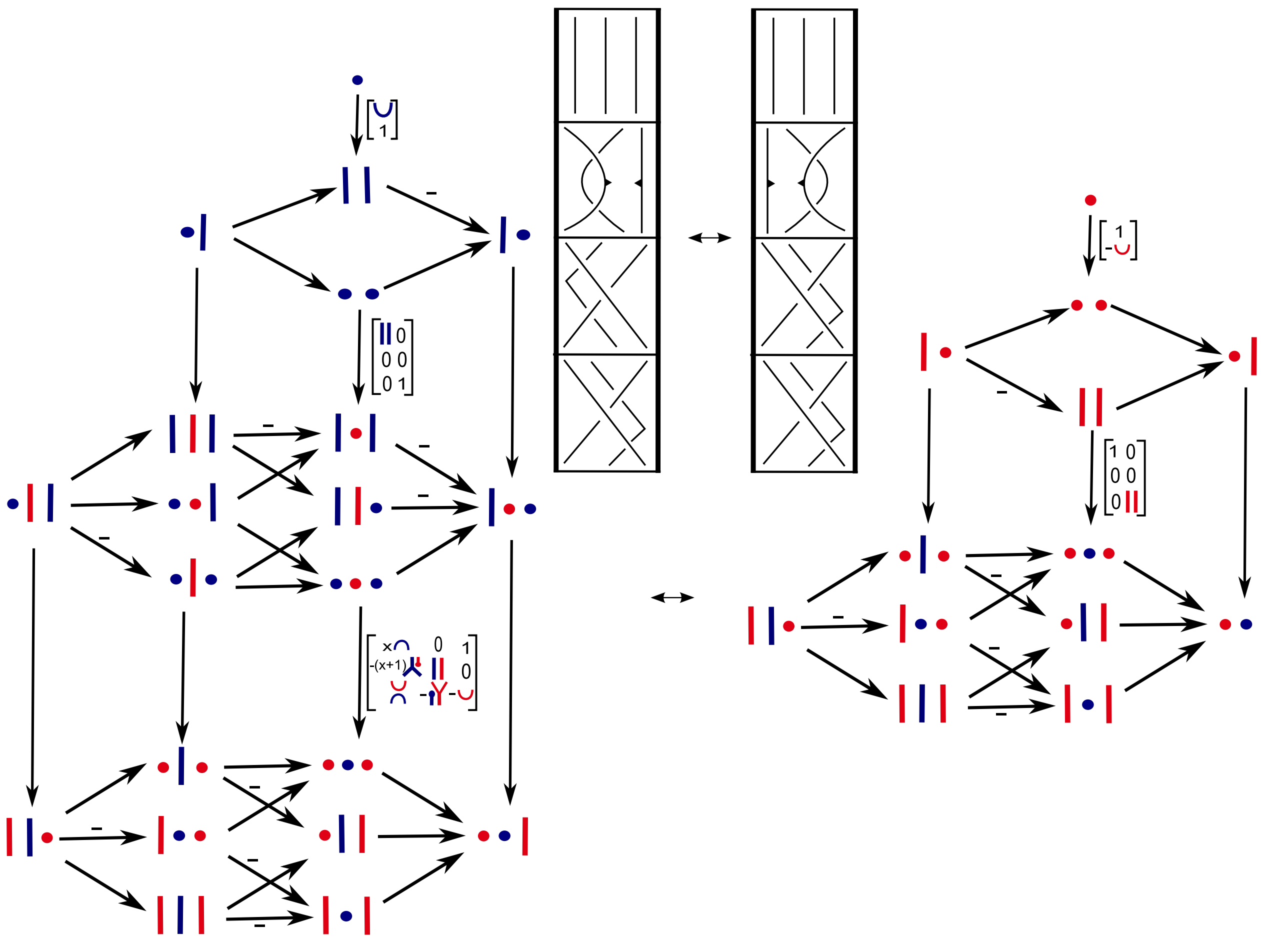


Figure 3.23: Movie Move 14

3.5 Additional Comments

3.5.1 The Benefits of Brute Force

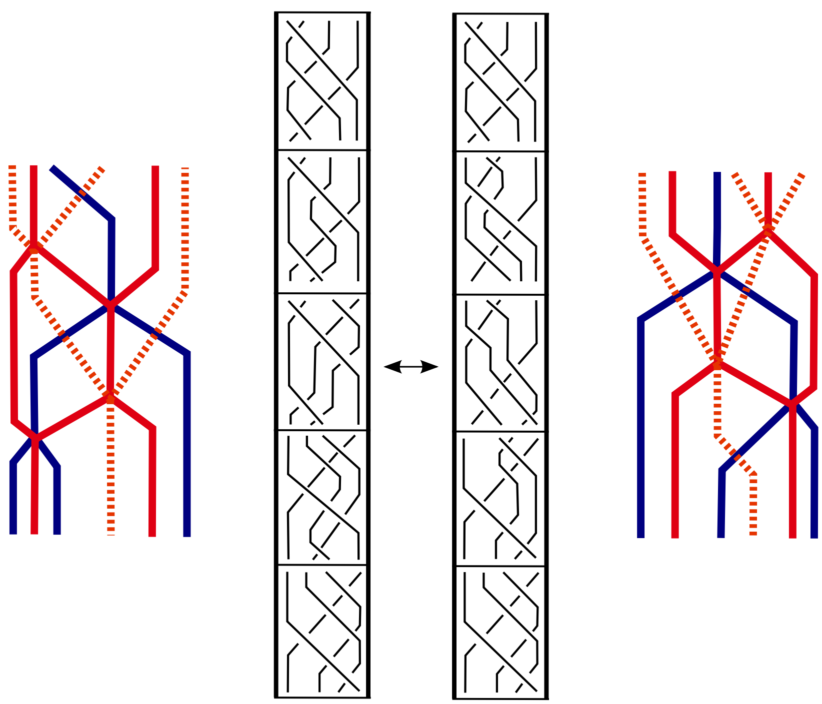
We have now shown that there is a functor from the braid cobordism category into the homotopy category of complexes in $\mathcal{SC}_2$. Our method of proof used homotopically isolated summands, and hence relied on the fact that $\mathrm{Hom}(B_\emptyset, B_\emptyset)$ was 1-dimensional. This is a trivial fact in the context of R-bimodules, amounting to the statement that $\mathrm{HOM}(R,R) = R$. However, it is a non-trivial fact to prove for the graphical definition of $\mathcal{SC}_1$, requiring the more complicated graphical proofs in (1). Moreover, $\mathrm{Hom}(B_\emptyset, B_\emptyset)$ need not be 1-dimensional in some arbitrary category $\mathcal{C}$ of which $\mathcal{SC}_1$ is a (non-full) subcategory, and we may be interested in such categories $\mathcal{C}$. For instance, it would be interesting to define such a category $\mathcal{C}$ for which one would have all birth and death maps nontrivial (although the authors have yet to find an *interesting* extension of this type).

Our method of proof, however, is irrelevant and the truth of Theorem 3.1 does not depend on it. One could avoid any machinery by checking each movie move explicitly (in fact, the only ones that remain to be checked are MM8, MM9, and MM10). Checking even a single variant of MM10 by brute force is extremely tedious, since each complex has 64 summands, but it could be done. In addition, we have actually proven slightly more: for any additive monoidal category $\mathcal{C}$ having objects B_i and morphisms satisfying the $\mathcal{SC}_1$ relations, we can define a functor from the braid cobordism category into the homotopy category of complexes in $\mathcal{C}$. This is an obvious corollary, since that same data gives a functor from $\mathcal{SC}_2$ to $\mathcal{C}$. If one chose to change the birth and death maps, the proof for movie moves 1 through 10 would be unchanged, and one would only need to check 11 through 14.

One other benefit to (theoretically) checking everything by hand is in knowing precisely which coefficients are required, and thus understanding the dependence on the base ring $\Bbbk$. In all the movie moves we check in this paper, each differential, chain map, and homotopy has integral coefficients (or free variables which may be chosen to be integral). In fact, every nonzero coefficient that didn't involve a free variable was ± 1, and free variables may be chosen such that every coefficient is 1,0, or -1. From our other calculations, the same should be true for MM8 through MM10 as well (Khovanov and Thomas (24) already showed that Rouquier complexes lift over $\mathbb{Z}$ to a projective functor, which implies the existence of homotopy maps over $\mathbb{Z}$). The next section discusses the definition of this functor in a $\mathbb{Z}$-linear category.

As an additional bonus, checking the movie moves does provide some intuition as to why $\mathcal{SC}_1$ has the relations that it does. One might wonder why these particular relations should be correct: in (1) we know they are correct because they hold in the R-bimodule category and because they are sufficient to reduce all graphs to a simple form. There should be a more intuitive explanation.

As an illustrative example, consider the overcrossing-only variation of Movie Move 10 and the unique summand of lowest (leftmost) homological degree: it is a sequence of 6 lines. Then the left hand movie and the right hand movie correspond to the following maps on this summand:



Thus equality of these two movies on the highest term, modulo relation (3.17), is exactly relation (3.27).

Similarly, the highest terms in various other movie move variants utilize the other relations, as in the chart below.

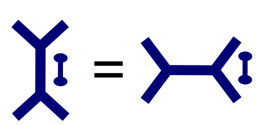
MM	Relation
1	(3.11)
2	(3.8)
3	(3.17)
4	(3.22)
5	(3.15)
8	(3.12)
9	(3.21)
10	(3.27)

We can view these relations heuristically as planar holograms encoding the equality of cobordisms given by the movie moves.

More relations are used to imply that certain maps are chain maps, or that homotopies work out correctly. For example, relation (3.18) is needed for the slide generator to be a chain map. One can go even further, although we shall be purposely vague: so long as one disallows certain possibilities (like degree ≤ 0 maps from a red line to a blue line, or negative degree endomorphisms of indecomposable objects) then our graphical generators must exist a priori, and must satisfy a large number of the relations above.

Type II movie moves (11 through 14) do not contribute any relations or requirements not already forced by Type I movie moves (although they do fix the sign of various generators).

Almost every relation in the calculus is used in a brute force check of functoriality (including the brute force checks of MM8-10). However, there are two exceptions: (3.13) and (3.25). Both these relations are in degree -2, and degree -2 does not appear in chain maps or homotopies, so they could not have appeared. Nonetheless, they are effectively implied by the remainder of the relations. It is not hard to use the rest of the one color relations to show that



Hence, (3.13) will hold, so long as R acts freely on morphisms. Under this mild assumption, all the relations are required. While no proof is presented here, it is safe to say that the category $\mathcal{SC}_1$ is universal amongst all categories for which Rouquier complexes could be defined functorially up to Type I movie moves (under suitable conditions on color symmetry and torsion-free double dot actions), and that these relations are effectively predetermined.

3.5.2 Working over $\mathbb{Z}$

Knot theorists should be interested in a $\mathbb{Z}$-linear version of the Soergel bimodule story, because it could theoretically yield a functorial link homology theory over $\mathbb{Z}$. We describe the $\mathbb{Z}$-linear version below. Because defining things over $\mathbb{Z}$ is not really the focus of this paper, and because a thorough discussion would require poring over (1) for coefficients, we do not provide rigorous proofs of the statements in this section.

Ignoring the second equality in (3.26), which is equivalent to (3.16) after multiplication by 2, every relation given has coefficients in $\mathbb{Z}$. One could use these relations to define a $\mathbb{Z}$-linear version of $\mathcal{SC}_1$ and $\mathcal{SC}_2$, and then use base extension to define the category over any other ring. The functor can easily be defined over $\mathbb{Z}$, as we have demonstrated, and all the brute force checks work without resorting to other coefficients. Theorem 3.1 still holds for the $\mathbb{Z}$-linear version of $\mathcal{SC}_2$.

In fact, the same method of proof (using homotopically isolated summands) will work over $\mathbb{Z}$ in most contexts. One begins by checking the isomorphisms (3.4) through (3.6). The only one which is in doubt is $B_i \otimes B_i \cong B_i\{-1\} \oplus B_i\{1\}$. So long as, for each i, there is an adjacent color in I, we may use (3.29) to check this isomorphism. Otherwise, we are forced to use (3.28), which does not have integral coefficients.

For now, assume that adjacent colors are present; we will discuss the other case below. One still has a map of algebras from $\mathcal{H}$ to the additive Grothendieck group of $\mathcal{SC}_1$. A close examination of the methods used in the last chapter of (1) will show that the graphical proofs which classify $\mathrm{HOM}(\emptyset, \underline{\boldsymbol{i}})$ still work over $\mathbb{Z}$ in this context. Boundary dots with a polynomial will be a spanning set for morphisms. One can still define a functor into a bimodule category to show that this spanning set is in fact a basis. Therefore, the Hom space pairing on $\mathcal{SC}_1$ will induce a semi-linear pairing on $\mathcal{H}$, and it will be the same pairing as before. Hom spaces will be free $\mathbb{Z}$-modules of the appropriate graded rank, and this knowledge suffices to use all the homotopically isolated arguments.

Remark 3.21. This statement does *not* imply that $\mathcal{SC}$ will categorify the Hecke algebra when defined over $\mathbb{Z}$. There may be missing idempotents, or extra non-isomorphic idempotents, so that the Grothendieck ring of the idempotent completion may be too big or small.

If adjacent colors are not present, the easiest thing to do to prove Theorem 3.1 is to include $\mathcal{SC}_1(I)$ into a larger $\mathcal{SC}_1(I')$ for which adjacent colors are present. Since this inclusion is faithful, all movie move checks which hold for I' will hold for I. Alternatively, one could use an extension of the category $\mathcal{SC}_1(I)$, extending the generating set by adding more polynomials, either as originally done in (1), or by formally adding $\frac{1}{2}$ times the double dot. Both of these should give an integral version of the category where the isomorphism (3.4) holds, and where the graphical proofs of (1) still work. Finally, if one does not mind ignoring 2-torsion, defining the category over $\mathbb{Z}[\frac{1}{2}]$ will also work.

Chapter 4

Intergral HOMFLY-PT and $sl(n)$-link homology

4.1 Background for diagrammatics of Soergel bimodules and Rouquier Complexes

We extend the work in the last chapter on functoriality of Rouquier complexes to the context of HOMFLY-PT and $sl(n)$-link homology. As there has yet to be seen an integral version of either HOMFLY-PT or $sl(n)$ homology for $n > 3$, with the original Khovanov homology being defined over $\mathbb{Z}$ and torsion playing an interesting role, a natural question arose as to whether this graphical calculus could be used to define these. The definition of such integral theories is precisely the purpose of this chapter. The one immediate disadvantage to the graphical approach is that at the present moment there does not exist a diagrammatic calculus for the Hochschild homology of Soergel bimodules. Hence, to define integral HOMFLY-PT homology, our path takes a rather roundabout way, jumping between matrix factorizations and diagrammatic Rouquier complexes whenever one is deemed more advantageous than the other. For the $sl(n)$ version of the story, we add the Rasmussen spectral sequence into the mix and essentially repeat his construction in our context.

The organization of the chapter is the following: in section 4.2 we give a brief account of the necessary tools (matrix factorizations, Soergel bimodules, Hochschild homology, Rouquier complexes, and corresponding diagrammatics) - the emphasis here is brevity and we refer the reader to more original sources for particulars and details; in sections 4.3 and 4.4 we describe the integral HOMFLY-PT complex and prove the Reidemeister moves, utilizing all of the background in 4.2; section 4.5 is devoted to the Rasmussen spectral sequence and integral $sl(n)$-link homology, and we conclude it with some remarks and questions.

Throughout this chapter we will refer to a positive crossing as the one labelled D_+ and negative crossing as the one labelled D_- in figure 4.1. For resolutions of a crossing we will refer to D_o and D_s of figure 4.1 as the "oriented" and "singular" resolutions, respectively. We will use the following conventions for the HOMLFY-PT polynomial

$$aP(D_-) - a^{-1}P(D_+) = (q - q^{-1})P(D_o),$$

with P of the unknot being 1. Substituting $a = q^n$ we arrive at the quantum $sl(n)$-link polynomial.

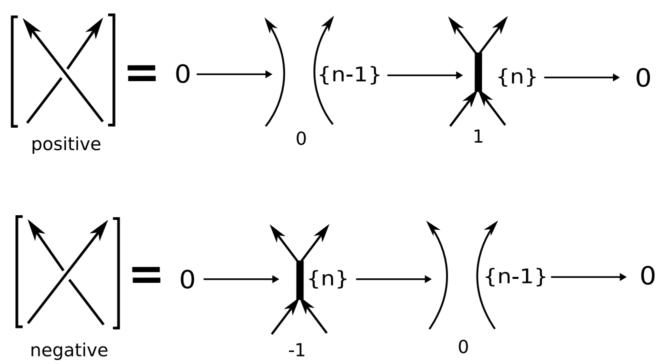


Figure 4.1: Crossings and resolutions

4.2 The toolkit

We will require some knowledge of matrix factorizations, Soergel bimodules and Rouquier complexes, as well as the corresponding diagrammatic calculus of Elias and Khovanov (1). In this section the reader will find a brief survey of the necessary tools, and for more details we refer him to the following papers: for matrix factorizations (23), (33), for Soergel bimodules and Rouquier complexes and diagrammatics (1), (2), (21), (36), and for Hochschild homology (15), (21).

4.2.1 Matrix factorizations

Let R be a Noetherian commutative ring, $w \in R$, and C^*, $* \in \mathbb{Z}$, a free graded R-module. A *$\mathbb{Z}$-graded matrix factorization with potential* w consists of C^* and a pair of differentials $d_\pm : C^* \to C^{*\pm 1}$, such that $(d_+ + d_-)^2 = wId_{C^*}$.

A morphism of two matrix factorizations C^* and D^* is a homomorphism of graded R-modules $f : C^* \to D^*$ that commutes with both d_+ and d_-. The tensor product $C^* \otimes D^*$ is taken as the regular tensor product of complexes, and is itself a matrix factorization with differentials d_+ and d_-. A useful and easy exercise is the following:

Lemma 4.1. *Given two matrix factorizations C^* and D^* with potentials w_c and w_d, respectively, the tensor product $C^* \otimes D^*$ is a matrix factorization with potential $w_c + w_d$.*

Remark 4.2. Following Rasmussen (33), we work with $\mathbb{Z}$-graded, rather than $\mathbb{Z}/2\mathbb{Z}$-graded, matrix factorizations as in (22). The $\mathbb{Z}$-grading implies that $(d_+ + d_-)^2 = wId_{C^*}$ is equivalent to

$$d_+^2 = d_-^2 = 0$$

and

$$d_+d_- + d_-d_+ = wId_{C^*}.$$

In the case that $w = 0$, we acquire a new $\mathbb{Z}/2\mathbb{Z}$-graded chain complex structure with differential $d_+ + d_-$. Suppressing the underlying ring R and potential w, we will denote the category of graded matrix factorizations by mf.

We also need the notion of complexes of matrix factorizations. If we visualize a collection of matrix factorizations as sitting horizontally in the plane at each integer level, with differentials d_+ and d_- running right and left, respectively, we can think of morphisms $\{d_v\}$ between these as running in the vertical direction. If $d_v^2 = 0$ we get a complex, i.e. all together we have that

$$d_\pm : C^{i,j} \to C^{i\pm 1,j}, \quad d_v : C^{i,j} \to C^{i,j+1},$$

where we think of i as the *horizontal* grading and j as the *vertical* grading, and will denote these as gr_h and gr_v, respectively.

In addition we will be taking tensor products of complexes of matrix factorizations (in the obvious way) and, just to add to the confusion we will also have homotopies of these complexes as well homotopies of matrix factorizations themselves. These notions will land us in different categories to which we now give some notation.

- *hmf* will denote the *homotopy category of matrix factorizations*
- $\mathcal{KOM}(mf)$ the *category of complexes of matrix factorizations*
- $\mathcal{KOM}_h(mf)$ *homotopy category of complexes of matrix factorizations*
- $\mathcal{KOM}_h(hmf)$ the obvious conglomerate.

4.2.2 Diagrammatics of Soergel bimodules

The diagrammatic category of Soergel bimodules $\mathcal{SC}_1$ was described in detail in the last chapter. We refer to the results there, but restate a few key facts and add some consequences which were only implicit.

In addition to the bimodules $B_{\underline{i}}$ above, we will require the use of the bimodule $R\otimes_{R^{i,i+1}} R\{-3\}$, where $R^{i,i+1}$ is the ring of invariants under the transpositions $(i, i+1)$ and $(i+1, i+2)$, and will use a black squiggly line, as in equation 4.4 below, to represent it. This bimodule comes into play in the isomorphisms

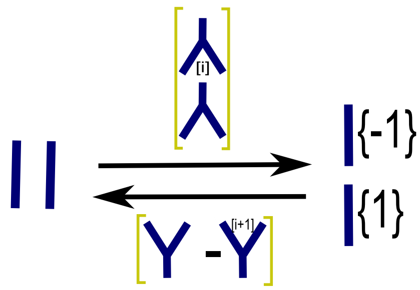
$$B_i \otimes B_{i+1} \otimes B_i \cong B_i \oplus (R \otimes_{R^{i,i+1}} R\{-3\}) \tag{4.1}$$

and

$$B_{i+1} \otimes B_i \otimes B_{i+1} \cong B_{i+1} \oplus (R \otimes_{R^{i,i+1}} R\{-3\}), \tag{4.2}$$

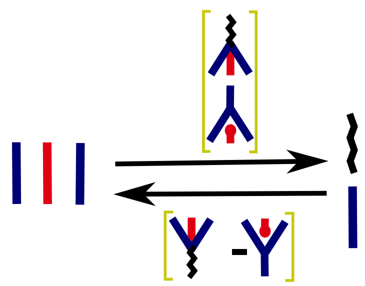
which we will use in the proof of Reidemeister move III.

Recall that our graphs are invariant under isotopy and in addition we have the following isomorphisms or “decompositions”:



(4.3)

Note that this relation is precisely that of $B_i \otimes B_i \cong B_i\{1\} \oplus B_I\{-1\}$ described diagrammatically.



(4.4)

Here we have the graphical counterpart of 4.1 and 4.2.

Primarily we will work in another category denoted $\mathcal{SC}_2$, the category formally containing all direct sums and grading shifts of objects in $\mathcal{SC}_1$, but whose morphisms are forced to be degree 0. In addition, we let $\mathcal{SC}$ be the Karoubi envelope, or idempotent completion, of the category $\mathcal{SC}_2$. Recall that the Karoubi envelope of a category $\mathcal{C}$ has as objects pairs (B, e) where B is an object in $\mathcal{C}$ and e an idempotent endomorphism of B. This object acts as though it were the "image" of this projection e, and in an additive category behaves like a direct summand. For more information on Karoubi envelopes, see Wikipedia. It is really here that the object $R \otimes_{R^{i,i+1}} R\{-3\}$ of 4.1 and 4.2 resides. In practice all our calculations will be done in $\mathcal{SC}_2$, but since this includes fully faithfully into $\mathcal{SC}$ they will be valid there as well.

4.2.3 Hochschild (co)homology

Let A be a k algebra and M an A-A-bimodule, or equivalently a left $A \otimes A^{op}$-module or a right $A^{op} \otimes A$-module. The definitions of the *Hochschild (co)homology* groups $HH_*(A, M)$ ($HH^*(A, M)$) are the following:

$$HH_*(A,M) := Tor_*^{A\otimes A^{op}}(M,A) \qquad HH^*(A,M) := Ext^*_{A\otimes A^{op}}(A,M). \tag{4.5}$$

To compute this we take a projective resolution of the A-bimodule A, with the natural left and right action, by projective A-bimodules

$$\cdots \to P_2 \to P_1 \to P_0 \to 0,$$

and tensor this with M over $A \otimes A^{op}$ to get

$$\cdots \to P_2 \otimes_{A\otimes A^{op}} M \to P_1 \otimes_{A\otimes A^{op}} M \to P_0 \otimes_{A\otimes A^{op}} M \to 0.$$

The homology of this complex is isomorphic to $HH_*(A,M)$.

Example: For any bimodule M, we have

$$HH_0(A,M) \cong M/[A,M] \qquad HH^0(A,M) \cong M^A,$$

where $[A,M]$ is the subspace of M generated by all elements of the form $am - ma$, $a \in A$ and $m \in M$, and $M^A = \{m \in M \mid am = ma \text{ for all a} \in A\}$. We leave this as an exercise or refer the reader to (15).

If we take the polynomial algebra $A = k[x_1, \ldots, x_n]$, with k commutative, then we can use a much smaller, "Koszul," resolution of A by free $A \otimes A$-modules. This is gotten by taking the tensor product of the following complexes

$$0 \longrightarrow A \otimes A \xrightarrow{x_i\otimes 1 - 1\otimes x_i} A \otimes A \longrightarrow 0,$$

for $1 \geq i \geq n$. This resolution has length n, and its total space is naturally isomorphic to the exterior algebra on n generators tensored with $A \otimes A$. Hence, we get that the Hochschild homology of a bimodule M over A is made up of 2^n copies of M, with the differentials coming from multiplication by $x_i \otimes 1 - 1 \otimes x_i$, i.e.

$$0 \to C_n(M) \to \cdots \to C_1(M) \to C_0(M) \to 0,$$

with

$$C_j(M) = \bigoplus_{I \subset \{1,\ldots,n\}, |I|=j} M \otimes_{\mathbb{Z}} \mathbb{Z}[I],$$

where $\mathbb{Z}[I]$ is the rank 1 free abelian group generated by the symbol $[I]$ (i.e. it's there to keep track where exactly we are in the complex). Here, the differential takes the form

$$d(m \otimes [I]) = \sum_{i \in I} \pm(x_i m - m x_i) \otimes [I \backslash \{i\}],$$

and the sign is taken as negative if I contains an odd number of elements less than i.

Remark 4.3. For the polynomial algebra, the Hochschild homology and cohomology are isomorphic,

$$HH_i(A, M) \cong HH^{n-i}(A, M),$$

for any bimodule M. This comes from self-duality of the Koszul resolution for such algebras. Hence, we will be free to use either homology or cohomology groups in the constructions below.

For us, taking Hochschild homology will come into play when looking at closed braid diagrams. To a given resolution of a braid diagram we will assign a Soergel bimodule; "closing off" this diagram will correspond to taking Hochschild homology of the associated bimodule. More details of this below in section 4.3.2.

4.3 The integral HOMFLY-PT complex

4.3.1 The matrix factorization construction

As stated above we will work with $\mathbb{Z}$-graded, rather than $\mathbb{Z}/2\mathbb{Z}$-graded, matrix factorizations and follow closely the conventions laid out in (33). We begin by first assigning the appropriate complex to a single crossing and then extend this to general braids.

Gradings: Our complex will be triply graded, coming from the internal or "quantum" grading of the underlying ring, the homological grading of the matrix factorizations, and finally an overall homological grading of the entire complex. It will be convenient to visualize our complexes in the plane with the latter two homological gradings lying in the horizontal and vertical directions, respectively. We will denoted these gradings by $(i, j, k) = (q, 2gr_h, 2gr_v)$ and their shifts by curly brackets, i.e. $\{a, b, c\}$ will indicate a shift in the quantum grading by a, in the horizontal grading by b, and in the vertical grading by c. Note that following the conventions in (33) we have doubled the latter two gradings.

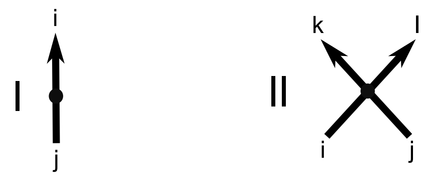


Figure 4.2:

{Edge ring} Given a diagram D with vertices labelled by $x_1, \ldots, x_n$, define the *edge ring* of D as $R(D) := \mathbb{Z}[x_1, \ldots, x_n]/ < rel(v_i) >$, where i runs over all internal vertices, or marks, with the defining relations being $x_i - x_j$ for type I and $x_k + x_l - x_i - x_j$ for type II vertices (see figure 4.2). Consider the two types of crossings D_+ and D_-, as in figure 4.1, with outgoing edges labeled by k, l, and incoming edges labelled by i, j . Let

$$R_c := \mathbb{Z}[x_i, x_j, x_k, x_l]/(x_k + x_l - x_i - x_j) \cong \mathbb{Z}[x_i, x_j, x_k]$$

be the underlying ring associated to a crossing. To the positive crossing D_+ we assign the following complex:

$$\begin{tikzcd}
R_c\{0,-2,0\} \arrow[r, "(x_k - x_i)"] & R_c\{0,0,0\} \\
R_c\{2,-2,-2\} \arrow[u, "(x_j - x_k)"] \arrow[r, "-(x_k - x_i)(x_k - x_j)"] & R_c\{0,0,-2\} \arrow[u, "1"']
\end{tikzcd}$$

To the negative crossing D_- we assign the following complex:

$$\begin{tikzcd}
R_c\{0,-2,2\} \arrow[r, "-(x_k - x_i)(x_k - x_j)"] & R_c\{-2,0,2\} \\
R_c\{0,-2,0\} \arrow[u, "1"] \arrow[r, "(x_k - x_i)"] & R_c\{0,0,0\} \arrow[u, "(x_j - x_k)"']
\end{tikzcd}$$

A few useful things to note: The horizontal and vertical differentials d_+ and d_v are homogeneous of degrees $(2,2,0)$ and $(0,0,2)$, respectively. For those more familiar with (22) and hoping to reconcile the differences, note that in R_c multiplication by $x_k x_l - x_i x_j = -(x_k - x_i)(x_k - x_j)$, so up to some grading shifts we are really working with the same underlying complex as in the original construction, but of course now over $\mathbb{Z}$, not $\mathbb{Q}$.

To write down the complex for a general braid we tensor the above for every crossing, keeping track of markings, and then replace the underlying ring with a copy of the edge ring $R(D)$. More precisely, given a diagram D of a braid let

$$C(D) := \bigotimes_{crossings} (C(D_c) \otimes_{R_c} R(D)).$$

{HOMFLY-PT homology }Given a braid diagram D of a link L we define its HOMFLY-PT homology to be the group

$$H(L) := H(H(C(D), d_+), d_v^*)\{-w+b, w+b-1, w-b+1\},$$

where w and b are the writhe and the number of strands of D, respectively.

Remark 4.4. In (33), this is what J. Rasmussen calls the "middle HOMFLY homology." The relation between this link homology theory and the HOMFLY-PT polynomial is that for any link $L \subset S^3$

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i dim H^{i,j,k}(L) = \frac{-P(L)}{q - q^{-1}}.$$

The reduced complex: There is a natural subcomplex $\overline{C}(D) \subset C(D)$ defined as follows: let $\overline{R}(D) \subset R(D)$ to be the subring generated by $x_i - x_j$ where i, j run over all edges of D and let $\overline{C}(D)$ be the subcomplex gotten by replacing in $C(D)$ each copy of $R(D)$ by one of $\overline{R}(D)$. A quick glance at the complexes $C(D_+)$ and $C(D_-)$ will reassure the reader that this is indeed a subcomplex, as the coefficients of both d_v and d_+ lie in $\overline{R}(D)$. We will refer to $\overline{C}(D)$ as the *reduced complex* for D.

- If i is an edge of D we can also define the complex $\overline{C}(D,i) := C(D)/(x_i)$. It is not hard to see that $\overline{C}(D,i) \cong \overline{C}(D)$ and is, hence, independent of the choice of edge i. See (33) section 2.8 for a discussion as well as (22).

Below we will work primarily with the reduced complex $\overline{C}(D)$, and will stick with the grading conventions of (33), which are different than that of (22).

{reduced homology}Given a braid diagram D of a link L we define its reduced HOMFLY-PT homology to be the group

$$\overline{H}(L) := H(H(\overline{C}(D), d_+), d_v^*)\{-w+b-1, w+b-1, w-b+1\},$$

where w and b are the writhe and the number of strands of D, respectively.

Remark 4.5. For any link $L \subset S^3$ we have

$$\sum_{i,j,k} (-1)^{(k-j)/2} a^j q^i dim \overline{H}^{i,j,k}(L) = P(L).$$

We can look at the complex $C(D)$ in two essential ways: either as the tensor product, over appropriate rings, of $C(D_+)$ and $C(D_-)$ for every crossing in our diagram D (as described above), or as a tensor product of corresponding complexes over all resolutions of the diagram. Although this is really just a matter of point of view, the latter approach is what we find in the original construction of Khovanov and Rozansky, as well as in the Soergel bimodule construction to be described below. To clarify this approach, consider the oriented D_o and singular D_s resolution of a crossing as in diagram 4.1. Assign to D_o the complex

$$0 \longrightarrow R_c \xrightarrow{(x_k - x_i)} R_c \longrightarrow 0$$

and to D_s the complex

$$0 \longrightarrow R_c \xrightarrow{-(x_k - x_i)(x_k - x_j)} R_c \longrightarrow 0.$$

Then we have

$$C(D_+) : \quad 0 \to C(D_s) \longrightarrow C(D_o) \to 0,$$

$$C(D_-) : \quad 0 \to C(D_o) \longrightarrow C(D_s) \to 0,$$

where the maps are given by d_v as defined above. [For simplicity we leave out the internal grading shifts.] Let a resolution of a link diagram D be a resolution of each crossing in either of the two ways above, and let the complex assigned to each resolution be the tensor product of the corresponding complexes for each resolved crossing. Then, modulo grading shifts, our total complex can be viewed as

$$C(D) = \bigoplus_{resolutions} C(D_{res}),$$

where D_{res} is the diagram of a given resolution. This closely mimics the "state-sum model" for the Jones polynomial, due to Kauffman (16), or the MOY calculus of (13) for other quantum polynomials.

4.3.2 The Soergel bimodule construction

We now turn to the Soergel bimodule construction for the HOMLFY-PT homology of (21). Recall from section 4.2.2 that the Soergel bimodule $B_i = R \otimes_{R^i} R\{-1\}$ where $R = \mathbb{Z}[x_1 - x_2, \ldots, x_{n-1} - x_n]$ is the ring generated by consecutive differences in variables $x_1, \ldots, x_n$ (n is the number of strands in the braid diagram), and $R^i \subset R$ is the subring of S_2-invariants corresponding to the permutation action $x_i \leftrightarrow x_{i+1}$. Furthermore define the map $B_i \to R$ by $1 \otimes 1 \longmapsto 1$, and the map $R \to B_i$ by $1 \longmapsto (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})$. We resolve a crossing in position $[i, i+1]$ in the either of the two ways, as in figure 4.1, assigning R to the oriented resolution and B_i to the singular resolution. For a positive crossing we have the complex

$$C(D_+): \quad 0 \to R\{2\} \longrightarrow B_i\{1\} \to 0,$$

and for a negative crossing the complex

$$C(D_-): \quad 0 \to B_i\{-1\} \longrightarrow R\{-2\} \to 0.$$

We place B_i in homological grading 0 and increase/decrease by 1, i.e. in the complex for D_+, $R\{2\}$ is in homological grading -1. Note, this grading convention differs from (21), and is the convention used in (2). The complexes above are known as Rouquier complexes, due to R. Rouquier who studied braid group actions with relation to the category of Soergel bimodules; for more information we refer the reader to (2), (21), and (36).

Given a braid diagram D we tensor the above complexes for each crossing, arriving at a total complex of length k, where k is the number of crossings of D, or equivalently

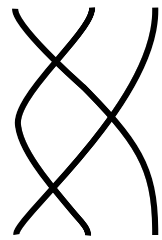


Figure 4.3:

the length of the corresponding braid word. Each entry in the complex can be thought of as a resolution of the diagram consisting of the tensor product of the appropriate Soergel bimodules. For example, to the graph in 4.3.2 we assign the bimodule $B_1 \otimes B_2 \otimes B_1$. That is, modulo grading shifts, we can view our total complex as

$$C(D) = \bigoplus_{resolutions} C(D_{res}).$$

To proceed, we take Hochschild homology $HH(C(D_{res}))$ for each resolution of D and arrive at the complex

$$HH(C(D)) = \bigoplus_{resolutions} HH(C(D_{res})),$$

with the induced differentials. Finally, taking homology of $HH(C(D))$ with respect to these differentials gives us our link homology.

{reduced homology} Given a braid diagram D of a link L we define its reduced HOMFLY-PT homology to be the group

$$H(HH(C(D))).$$

Of course, now that we have defined reduced HOMFLY-PT homology in two different ways, it would be nice to reconcile the fact that they are indeed the same.

Claim 4.6. *Up to grading shifts the two definitions of reduced HOMFLY-PT homology agree, i.e.* $H(H(\overline{C}(D), d_+), d_v^*) \cong H(HH(C(D)))$ *for a diagram D of a link L.*

Proof. The proof in (21) works without any changes for matrix factorizations and Soergel bimodules over $\mathbb{Z}$. We sketch it here for completeness and the fact that we will be referring to some of its details a bit later. Lets first look at the matrix factorization $C(D_s)$

(unreduced version) associated to a singular resolution D_s. Now $C(D_s)$ can be though of as a Koszul complex of the sequence $(x_k + x_l - x_i - x_j, x_k x_l - x_i x_j)$ in the polynomial ring $\mathbb{Z}[x_i, x_j, x_k, x_l]$ (don't forget that in R_c multiplication by $x_k x_l - x_i x_j = -(x_k - x_i)(x_k - x_j)$). Now this sequence is regular and the complex has cohomology in the right-most degree. The cohomology is the quotient ring

$$\mathbb{Z}[x_i, x_j, x_k, x_l]/(x_i + x_j - x_k - x_l, x_k x_l - x_i x_j).$$

This is naturally isomorphic to the Soergel bimodule B'_i (notice that this is the "unreduced" Soergel bimodule) over the polynomial ring $\mathbb{Z}[x_i, x_j]$. The left and right action of R' on B'_i corresponds to multiplication by x_i, x_j and x_k, x_l, respectively. Quotienning out by $x_k + x_l - x_i - x_j$ and $x_k x_l - x_i x_j$ agrees with the definition of B'_i as the tensor product $R' \otimes_{R'_i} R'$ over the subalgebra R' of symmetric polynomials in x_1, x_2.

Now lets consider a general resolution D_{res}. The matrix factorization for D_{res} is, once again, just a Koszul complex corresponding to a sequence of two types of elements. The first ones are as above, i.e. they are of the form $x_k + x_l - x_i - x_j$ and $x_k x_l - x_i x_j$ and come from the singular resolutions D_s, and the remaining are of the form $x_i - x_j$ that come from "closing off" our braid diagram D, which in turn means equating the corresponding marks at the top and bottom the diagram. Now it is pretty easy to see that the polynomials of the first type, coming from the D_s's form a regular sequence and we can quotient out by them immediately, just like above. The quotient ring we get is naturally isomorphic to the Soergel bimodule $B'(D_{res})$ associated to the resolution D_{res}. At this point all we have left is to deal with the remaining elements of the form $x_i - x_j$ coming from closing off D; to be more concrete, the Koszul complex we started with for D_{res} is quasi-isomorphic to the Koszul complex of the ring $B'(D_{res})$ corresponding to these remaining elements. This in turn precisely computes the Hochschild homology of $B'(D_{res})$.

Finally if we downsize from B'_i to B_i and from $C(D_{res})$ to $\overline{C}(D_{res})$ we get the required isomorphism. For more details we refer the reader to (21). $\square$

Gradings et al: We come to the usual rigmarole of grading conventions, which seems to be evepresent in link homology. Perhaps when using the Rouquier complexes above we could have picked conventions that more closely matched those of 4.3.1. However, we chose not to for a couple of reasons: first there would inevitably be some grading conversion to be done either way due to the inherent difference in the nature of the constructions, and second we use Rouquier complexes to aid us in just a few results (namely the proof of Reidemeister moves II and III), and leave them shortly after attaining these; hence, it is convenient for us, as well as for the reader familiar with the Soergel bimodule construction of (21) and the diagrammatic construction of (1), to adhere to the conventions of the former and the subsequent results in (2). For completeness, we describe the conversion rules. Recall that in the matrix factorization construction of 4.3.1 we denoted the gradings as $(i, j, k) = (q, 2gr_h, 2gr_v)$.

- To get the cohomological grading in the Soergel construction take $(j-k)/2$ from 4.3.1.
- The Hochschild here matches the "horizontal" or j grading of 4.3.1.
- To get the "quantum" grading i of 4.3.1 of an element x, take Hochschild grading of x minus $\deg(x)$, i.e. $\deg(x) = j(x) - i(x)$.

4.3.2.1 Diagrammatic Rouquier complexes

We now restate the last section in the diagrammatic landuage of (2), i.e. that of chapter 3, as outlined above in 4.2.2. The main advantage of doing this is the inherent ability of the graphical calculus developed by Elias and Khovanov in (1) to hide and, hence, simplify the complexity of the calculations at hand. Recall that we work in the integral version of Soergel category $\mathcal{SC}_2$ as defined in section 2.3 of (2), which allows for constructions over $\mathbb{Z}$ without adjoining inverses (see section 5.2 in (2) for a discussion of these facts). Recall, that an object of $\mathcal{SC}_2$ is given by a sequence of indices $\underline{\boldsymbol{i}}$, visualized as d points on the real line and morhisms are given by pictures or graphs embedded in the strip $\mathbb{R} \times [0, 1]$. We think of the indices as "colors," and depict them accordingly. The Soergel bimodule B_i is represented by a vertical line of "color" i (i.e. by the identity morphism from B_i to itself) and the maps we find in the Rouquier complexes above, section 4.3.2, are given by those referred to as "start-dot" and "end-dot." More precisely, the complexes $C(D_-)$ and $C(D_+)$ become

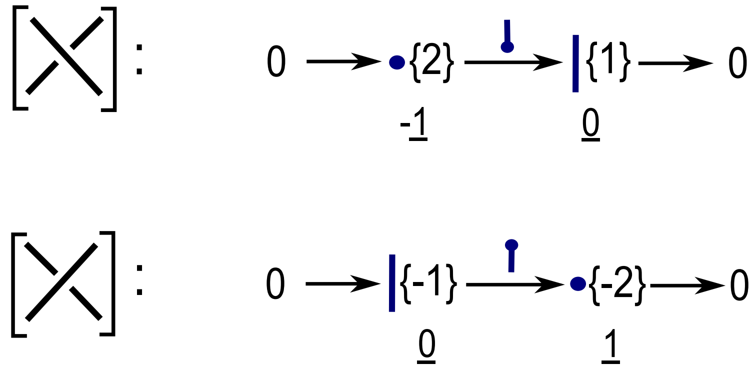


Figure 4.4: Diagrammatic Rouquier complex for right and left crossings

We refer the reader to the "Conventions" section 3.2.6 of the last chapter for details of how to go from a braid to diagrammatic Rouquier complex.

4.4 Checking the Reidemeister moves

We will use the matrix factorization construction of section 4.3.1 to check Reidemeister move I, as it is not very difficult to verify even over $\mathbb{Z}$ that this goes through, and the diagrammatic calculus of section 4.3.2.1 for the remaining moves. There are two main reasons for the interplay: first, checking Reidemeister II and III over $\mathbb{Z}$ using the matrix factorization approach is rather computationally intensive (it was already quite so over $\mathbb{Q}$ in (22) with all the algebraic advantages of working over a field at hand); second, at this moment there does not exist a full diagrammatic description of Hochschild homology of Soergel bimodules, which prevents us from using a pictorial calculus to compute link homology from closed braid diagrams. Of course, for Reidemeister II and III we could have used the computations of (2), where we prove the stronger result that Rouquier complexes are functorial over braid cobordisms, but the proofs we exhibit below use essentially the same strategy as the original paper (22), but are so much simpler and more concise that they underline well the usefulness of the diagrammatic calculus for computations. With that said, we digress...

A small lemma from linear algebra, which Bar-Natan refers to as "Gaussian Elimination for Complexes" in (5), will be very helpfull to us.

Lemma 4.7. *If $\phi : B \to D$ is an isomorphism (in some additive category $\mathcal{C}$), then the four term complex segment below*

$$\cdots\ [A] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}B\\ C\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix}D\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]\ \cdots \tag{4.6}$$

is isomorphic to the (direct sum) complex segment

$$\cdots\ [A] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix}B\\ C\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix}D\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]\ \cdots. \tag{4.7}$$

Both of these complexes are homotopy equivalent to the (simpler) complex segment

$$\cdots\ [A] \xrightarrow{(\beta)} [C] \xrightarrow{\left(\epsilon-\gamma\phi^{-1}\delta\right)} [E] \xrightarrow{(\nu)} [F]\ \cdots. \tag{4.8}$$

Here the capital letters are arbitrary columns of objects in $\mathcal{C}$ and all Greek letters are arbitrary matrices representing morphisms with the appropriate dimensions, domains and ranges (all the matrices are block matrices); $\phi : B \to D$ is an isomorphism, i.e. it is invertible.

Proof: The matrices in complexes (1) and (2) differ by a change of bases, and hence the complexes are isomorphic. (2) and (3) differ by the removal of a contractible summand; hence, they are homotopy equivalent. $\square$

4.4.1 Reidemeister I

Proof. The complex $C(D_{I_a})$ for the left-hand side braid in Reidemester Ia, see figure 4.5, has the form

$$\begin{tikzcd}
\mathbb{Z}[x_1,x_2]\{0,-2,0\} \arrow[r, "0"] & \mathbb{Z}[x_1,x_2]\{0,0,0\} \\
\mathbb{Z}[x_1,x_2]\{2,-2,-2\} \arrow[u, "(x_2-x_1)"] \arrow[r, "0"] & \mathbb{Z}[x_1,x_2]\{0,0,-2\} \arrow[u, "1"']
\end{tikzcd}$$

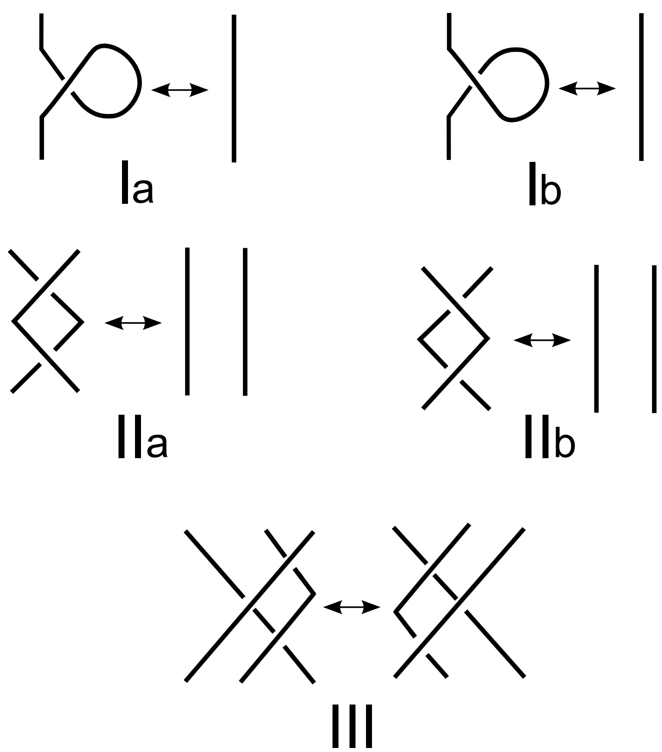


Figure 4.5: The Reidemeister moves

Up to homotopy, the right-hand side of the complex dissappears and only the top left corner survives after quotioning out by the relation $x_2 - x_1$. Note that the overall degree shifts of the total complex make sure that the left-over entry sits in the correct tri-grading.

Similarly, the complex $C(D_{I_b})$ for the left-hand side braid in Reidemester Ib, has the form

$$\begin{tikzcd}
\mathbb{Z}[x_1,x_2]\{0,-2,2\} \arrow[r, "0"] & \mathbb{Z}[x_1,x_2]\{-2,0,2\} \\
\mathbb{Z}[x_1,x_2]\{0,-2,0\} \arrow[u, "1"] \arrow[r, "0"] & \mathbb{Z}[x_1,x_2]\{0,0,0\} \arrow[u, "(x_2-x_1)"']
\end{tikzcd}$$

The left-hand side is annihilated and the upper-right corner remains modulo the relation $x_2 - x_1$. □

4.4.2 Reidemeister II

Proof. Let's first consider the braid diagrams for Reidemeister type IIa in figure 4.5. Recall the decomposition $B_i \otimes B_i \cong B_i\{-1\} \oplus B_i\{1\}$ in $\mathcal{SC}_2$ and its pictorial counterpart 4.3. The complex we are interested in is

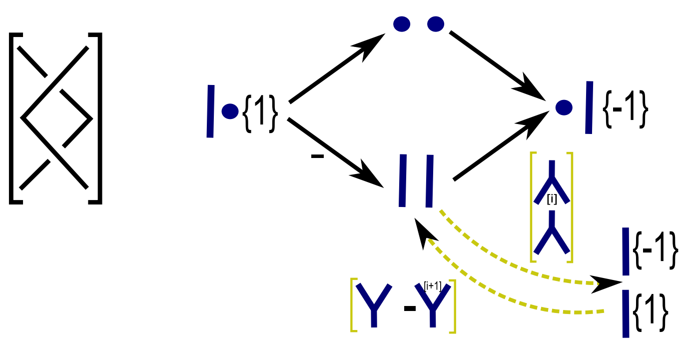


Figure 4.6: Reidemeister IIa complex with decomposition 4.3

Inserting the decomposed $B_i \otimes B_i$ and the corresponding maps, we find two isomorphisms staring at us; we pick the left most one and mark it for removal.

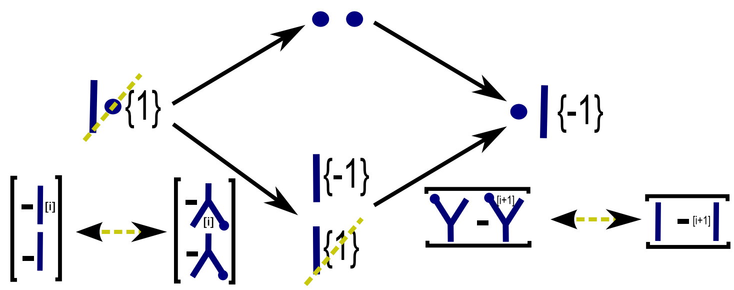


Figure 4.7: Reidemeister IIa complex, removing one of the acyclic subcomplexes

After changing basis and removing the acyclic complex, as in Lemma 4.7, we arrive at the complex below with two more entries marked for removal.

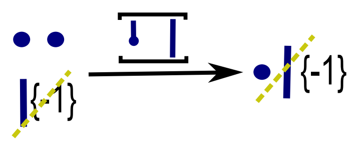


Figure 4.8: Reidemeister IIa complex, removing a second acyclic subcomplex

With the marked acyclic subcomplex removed, we arrive at our desired result, the complex assigned to the no crossing braid of two strands as in figure 4.5. The computation for Reidemeister IIb is virtually identical. $\square$

4.4.3 Reidemeister III

Proof. Luckily, we only have to check one version of Reidemeister move III, but as the reader will see below even that is pretty easy and not much harder than that of Reidemeister II above. We follow closely the structure of the proof in (22), utilizing the bimodule $R \otimes_{R^{i,i+1}} R\{-3\}$ and decomposition 4.4 to reduce the complex for one of the RIII braids to that which is invariant under the move or, equivalently in our case, invariant under color flip. We start with the braid on the left-hand side of III in figure 4.5; the corresponding complex, with decomposition 4.3 and 4.4 given by dashed/yellow arrows, is

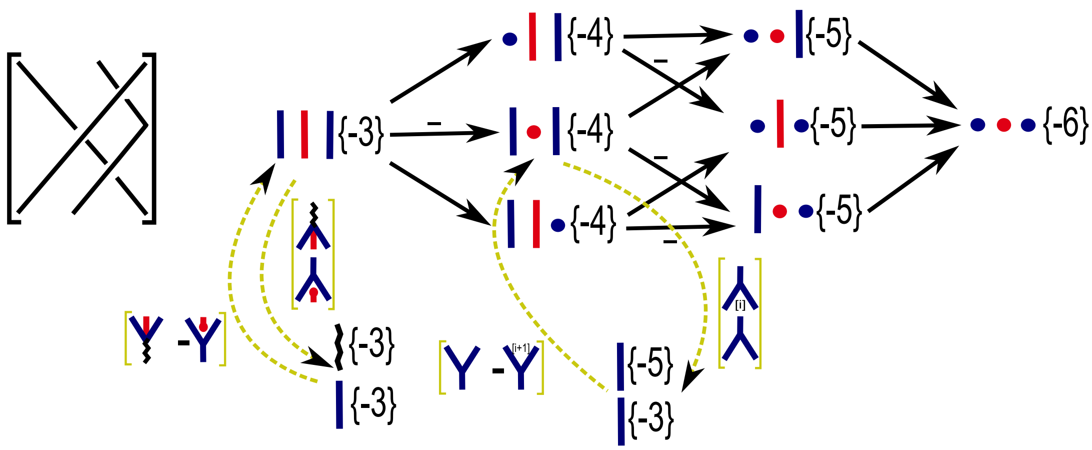


Figure 4.9: Reidemeister III complex with decompositions 4.3 and 4.4

We insert the decomposed bimodules and the appropriate maps; then we change bases as in Lemma 4.7 (the higher matrix of the two is before base-change, and the lower is after).

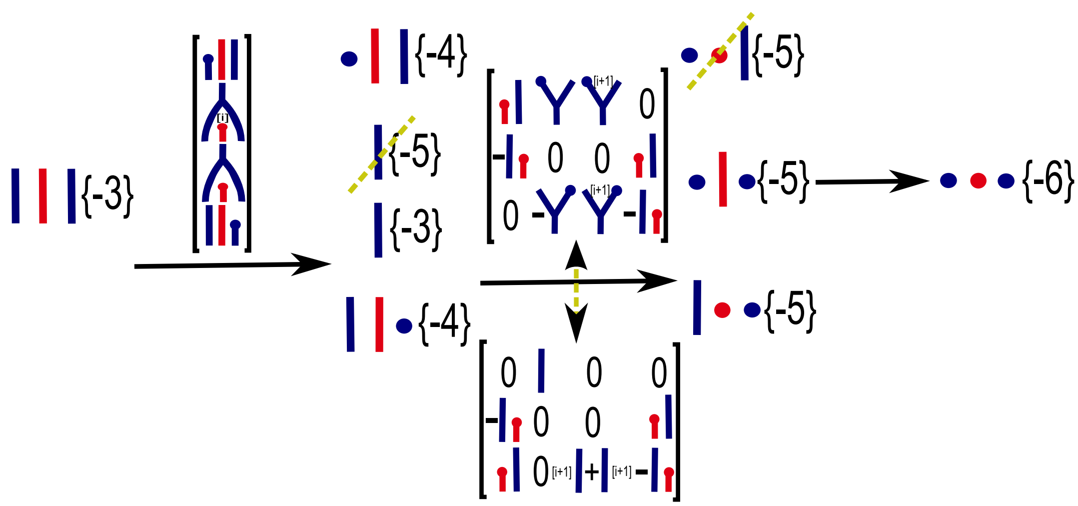


Figure 4.10: Reidemeister III complex, with an acyclic subcomplex marked for removal

We strike out the acyclic subcomplex and mark another one for removal; yet again we change bases (the lower matrix is the one after base change).

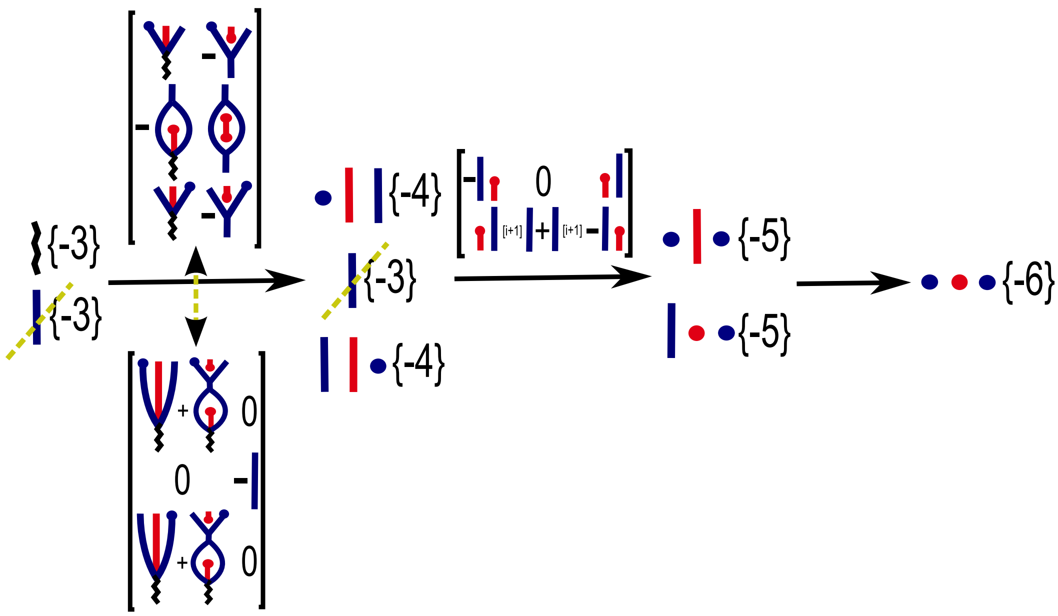
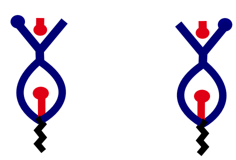


Figure 4.11: Reidemeister III complex, with another acyclic subcomplex marked for removal

Now we are almost done; if we can prove that the maps



are invariant under color change, we would arrive at a complex that is invariant under Reidemeister move III. To do this we must stop for a second, go back to the source and examine the original, algebraic, definitions of the morphisms in (1); upon doing so we are relieved to see that the maps we are interested in are actually equal to zero (they are defined by sending $1 \otimes 1 \longmapsto 1 \otimes 1 \otimes 1 \otimes 1 \longmapsto 1 \otimes 1 \otimes 1 \longmapsto 0$). In all, we have arrived at

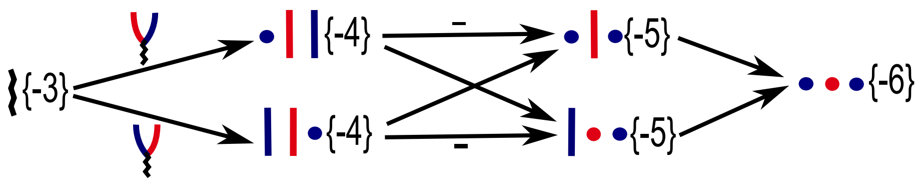


Figure 4.12: Reidemeister III complex - the end result, after removal of all acyclic subcomplexes

Repeating the calculation for the braid on the right-hand side of RIII, figure 4.5, amounts to the above calculation with the colors switched - a quick glance will convince the reader that the end result is the same complex rotated about the x-axis. □

4.4.4 Observations

Having seen this interplay between the different constructions, perhaps it is a good moment to highlight exactly what categories we do need to work in so as to arrive at a genuine link invariant, or a braid invariant at that. Well, let us start with the latter: we can take the category of complexes of Soergel bimodules $\mathcal{KOM}(\mathcal{SC})$ (either the diagrammatic or "original" version) and construct Rouquier complexes; if we mod out by homotopies and work in $\mathcal{KOM}_h(\mathcal{SC})$, we arrive at something that is not only an invariant of braids but of braid cobordisms as well (over $\mathbb{Z}$ or $\mathbb{Q}$ if we wish). Now if we repeat the construction in the category of complexes of graded matrix factorizations $\mathcal{KOM}(mf)$, we have some choices of homotopies to quotient out by. First, we can quotient out by the homotopies in the category of graded matrix factorizations and work in $\mathcal{KOM}(hmf)$ and second, we can quotient in the category of the complexes and work in $\mathcal{KOM}_h(mf)$, or we can do both and work in $\mathcal{KOM}_h(hmf)$. It is immediate that working in $\mathcal{KOM}_h(mf)$ is necessary, but one could hope that it is also sufficient. A close look at the argument of Claim 4.6, where the two constructions are proven equivalent, shows that if we start with the Koszul complex associated to the resolution of a braid D_{res} the polynomial relations coming from the singular vertices in D_{res} form a regular sequence and, hence, the homology of this complex is the quotient of the edge ring $R(D_{res})$ by these relations and is supported in the right-most degree. It is this quotient that is isomorphic to the corresponding Soergel bimodule, i.e. the Koszul complex is quasi-isomorphic, as a bimodule, to $B'(D_{res})$. Hence, we really do need to work in $\mathcal{KOM}_h(hmf)$, to have a braid invariant or an invariant of braid cobordisms, or a link invariant.

Anyone, who has suffered through the proofs of, say, Reidemeister III in (22) would probably find the above a relief. Of course, much of the ease in computation using this pictorial language is founded upon the intimate understanding and knowledge of hom spaces between objects in $\mathcal{SC}$, which is something that is only available to us due to the labors Elias and Khovanov in (1). With that said, it would not be surprising if this diagrammatic calculus would aid other calculations of link homology in the future.

All in all we have arrived at an integral version of HOMFLY-PT link homology; combining with the results of (2) we have the following:

Theorem 4.8. *Given a link $L \subset S^3$, the groups $H(L)$ and $\overline{H}(L)$ are invariants of L and when tensored with $\mathbb{Q}$ are isomorphic to the unreduced and reduced versions, respectively, of the Khovanov-Rozansky HOMFLY-PT link homology. Moreover, these integral homology theories give rise to functors from the category of braid cobordisms to the category of complexes of graded R-bimodules.*

4.5 Rasmussen's spectral sequence and integral $sl(n)$-link homology

It is time for us to look more closely at Rasmussen's spectral sequence from HOMFLY-PT to $sl(n)$-link homology. For this we need an extra "horizontal" differential d_- in our complex, and here is the first time we encounter matrix factorizations with a non-zero potential; as before, to a link diagram D we will associate the tensor product of complexes of matrix factorizations with potential for each crossing. These will be complexes over the ring

$$R_c = \mathbb{Z}[x_i, x_j, x_k, x_l]/(x_k + x_l - x_i - x_j) \cong \mathbb{Z}[x_i, x_j, x_k],$$

with total potential

$$W_p[x_i, x_j, x_k, x_l] = p(x_k) + p(x_l) - p(x_i) - p(x_j),$$

where the $p(x) \in \mathbb{Z}[x]$. We do not specify the potential $p(x)$ at the moment as the spectral sequence works for any choice; later on when looking at $sl(n)$-link homology we will set $p(x) = x^{n+1}$.

To define d_-, let $p_i = W_p/(x_k - x_i)$ and $p_{ij} = -W_p/(x_k - x_i)(x_k - x_j)$ (recall that in R_c, $(x_k - x_i)(x_k - x_j) = x_i x_j - x_k x_l$, and note that these polynomials are actually in R_c).

To the positive crossing D_+ we assign the following complex:

$$\begin{tikzcd}
R_c\{0,-2,0\} \arrow[r, shift left, "(x_k - x_i)"] & R_c\{0,0,0\} \arrow[l, shift left, "p_i"] \\
R_c\{2,-2,-2\} \arrow[u, "(x_j - x_k)"] \arrow[r, shift left, "-(x_k-x_i)(x_k-x_j)"] & R_c\{0,0,-2\} \arrow[u, "1"'] \arrow[l, shift left, "p_{ij}"]
\end{tikzcd}$$

To the negative crossing D_- we assign the following complex:

$$\begin{tikzcd}
R_c\{0,-2,2\} \arrow[r, shift left, "-(x_k-x_i)(x_k-x_j)"] & R_c\{-2,0,2\} \arrow[l, shift left, "p_{ij}"] \\
R_c\{0,-2,0\} \arrow[u, "1"] \arrow[r, shift left, "(x_k - x_i)"] & R_c\{0,0,0\} \arrow[u, "(x_j - x_k)"'] \arrow[l, shift left, "p_i"]
\end{tikzcd}$$

The total complex for a link L with diagram D will be defined analagously to the one above, i.e.

$$C_p(D) := \bigotimes_{crossings} (C(D_c) \otimes_{R_c} R(D)),$$

as will be the reduced $\overline{H}_p(L, i)$ and unreduced $H_p(L)$ versions of link homology.

The main result of (33) is the following:

Theorem 4.9. $\{$*Rasmussen, (33)*$\}$ *Suppose $L \subset S^3$ is a link, and let i be a marked component of L. For each $p(x) \in \mathbb{Q}[x]$, there is a spectral sequence $E_k(p)$ with $E_1(p) \cong \overline{H}(L)$ and $E_\infty(p) \cong \overline{H}_p(L, i)$. For all $k > 0$, the isomorphism type of $E_k(p)$ is an invariant of the pair (L, i).*

In particular setting $p(x) = x^{n+1}$ one would arrive at a spectral sequence from the HOMFLY-PT to the $sl(n)$-link homology. Rasmussen's result pertains to rational link homology with matrix factorizations defined over the ring $\mathbb{Q}[x_1, \dots, x_n]$ and potentials polynomials in $\mathbb{Q}[x]$. We will essentially repeat his construction in our setting and, for the benefit of those familiar with the results of (33), will stay as close as possible to the notation and conventions therein. This will be a rather condensed version of the story and we refer the reader to the original paper for more details.

We will work primarily with reduced link homology (although all the results follow through for both versions) and with closed link diagrams, where all three differentials d_v, d_+, and d_- anticommute. We have some choices as to the order of running the differentials, so let us define

$$\overline{H}^+(D, i) = H(\overline{C}(D, i), d_+).$$

Here, $\overline{H}^+(D, i)$ inherits a pair of anticommuting differentials d_-^* and d_v^*, where d_-^* lowers gr_h by 1 while preserving gr_v and d_v^* raises gr_v by 1 while preserving gr_h. Hence, $(\overline{H}_p^+(D, i), d_v^*, d_-^*)$ defines a double complex with total differential $d_{v-} := d_v^* + d_-^*$.

Let $E_k(p)$ be the spectral sequence induced by the horizontal filtration on the complex $(\overline{H}_p^+(D, i), d_{v-})$.

After shifting the triple grading of $E_k(p)$ by $\{-w + b - 1, w + b - 1, w - b + 1\}$ it is immediate that the first page of the spectral sequence is isomorphic to $\overline{H}(L, i)$ (the part of the differential $d_v^* + d_-^*$ which preserves horizontal grading on $E_0(p) = \overline{H}^+(D, i)\{-w + b - 1, w + b - 1, w - b + 1\}$ is precisely d_v^*, i.e. $d_0(p) = d_v^*$ and

$$E_1(p) = H(\overline{H}^+(D, i), d_v^*)\{-w + b - 1, w + b - 1, w - b + 1\} \cong \overline{H}(L, i),$$

where D is a diagram for L). It also follows that $d_k(p)$ is homogenous of degree $-k$ with respect to gr_h and degree $1 - k$ with respect to gr_v, and in the case that $p(x) = x^{n+1}$ it is also homogeneous of degree $2nk$ with respect to the q-grading.

Claim 4.10. *Suppose $L \subset S^3$ is a link, and let i be a marked component of L. For each $p(x) \in \mathbb{Z}[x]$, the spectral sequence $E_k(p)$ has $E_1(p) \cong \overline{H}(L, i)$ and $E_\infty(p) \cong \overline{H}_p(L, i)$. For all $k > 0$, the isomorphism type of $E_k(p)$ is an invariant of the pair (L, i).*

Proof. We argue as in (33) section 5.4. Suppose that we have two closed diagrams D_j and D'_j that are related by the j'th Reidemeister move, and suppose that there is a morphism

$$\sigma_j : \overline{H}_p^+(D_j, i) \to \overline{H}_p^+(D'_j, i)$$

in the category $\mathcal{KOM}(mf)$ that extends to a homotopy equivalence in the category of modules over the edge ring R. Then σ_j induces a morphism of spectral sequences $(\sigma_j)_k$: $E_k(D_j, i, p) \to E_k(D'_j, i, p)$ which is an isomorphism for $k > 0$. See (33) for more details and discussion. Hence, in practice we have to exhibit morphisms and prove invariance for the first page of the spectral sequence, i.e. for the HOMLFY-PT homology, which is basically already done. However, we ought to be a bit careful, of course, as here we are working with $\overline{H}_p^+(D, i)$ and not with the complex $\overline{C}(D, i)$ defined in section 4.4.

Reidemeister I is done, as in this case $d_+ = 0$ and, hence, the complex $\overline{H}_p^+(D, i) = \overline{C}_p(D, i)$ and the same argument as the one in section 4.4.1 works here.

For Reidemesiter II and III, we have to observe that for a closed diagram we have morphisms $\sigma_j : \overline{C}_p(D_j, i) \to \overline{C}_p(D'_j, i)$ for $j = II, III$, which are homotopy equivalences of complexes (these can be extrapolated from section 4.4 above, or from (2), where all chain maps are exhibited concretely). Therefore we get induced maps $(\sigma_j)_k$ on the spectral sequence with the property that $(\sigma_j)_1 = \sigma_{j*}$ is an isomorphism.

To get the last part of the claim, i.e. that the reduced homology depends only on the link component and not on the edge therein we refer the reader to (33), as the arguments from there are valid verbatum. □

Setting $p(x) = x^{n+1}$, we get that the differentials $d_k(p)$ preserve $q + 2ngr_h$ and, hence, the graded Euler characteristic of $H(\overline{H}_p^+(D, i), d_{v-})$ with respect to this quantity is the same as that of $E_1(x^{n+1})$. Tensoring with $\mathbb{Q}$, to get rid of torsion elements, and computing we see that the Euler characteristic of the $E_\infty(x^{n+1})$ is the quantum $sl(n)$-link polynomial $P_L(q^n, q)$ of L. See (33) section 5.1 for details. We have arrived at:

Theorem 4.11. *The $E_\infty(x^{n+1})$ of the spectral sequence defined in 4.5 is an invariant of L and categorifies the quantum $sl(n)$-link polynomial $P_L(q^n, q)$.*

Remark 4.12. Well, we have a categorification over $\mathbb{Z}$ of the quantum $sl(n)$-link polynomial, but what homology theory exactly are we dealing with? Is it isomorphic to

$H(H(H(\overline{C}_{x^{n+1}}(D,i),d_+),d_-^*),d_v^*)$ or to $H(H(\overline{C}_{x^{n+1}}(D,i),d_+ + d_-),d_v^*)$ and are these two isomorphic here? The answer is not immediate. In (33), Rasmussen bases the corresponding results on a lemma that utilizes the Kunneth formula, which is much more manageable in this context when looked at over $\mathbb{Q}$. Of course, for certain classes of knots things are easier. For example, if we take the class of knots that are *KR-thin*, then the spectral sequence converges at the E_1 term, as this statement only depends on the degrees of the differentials, and we have that $E_\infty(x^{n+1}) \cong H(H(\overline{C}_{x^{n+1}}(D,i),d_+),d_v^*)$. However, that's a bit of a 'red herring' as stated.

Chapter 5

A particular example in $sl(n)$-link homology

5.1 Introduction

For the duration of this chapter we return to the original Khovanov-Rozansky $sl(n)$-link homology and explore the complex associated to a particular class of tangles. Using ideas from (4) we show that for these tangles, and hence for knots and links composed from them, the Khovanov-Rozansky complex reduces to one that is quite simple, i.e. one without any "thick" edges. In particular we consider the tangle in figure 5.1 and show that its associated complex is homotopic to the one below, with some grading shifts and basic maps which we leave out for now.

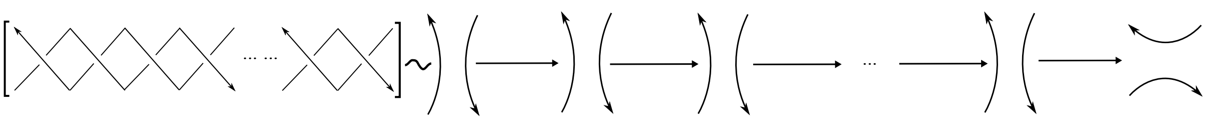


Figure 5.1: Our main tangle and its reduced complex

The complexes for these knots and links are entirely "local," and to calculate the homology we only need to exploit the Frobenius structure of the underlying algebra assigned to the unknot. Hence, here the calculations and complexity is similar to that of sl_2-homology. We also discuss a general algorithm, basically the one described in (4), to compute these homology groups in a more time-efficient manner. We compare our results with similar computations in the version of sl_3-homology found in (17), which we refer to as the "foam" version (foams are certain types of cobordisms described in this paper), and giving an explicit isomorphism between the two versions. A very similar calculation in the sl_3-homology, that for the $(2,n)$ torus knots, was first done in (30). Althought the construction of $sl(n)$ homology is essentially a specialization of its equivariant counterpart, for the sake of clarity and completeness, we restate explicitly the relevant details. The bulk of the chapter deals with the main calculation, and we finish with a discussion of an algorithm to compute such links, compare the results with similar computations in the "foam" version of $sl(3)$ homology, and discuss general properties of relating to our examples.

5.2 A Review of Khovanov-Rozansky Homology

Our graphs are embedded in a disk and have two types of edges, unoriented and oriented. Unoriented edges are called "thick" and drawn accordingly; each vertex adjoining a thick edge has either two oriented edges leaving it or two entering. In figure 5.2 left x_1, x_2 are outgoing and x_3, x_4 are incoming. As before oriented edges are allowed to have marks and we also allow closed loops; points of the boundary are also referred to as marks. To such a graph Γ we assign a matrix factorization in the following manner:

To a thick edge t as in figure 5.2 left we assign a factorization C_t with potential $\omega_t = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$ over the ring $R_t = \mathbb{Q}[x_1, x_2, x_3, x_4]$. Since $x^{n+1} + y^{n+1}$ lies in the ideal generated by $x+y$ and xy we can write it as a polynomial $g(x+y, xy)$. Hence, ω_t can be written as

$$\omega_t = (x_1 + x_2 - x_3 - x_4)u_1 + (x_1x_2 - x_3x_4)u_2$$

where

$$u_1 = \frac{x_1^{n+1} + x_2^{n+1} - g(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4},$$

$$u_2 = \frac{g(x_3 + x_4, x_1x_2) - x_3^{n+1} - x_4^{n+1}}{x_1x_2 - x_3x_4}.$$

C_t is the tensor product of graded factorizations

$$R_t \xrightarrow{u_1} R_t\{1-n\} \xrightarrow{x_1+x_2-x_3-x_4} R_t$$

and

$$R_t \xrightarrow{u_2} R_t\{3-n\} \xrightarrow{x_1x_2-x_3x_4} R_t.$$

To an arc α bounded by marks oriented from j to i we assign the factorization L_j^i

$$R_\alpha \xrightarrow{\pi_{ij}} R_\alpha \xrightarrow{x_i-x_j} R_\alpha,$$

where $R_\alpha = \mathbb{Q}[x_i, x_j]$ and

$$\pi_{ij} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}.$$

Finally, to an oriented loop with no marks we assign the complex $0 \to A \to 0 = A\langle 1\rangle$ where $A = \mathbb{Q}[x]/(x^n)$. [Note: to a loop with marks we assign the tensor product of L_j^i's as above, but this turns out to be isomorphic to $A\langle 1\rangle$ in the homotopy category.]

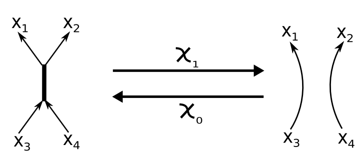


Figure 5.2: Maps χ_0 and χ_1

We define $C(\Gamma)$ to be the tensor product of C_t over all thick edges t, L_j^i over all edges α from i to j, and $A\langle 1\rangle$ over all oriented markless loops. This tensor product is taken over appropriate rings such that $C[\Gamma]$ is a free module over $R = \mathbb{Q}[x_i]$ where the x_i's are marks. $C(\Gamma)$ becomes a $\mathbb{Z}\oplus\mathbb{Z}_2$-graded complex with the $\mathbb{Z}_2$-grading coming from the factorization. It has potential $\omega = \sum_{i\in\partial\Gamma} \pm x_i^{n+1}$, where $\partial\Gamma$ is the set of all boundary marks and the $+$, $-$ is determined by whether the direction of the edge corresponding to x_i is towards or away from the boundary. [Note: if Γ is a closed graph the potential is zero.]

The maps χ_0 and χ_1

We now define maps between matrix factorizations associated to the thick edge and two disjoint arcs as in figure 5.2. Let Γ^0 correspond to the two disjoint arcs and Γ^1 to the thick edge.

$C(\Gamma^0)$ is the tensor product of L_4^1 and L_3^2. If we assign labels a, b to L_4^1, L_3^2 respectively, the tensor product can be written as

$$\begin{pmatrix} R(\varnothing) \\ R(ab)\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R(a)\{1-n\} \\ R(b)\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R(\varnothing) \\ R(ab)\{2-2n\} \end{pmatrix},$$

where

$$P_0 = \begin{pmatrix} \pi_{14} & x_2 - x_3 \\ \pi_{23} & x_4 - x_1 \end{pmatrix}, P_1 = \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ \pi_{23} & -\pi_{14} \end{pmatrix},$$

$$\pi_{ij} = \sum_{k=0}^{n} x_i^k x_j^{n-k}.$$

Assigning labels a' and b' to the two factorizations in $C(\Gamma^1)$, we have that $C(\Gamma^1)$ is given by

$$\begin{pmatrix} R(\varnothing)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R(a')\{n\} \\ R(b')\{2-n\} \end{pmatrix} \xrightarrow{Q_2} \begin{pmatrix} R(\varnothing)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} u_1 & x_1x_2 - x_3x_4 \\ u_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix}, Q_2 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ u_2 & -u_1 \end{pmatrix}.$$

A map between $C(\Gamma^0)$ and $C(\Gamma^1)$ can be given by a pair of 2×2 matrices. Define $\chi_0 : C(\Gamma^0) \to C(\Gamma^1)$ by

$$U_0 = \begin{pmatrix} x_1 - x_3 & 0 \\ \frac{u_1 + x_1u_2 - \pi_{23}}{x_1 - x_4} & 1 \end{pmatrix}, U_1 = \begin{pmatrix} x_1 & -x_3 \\ -1 & 1 \end{pmatrix},$$

and $\chi_1 : C(\Gamma^1) \to C(\Gamma^0)$ by

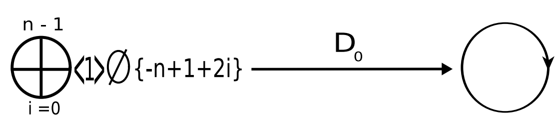
$$V_0 = \begin{pmatrix} 1 & 0 \\ \frac{u_1+x_1u_2-\pi_{23}}{x_4-x_1} & x_1 - x_3 \end{pmatrix}, V_1 = \begin{pmatrix} 1 & x_3 \\ 1 & x_1 \end{pmatrix}.$$

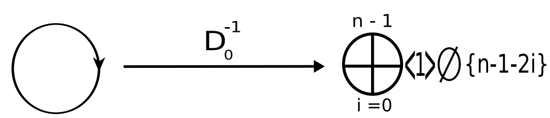
These maps have degree 1. Computing we see that the composition $\chi_1\chi_0 = (x_1 - x_3)I$, where I is the identity matrix, i.e. $\chi_1\chi_0$ is multiplication by $x_1 - x_3$. Similarly $\chi_0\chi_1 = (x_4 - x_2)I$. [Note: these are specializations of the maps χ_0 and χ_1 given in (22), with $\lambda = 0$ and $\mu = 1$. As these maps are homotopic for any rational value of λ and μ we are free to do so.]

Define the trace $\varepsilon : \mathbb{Q}[x]/(x^n) \longrightarrow \mathbb{Q}$ as $\varepsilon(x^i) = 0$ for $i \neq n-1$ and $\varepsilon(x^{n-1}) = 1$. The unit $\iota : \mathbb{Q} \longrightarrow \mathbb{Q}[x]/(x^n)$ is defined by $\iota(1) = 1$.

As before, the relations between $C(\Gamma)$'s mimic the graph skein relations, see for example (22), and we list the ones needed below.

Direct Sum Decomposition 0:





where $D_0 = \sum_{i=0}^{n-1} x^i\iota$ and $D_0^{-1} = \sum_{i=0}^{n-1} \varepsilon x^{n-1-i}$.

By the pictures above, we really mean the complexes assigned to them, i.e. $\emptyset\langle 1\rangle$ is the complex with $\mathbb{Q}$ sitting in homological grading 1 and the unknot is the complex $A\langle 1\rangle$ as above. The map $x^i\iota$ is a composition of maps

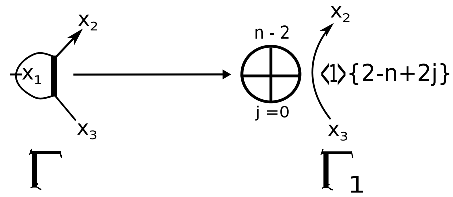
$$A\langle 1\rangle \xrightarrow{x^i} \langle 1\rangle \xrightarrow{\iota} \emptyset\langle 1\rangle,$$

where x^i is multiplication and ι is the unit map, i.e. $x^i\iota$ is the map

$$\mathbb{Q}[x]/(x^n) \xrightarrow{x^i} \mathbb{Q}[x]/(x^n) \xrightarrow{\iota} \mathbb{Q}.$$

Similar with εx^{n-1-i}. It is easy to check that the above maps are grading preserving and their composition is the identity.

Direct Sum Decomposition I:



where $D_1 = \sum_{i=0}^{n-2} \beta x_1^{n-i-2}$ and $D_1^{-1} = \sum_{i=0}^{n-2}\sum_{j=0}^{i} x_1^j x_2^{i-j}\alpha$ with $\alpha := \chi_0 \circ \iota'$ and $\beta := \varepsilon' \circ \chi_1$. Here $\iota' = \iota \otimes Id$ and $\varepsilon' = \varepsilon \otimes Id$; the Id corresponds to the arc with endpoints labeled by x_2, x_3, i.e ι' is the map that includes the single arc diagram into one with the unknot and single arc disjoint, see figure 5.3. Similar with ε' in the right half of figure 5.4.

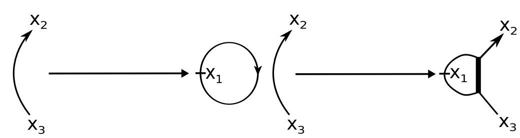


Figure 5.3: The map α in Direct Sum Decomposition I

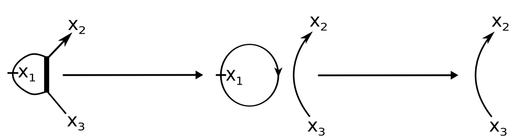


Figure 5.4: The map β in Direct Sum Decomposition I

Direct Sum Decomposition II:

where $D_2 = S \oplus \sum_{j=0}^{n-3} \beta_j$ and $\beta_j = \sum_{j=0}^{n-3} \beta \sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c$.

Here β is given by the composition of two χ_1's, corresponding to the two thick edges on the left-hand side above, and the trace map ε, see figure 5.5. Finally S is gotten by "merging" the thick edges together to form two disjoint horizontal arcs, as in the top right-hand corner above; an exact description of S won't really matter so we will not go into details and refer the interested reader to (22).

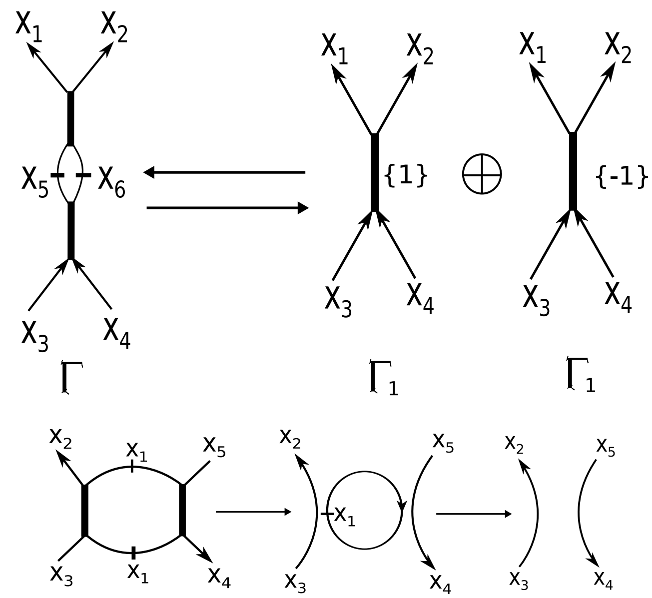


Figure 5.5: The map β in Direct Sum Decomposition II

Tangles and complexes

We resolve a crossing p in the two ways and assign to it a complex C^p depending on whether the crossing is positive or negative. To a diagram D representing a tangle L we assign the complex $C(D)$ of matrix factorization which is the tensor product of C^p, over all crossings p, of L^i_j over arcs $j \to i$, and of $A\langle 1\rangle$ over all crossingless markless circles in D. The tensor product is taken as before so that $C(D)$ is free and of finite rank as an R-module. This complex is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ graded.

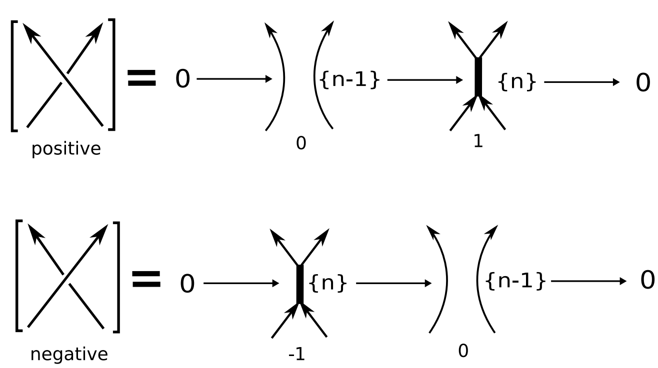


Figure 5.6: Complexes associated to pos/neg crossings; the numbers below the diagrams are cohomological degrees.

Theorem 5.1. *(Khovanov-Rozansky, (22)) The isomorphism class of $C(D)$ up to homotopy is an invariant of the tangle.*

If L is a link the cohomology groups are nontrivial only in degree equal to the number of components of L mod 2. Hence, the grading reduces to $\mathbb{Z} \oplus \mathbb{Z}$. The resulting cohomology groups are denoted by

$$H_n(D) = \bigoplus_{i,j \in \mathbb{Z}} H_n^{i,j}(D),$$

and the Euler characteristic of $H_n(D)$ is the quantum link polynomial $P_n(L)$, i.e.

$$P_n(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j dim_{\mathbb{Q}} H_n^{i,j}(D).$$

The isomorphism classes of $H_n^{i,j}(D)$ depend only on the link L and, hence, are invariants of the link.

5.3 The Basic Calculation

We first consider the complex associated to the tangle T in figure 5.7 with the appropriate maps χ_0 and χ_1 left out.

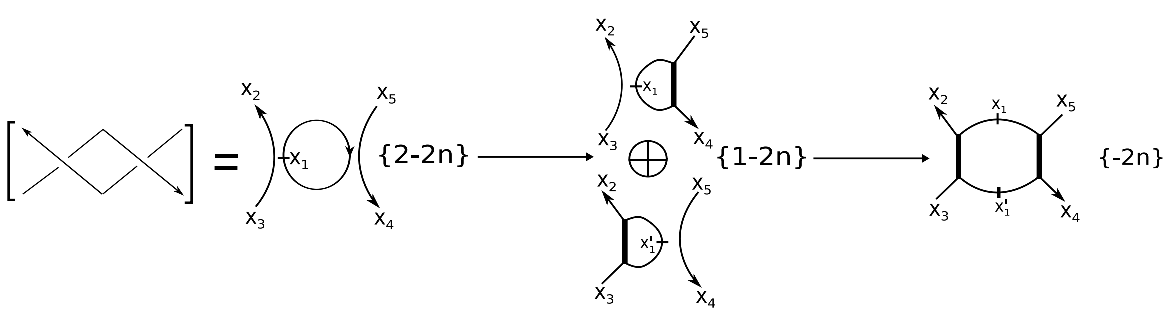


Figure 5.7: The tangle T and its complex

We first look at the following part of the complex and, for simplicity, leave out the overall grading shifts until later:

We apply direct sum decompositions 0 and I and end up with the following where the maps F_1 and F_2 are isomorphisms:

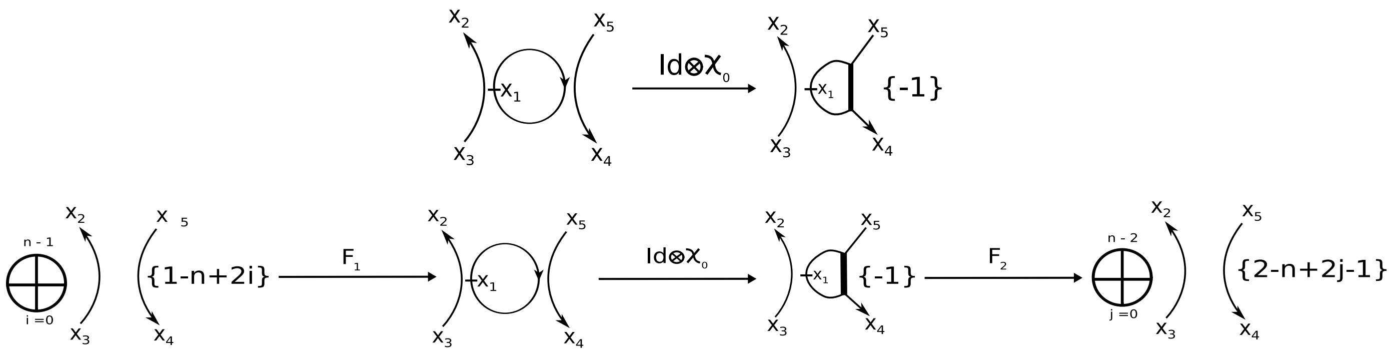


Figure 5.8: First part of the complex for T with decompositions

Explicitly, $F_1 = \sum_{i=0}^{n-1} Id \otimes x_1^i \iota \otimes Id$ and $F_2 = \sum_{j=0}^{n-2} Id \otimes \beta_j$

Composing the maps we get:

$$\begin{aligned}
F_2 \circ (Id \otimes \chi_0) \circ F_1 &= (\sum_{j=0}^{n-2} Id \otimes \beta_j) \circ (Id \otimes \chi_0) \circ (\sum_{i=0}^{n-1} Id \otimes x_1^i \iota \otimes Id) \\
&= (\sum_{j=0}^{n-2} Id \otimes \beta_j) \circ (\sum_{i=0}^{n-1} Id \otimes (\chi_0 \circ (x_1^i \iota \otimes Id))) \\
&= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} Id \otimes (\beta_j \circ \chi_0 \circ (x_1^i \iota \otimes Id)) \\
&= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} Id \otimes (\varepsilon'(x_1 - x_4) x_1^{n+i-j-2}) \\
&= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} Id \otimes (\varepsilon'(x_1^{n+i-j-1} - x_4 x_1^{n+i-j-2})) \\
&= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} Id \otimes \underbrace{(\varepsilon(x_1^{n+i-j-1}) - x_4 \varepsilon(x_1^{n+i-j-2}))}_{\Theta}.
\end{aligned}$$

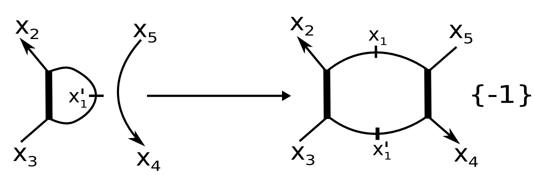
To go from line 3 to 4 and 4 to 5, recall that $\beta_j = \varepsilon' \circ \chi_1 x_1^{n-j-2}$ and $\chi_1 \circ \chi_0 = x_1 - x_4 = x_1 - x_5$. [Note: for lack of better notation, we use "$\sum$" to indicate both a map from a direct sum and an actual sum, as seen above indexed i and j respectively.]

Now $\Theta = Id$ if $i = j$, $-x_4$ if $i = j + 1$, and 0 otherwise, $F_2 \circ (Id \otimes \chi_0) \circ F_1$ is given by the following $(n-1) \times n - 1$ matrix:

$$\begin{bmatrix} Id & -x_4 & 0 & \dots & \dots & 0 \\ 0 & Id & -x_4 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & Id & -x_4 \end{bmatrix}$$

Using Gaussian Elimination for complexes 4.7 it is easy to see that, up to homotopy, only the top degree term survives. By degree, we mean with respect to the above grading shifts.

Now we look at the following subcomplex:



Including all the isomorphisms we have the complex in figure 5.9, with $G_1 = \sum_{i=0}^{n-2} \alpha_i \otimes Id$ and $G_2 = S \oplus \sum_{j=0}^{n-3} \beta_j$ (S is the saddle map).

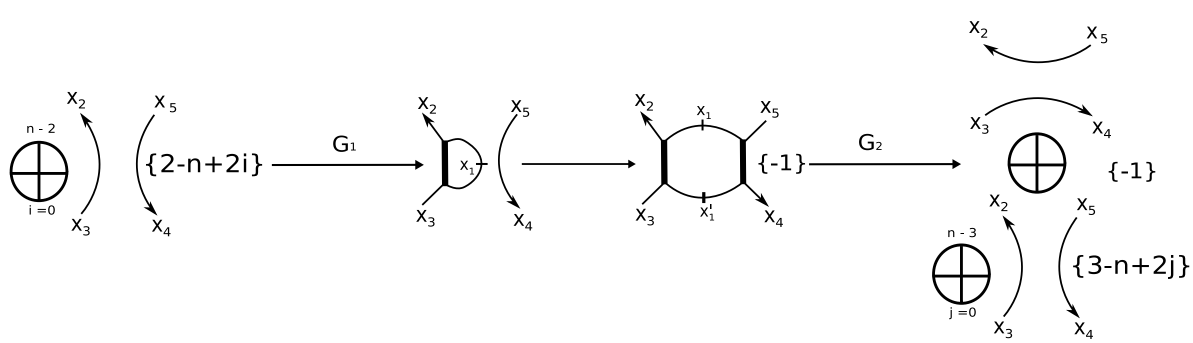


Figure 5.9: The second part of the complex for T with decompositions

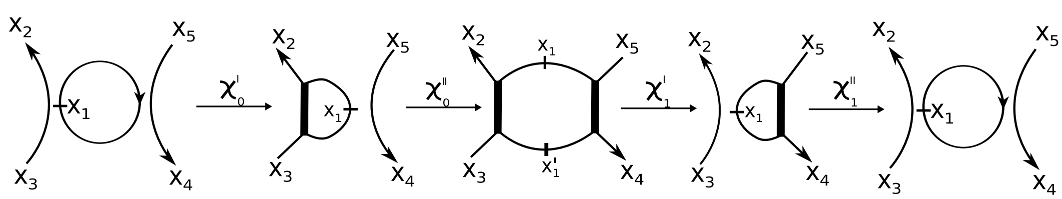
Composing these maps we get:

$$
\begin{aligned}
G_2 \circ \chi_0'' \circ G_1 &= (S \oplus \sum_{j=0}^{n-3} \beta_j) \circ \chi_0'' \circ (\sum_{i=0}^{n-2} \alpha_i \otimes Id) \\
&= \left(S \oplus \sum_{j=0}^{n-3} \beta \sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c \right) \circ \chi_0'' \circ \left(\sum_{i=0}^{n-2} \sum_{k=0}^{i} x_1^k x_2^{i-k} \alpha \otimes Id \right) \\
&= \left(S \oplus \sum_{j=0}^{n-3} \varepsilon' \circ \chi_1'' \circ \chi_1' \sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c \right) \circ \chi_0'' \circ \left(\sum_{i=0}^{n-2} \sum_{k=0}^{i} x_1^k x_2^{i-k} \chi_0' \circ \iota' \otimes Id \right) \\
&= \overline{S} \oplus \sum_{j=0}^{n-3} \sum_{i=0}^{n-2} \varepsilon' \circ \chi_1'' \circ \chi_1' \chi_0'' \circ \chi_0' \circ \left(\sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c) \right) \left(\sum_{k=0}^{i} x_1^k x_2^{i-k} \right) \iota' \\
&= \overline{S} \oplus \sum_{j=0}^{n-3} \sum_{i=0}^{n-2} \underbrace{\varepsilon' (x_1^2 - x_1 x_2 - x_1 x_4 + x_2 x_4) \left(\sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c) \right) \left(\sum_{k=0}^{i} x_1^k x_2^{i-k} \right) \iota'}_{\Omega}
\end{aligned}
$$

where

$$
\overline{S} = S \circ \chi_0'' \circ \left(\sum_{i=0}^{n-2} \sum_{k=0}^{i} x_1^k x_2^{i-k} \chi_0' \circ \iota' \otimes Id \right) \tag{5.1}
$$

To go from line 4 to 5 we recall what these χ's are:



The composition $\chi_1'' \circ \chi_0'' \circ \chi_1' \circ \chi_0' = (x_4 - x_1)(x_2 - x_1) = x_1^2 - x_1 x_2 - x_1 x_4 + x_2 x_4$, so now we just have to figure what happens with Ω.

Claim If $i < j$ then $\Omega = 0$ and if $i = j$ then $\Omega = Id$

Proof: This is just a simple check. The only thing to note is that $\Omega \neq 0$ iff one of the following occurs:

1)$c + k = n - 1$

2)$c + k + 1 = n - 1$

3)$c + k + 2 = n - 1$

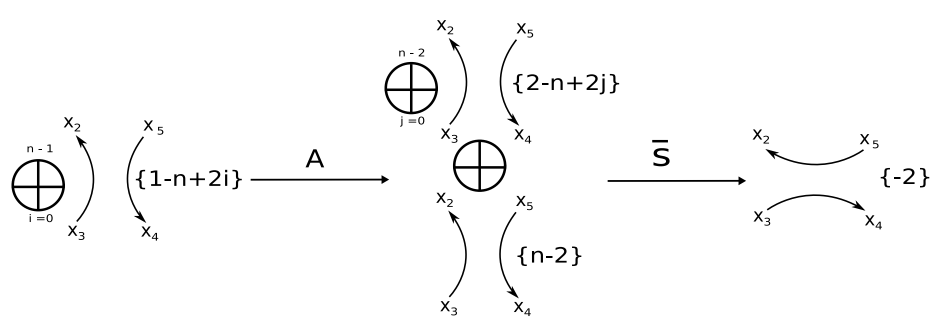
So $i < j \Rightarrow k < j$ so say $c + k = n - 1$. Then $a + b + c = a + b + n - 1 - k = n - 3 - j \Rightarrow a + b = -2 + k - j < 0$ contradiction, since a, b, c are nonnegative integers. The other two cases are similar.

From above we see that we need k at least equal to j. So if $i = j = k$ and $c + k + 2 = n - 1 \Rightarrow a + b + c = a + b + n - 3 - k = n - 3 - j \Rightarrow a + b = 0$ and $\Omega = Id$. The other two cases force $a + b < 0$. $\square$

So the matrix for Ω looks like:

$$\begin{bmatrix} Id & * & * & *** & *** & * & * \\ 0 & Id & * & *** & *** & * & * \\ \vdots & 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & \ddots & * & * \\ 0 & & \ldots & \ldots & 0 & Id & * \end{bmatrix}$$

Using Gaussian Elimination 4.7 we see that only the entry corresponding to $i = n - 2$ survives and the original complex is homotopic to:



where $A =$

$$\begin{bmatrix} Id & -x_4 & 0 & \dots & \dots & 0 & 0 \\ 0 & Id & -x_4 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & Id & -x_4 & 0 \\ 0 & & \dots & \dots & 0 & Id & -x_4 \\ 0 & & \dots & \dots & 0 & -Id & x_2 \end{bmatrix}$$

This is just our original matrix Θ but with one more row for the extra term, for which the entries are computed identically as we have already done. We reduce the complex in fig. 5.7, insert the overall grading shifts and arrive at our desired conclusion, i.e.:

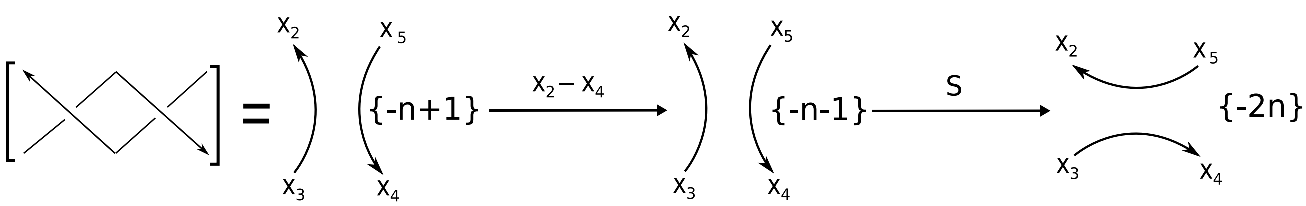


Figure 5.10: The reduced complex for tangle T

Note: to convince ourselves that the map S above is indeed the “saddle” map as prescribed, we need only to know that the hom-space of degree zero maps between the two right-most diagrams above is 1-dimensional, in the homotopy category, and then argue that the map is nonzero. This can be done by say closing off the two ends of the tangle above such that we have a non-standard diagram of the unknot and looking at the cohomology of the associated complex. We leave the details to the reader and refer to (23) for hom-space calculations.

5.4 Basic Tensor Product Calculation

We now consider our tangle T composed with itself, i.e. the tangle gotten by taking two copies of T and gluing the rightmost ends of one to the leftmost of the other. On the complex level this corresponds to taking the tensor product of the complex for T with itself while keeping track of the associated markings.

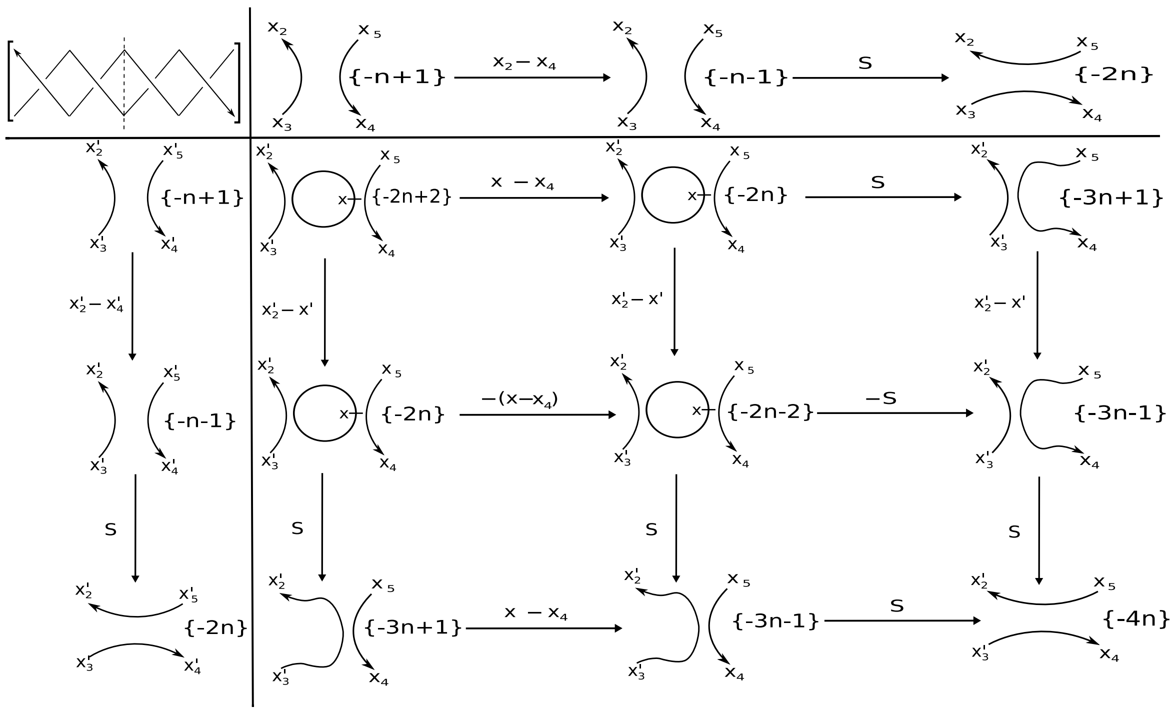


Figure 5.11: Complex for the tensor product

Note that when we take the tensor product we need to keep track of markings. For example: in the left most entry of the tensored complex $x_2 = x'_5 = x'_4 = x_3$, which we denote simply by x, etc.

As before, we decompose entries in the complex into direct sums of simpler objects, compute the differentials and reduce using Gaussian Elimination 4.7. In a number of instances we will restrict ourselves to the $n = 3$ case, as the general case works in exactly the same way with the computation more cumbersome.

We break the computation up based on homological grading.

Degree 0:

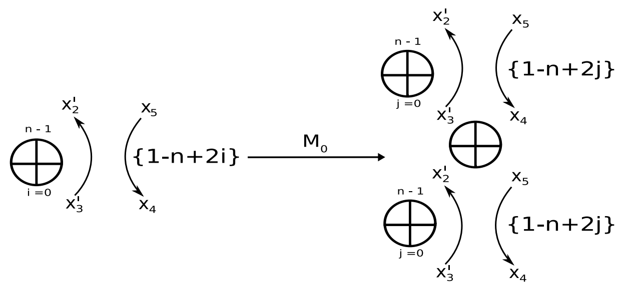


Figure 5.12: Calculating degree 0 to 1

where M_0 is:

$$\begin{bmatrix} \sum_{i,j=0}^{n-1} Id \otimes \varepsilon(x^{n+i-j} - x^{n-1+i-j}x_4)\iota \otimes Id \\ \sum_{i,j=0}^{n-1} Id \otimes \varepsilon(x_2' x^{n-1+i-j} - x^{n+i-j})\iota \otimes Id \end{bmatrix}$$

For $n = 3$ we have the following:

$$\begin{bmatrix} -x_4 & 0 & 0 \\ Id & -x_4 & 0 \\ 0 & Id & -x_4 \\ x_2' & 0 & 0 \\ -Id & x_2' & 0 \\ 0 & -Id & x_2' \end{bmatrix} \overset{reduce}{\rightsquigarrow} \begin{bmatrix} Id & -x_4 \\ -x_4^2 & 0 \\ x_2'x_4 & 0 \\ x_2' - x_4 & 0 \\ -Id & x_2' \end{bmatrix} \overset{reduce}{\rightsquigarrow} \begin{bmatrix} 0 \\ x_2'x_4^2 \\ x_2'x_4 - x_4^2 \\ x_2' - x_4 \end{bmatrix} = \overline{M}_0$$

[Note: we first permute the rows in the first half of the matrix s.t. the Id maps appear on the diagonal.]

The general case is exactly the same, i.e. in the left most matrix above, the upper and lower 3×3 matrices become expanded to similar $n \times n$ matrices. Hence, the complex reduces to:

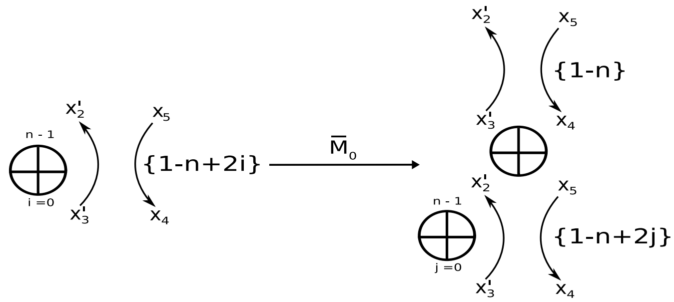


Figure 5.13: Degree 0 to 1

Degree 1:

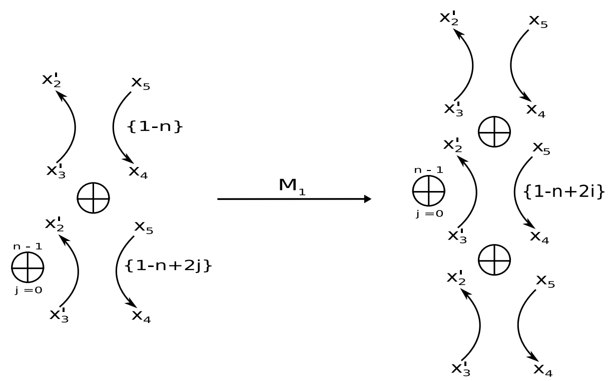


Figure 5.14: Calculating degree 1 to 2

with M_1=:

$$\begin{bmatrix} Id\otimes S\circ\iota\otimes Id & \{0\}_{1\times n} \\ M_1^a & M_1^b \\ \{0\}_{n\times 1} & M_1^c \end{bmatrix}$$

where

$$M_1^a=\sum_{j=0}^{n-1} Id\otimes\varepsilon(x_2'x^{n-1-j}-x^{n-j})\iota\otimes Id,$$

$$M_1^b=\sum_{i,j=0}^{n-1} Id\otimes\varepsilon(x_4x^{n-1-j+i}-x^{n-j+i})\iota\otimes Id,$$

$$M_1^c = \sum_{i=0}^{n-1} Id \otimes x^i S \circ \iota \otimes Id.$$

(Note: $x^i S \circ \iota$ here is equal to multiplication by $x_2'^i$) expanding we get:

$$\begin{bmatrix} Id & 0 & \dots & \dots & \dots & 0 \\ x_2' & x_4 & 0 & \dots & \dots & 0 \\ -Id & -Id & x_4 & 0 & \dots & \vdots \\ 0 & \dots & \ddots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & -Id & x_4 \\ 0 & Id & x_2' & \dots & \dots & x_2'^{n-1} \end{bmatrix} \overset{reduce}{\rightsquigarrow} \begin{bmatrix} x_4 & 0 & \dots & \dots & 0 \\ -Id & x_4 & 0 & \dots & \vdots \\ 0 & \ddots & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -Id & x_4 \\ Id & x_2' & \dots & \dots & x_2'^{n-1} \end{bmatrix} \overset{row-moves}{\rightsquigarrow}$$

$$\begin{bmatrix} -Id & x_4 & 0 & \dots & \vdots \\ 0 & -Id & x_4 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -Id & x_4 \\ x_4 & 0 & \dots & \dots & 0 \\ Id & x_2' & \dots & \dots & x_2'^{n-1} \end{bmatrix} \overset{reduce}{\rightsquigarrow} \begin{bmatrix} -Id & x_4 & \dots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -Id & x_4 \\ x_4^2 & \dots & \dots & 0 \\ (x_2' + x_4) & \dots & \dots & x_2'^{n-1} \end{bmatrix} \overset{reduce}{\rightsquigarrow} \begin{bmatrix} 0 \\ \sum_{i=0}^{n-1} x_2'^i x_4^{n-1-i} \end{bmatrix}$$

and we have the following:

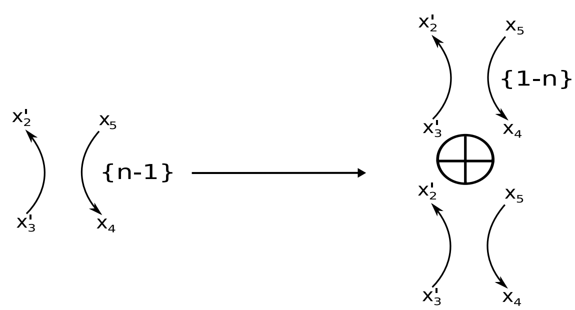


Figure 5.15: Calculating degree 1 to 2

Degree 2 and 3:

The complex now is pretty simple:

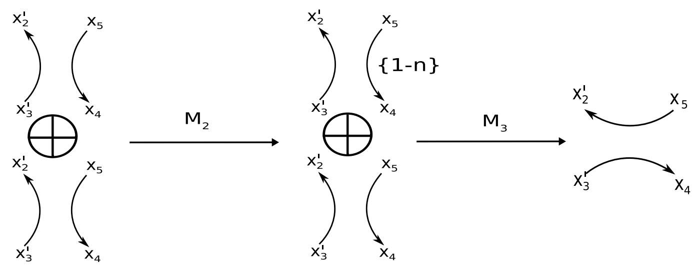


Figure 5.16: Calculating degree 2 and 3

$$M_2 = \left[\begin{array}{cc} -(Id \otimes S \circ \iota) \otimes Id & Id \otimes (S \circ \iota \otimes Id) \\ 0 & x_2' - x_4 \end{array} \right], \qquad M_3 = [S \ S] .$$

All we have to do is note that $Id \otimes S \circ \iota \otimes Id = Id$ reduce, insert the grading shifts and arrive at the desired conclusion, i.e.:

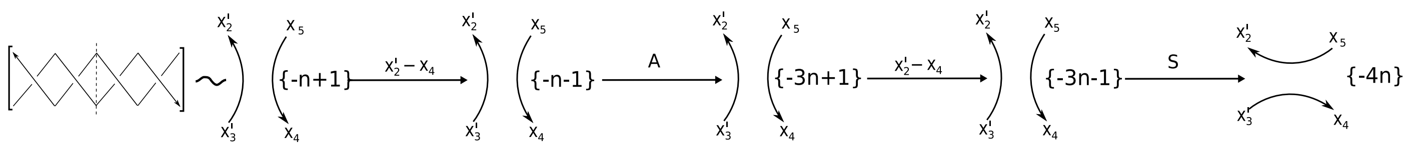


Figure 5.17: The tensor complex

with $A = \displaystyle\sum_{i=0}^{n-1} x_2'^i x_4^{n-1-i}$.

5.5 The General Case

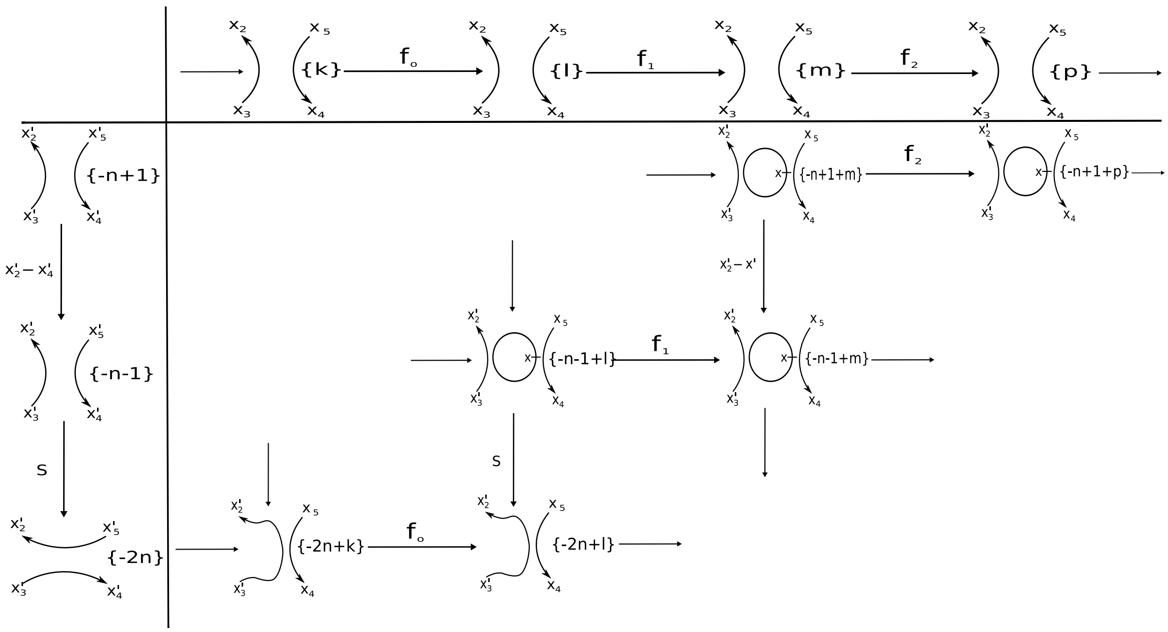


Figure 5.18: Tensoring the complex with another copy of the basic tangle T

We suppose by induction that the k-fold tensor product of our basic complex has the form as above in fig. 5.17 with alternating maps $x_2' - x_4$ and A, the last map being the saddle cobordism S, and investigate what happens when we add one more iteration. As before, this corresponds to tensoring with another copy of the reduced complex for tangle T, i.e. the one in fig. 5.10, but as we will see below "most" of this new complex is null-homotopic and it suffices to consider only the part depicted in fig. 5.18 directly above. Note that here the bottom row is a subcomplex which is isomorphic to that of the top tangle and we claim that, up to homotopy, this plus two more terms in leftmost homological degree is exactly what survives. The remaining calculation is left to clear up this statement and we begin by taking a look at the highlighted part of the complex depicted in fig. 5.18, i.e.:

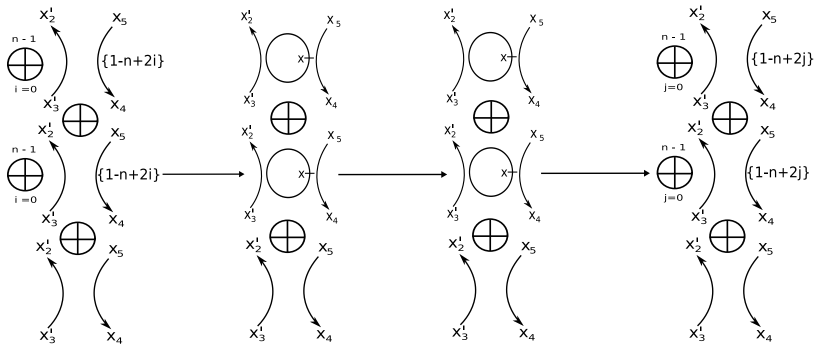


Figure 5.19: Decomposing the entries of the general tensor product

...of course we have once again decomposed the complex and left out the overall grading shifts until later.

The above composition of maps is:

$$\begin{bmatrix} M^a & \{0\}_{n\times n} & \{0\}_{n\times 1} \\ M^b & M^c & \{0\}_{n\times 1} \\ \{0\}_{1\times n} & M^d & f_0 \end{bmatrix}$$

$$M^a = \sum_{i,j=0}^{n-1} Id \otimes \varepsilon f_2 x^{n-1-j+i}\iota \otimes Id \qquad M^c = -\sum_{i,j=0}^{n-1} Id \otimes \varepsilon f_1 x^{n-1-j+i}\iota \otimes Id$$

$$M^b = \sum_{i,j=0}^{n-1} Id \otimes \varepsilon x^{n-1-j}(x_2' - x)x^i\iota \otimes Id \qquad M^d = \sum_{j=0}^{n-1} Id \otimes x^{n-1-j} S \circ \iota \otimes Id$$

Expanding, with $f_0 = f_2 = x - x_4$ and $f_1 = \sum_{m=0}^{n-1} x^m x_4^{n-1-m}$ we get the following submatrices:

$$M^a = \begin{bmatrix} -x_4 & 0 & \dots & \dots & 0 \\ Id & -x_4 & 0 & \dots & \vdots \\ 0 & Id & -x_4 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & Id & -x_4 \end{bmatrix} \qquad M^b = \begin{bmatrix} x_2' & 0 & \dots & \dots & 0 \\ -Id & x_2' & 0 & \dots & \vdots \\ 0 & -Id & x_2' & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -Id & x_2' \end{bmatrix}$$

$$M^c = -\begin{bmatrix} x^{n-1}x_4^{n-1} & 0 & \dots & \dots & 0 \\ * & x^{n-1}x_4^{n-1} & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ * & \dots & * & x^{n-1}x_4^{n-1} & 0 \\ Id & * & \dots & \dots & x^{n-1}x_4^{n-1} \end{bmatrix}$$

Now this might look like a mess to reduce, but the thing to notice is that, in the corresponding summand in our decomposition, the first matrix above kills off all but the topmost degree terms (with respect to the decomposition-induced grading shifts), whereas the Id map found in the left-bottom corner of the second kills off precisely the topmost degree term. As the maps alternate when we increase cohomological grading and none of the reductions affect the bottom row (this is easy to see due to the 0's found in the first row), up to homotopy the bottom row remains altered only by a grading shift.

As far as the beginning and the end of the complex is concerned we have already done those computations when we looked at the 2-fold tensor product. Hence, we arrive at our desired conclusion:

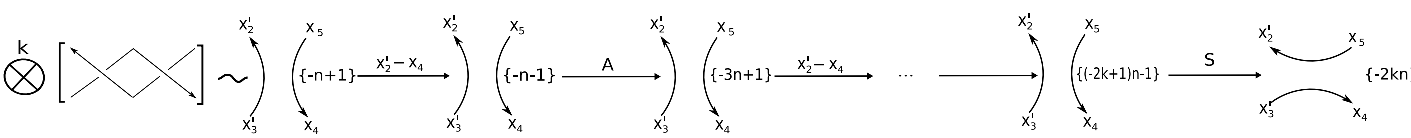


Figure 5.20: The complex of the k-fold tensor product

where $A = \sum_{i=0}^{n-1} x_2'^{i} x_4^{n-1-i}$.

Similarly we see that the tangle gotten by flipping all the crossings is

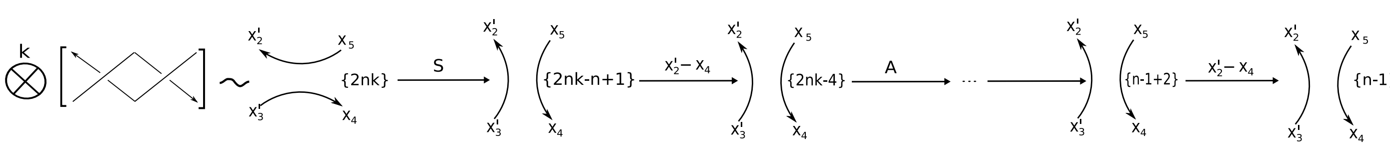


Figure 5.21: The complex of the k-fold tensor product

5.6 Remarks

Following (5) we can propose a similar “local” algorithm for computing Khovanov-Rozansky homology. Start with a knot or link diagram and reduce it locally using the Direct Sum Decompositions found. Then put all the pieces back together and end up with a complex where the objects are are just circles, which we can further reduce to a complex of empty sets with grading shifts, i.e. direct sums of $\mathbb{Q}$ the maps are matrices with rational entries. Since a multiplication map $\mathbb{Q} \to \mathbb{Q}$ is either an zero or an isomorphism we can use Gaussian Elimination, as above, to further reduce this complex to one where all the differentials are zero. The computational advantage of such an algorithm is described in more detail in (5). Unfortunately no such program exists to our knowledge.
Furthermore, for the examples of tangles we consider here the computational complexity is similar to that of sl_2-homology. As there are no more “thick edges” in any resolution, only Direct Sum Decomposition 0 is necessary to reduce the complex to $\mathbb{Q}$ vector spaces and matrices between them. Potentially a modification of the existing programs could allow to compute a large collection of examples composed from these tangles.

We have done a similar computation for the “foam” version of sl_3-homology introduced in (17). Here the nodes in the cube of resolutions are generated by maps from the empty graph to the one at the corresponding node, with some relations, and the maps are given by cobordisms between these trivalent graphs. The decompositions mimic the ones we find here, when specializing to $n = 3$, as do the relations on the maps. Reducing the complex as before we find that it is identical to the one found above when specialized to the $n = 3$ case. Hence, any link that can be decomposed into the above tangles has exactly the same homology groups for the ”foam” and matrix-factorization version. This provides a rather vast number of examples where the isomorphism between the two theories is completely explicit.

Bibliography

[1] Elias B. and M. Khovanov. Diagrammatics for Soergel categories. 2009, math.QA/0902.4700v1.

[2] Elias B. and D. Krasner. Rouquier complexes are functorial over braid cobordisms. 2009, arXiv:0906.4761v2.

[3] J. Baldwin and O. Plamenevskaya. Khovanov homology, open books, and tight contact structures. arXiv:0808.2336v2.

[4] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499 (electronic), 2005.

[5] Dror Bar-Natan. Fast Khovanov homology computations. *J. Knot Theory Ramifications*, 16(3), 2007.

[6] D.J. Benson. *Representations and Cohomology I. Basic representation theory of finite groups and associative algebras.* Number 30 in Cambridge studies in advanced mathematics. Cambridge U. Press, 1995.

[7] J. Carter and M. Saito. *Knotted surfaces and their diagrams,.* Number 55 in Math. Surv. and Mon. AMS, 1998.

[8] S. Cautis and J. Kamnitzer. Knot homology via derived categories of coherent sheaves. i. the sl(2) case. *Duke Math. J*, 142.

[9] S. Cautis and J. Kamnitzer. Knot homology via derived categories of coherent sheaves. ii. the sl(m) case. *Invent. Math.*, 174(1):165–232, 2008.

[10] S. Morrison D. Clark and K. Walker. Fixing the functoriality of Khovanov homology. arXiv:math/0701339v2.

[11] D. Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260:35–64, 1980.

[12] B. Gornik. Note on Khovanov link homology. 2004, math.QA/0402266.

[13] T. Otsuki H. Murakami and S. Yamada. HOMFLY polynomial via an invariant of colored plane graphs. *Enseign. Math.*, 44(3-4):325–360, 1998.

[14] V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, (12):103–111, 1985.

[15] L. Kadison. *New examples of Frobenius extensions.* Number 14 in University Lecture Series. AMS, 1999.

[16] Louis H. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.

[17] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[18] Mikhail Khovanov. sl(3) link homology. *Algebr. Geom. Topol.*, 4:1045–1081 (electronic), 2004.

[19] Mikhail Khovanov. An invariant of tangle cobordisms. *Trans. Amer. Math. Soc.*, 358(1):315–327 (electronic), 2006.

[20] Mikhail Khovanov. Link homology and Frobenius extensions. *Fund. Math.*, 190:179–190, 2006.

[21] Mikhail Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *Internat. J. Math.*, 18(8):869–885, 2007.

[22] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. *Fund. Math.*, 199(1):1–91, 2008.

[23] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. II. *Geom. Topol.*, 12(3):1387–1425, 2008.

[24] Mikhail Khovanov and Richard Thomas. Braid cobordisms, triangulated categories, and flag varieties. *Homology, Homotopy Appl.*, 9(2):19–94, 2007.

[25] E.S. Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197:554–586, 2005.

[26] A. Lauda M. Khovanov. A diagrammatic approach to categorification of quantum groups i. *Represent. Theory*, 13:309–347, 2009.

[27] M. Mackaay and P. Vaz. The reduced HOMFLY-PT homology for the Conway and the Kinoshita-Terasaka knots. *Enseign. Math.*, arXiv:0812.1957v1.

[28] Marco Mackaay and Pedro Vaz. The universal sl_3-link homology. *Algebr. Geom. Topol.*, 7:1135–1169, 2007.

[29] Marco Mackaay and Pedro Vaz. The foam and the matrix factorization sl_3 link homologies are equivalent. *Algebr. Geom. Topol.*, 8(1):309–342, 2008.

[30] S. Morrison and A. Nieh. On Khovanov's cobordism theory for su(3) knot homology. 2006, math/0612754.

[31] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[32] O. Plamenevskaya. Transverse knots and Khovanov homology. *Math. Res. Lett.*, 13(4):571–586, 2006.

[33] J. Rasmussen. Some differentials on Khovanov-Rozansky homology. arXiv:math/0607544v2.

[34] J. Rasmussen. Khovanov homology and the slice genus. 2004, math.GT/0402131.

[35] J. Rasmussen. Khovanov-Rozansky homology of two-bridge knots and links. *Duke Math. J.*, 136(3):551–583, 2007.

[36] R. Rouquier. Categorification of the braid groups. math.RT/0409593.

[37] W. Soergel. Gradings on representation categories,. *Proceedings of the ICM 1994 in Zürich*, pages 800–806.

[38] W. Soergel. Kazhdan-Lusztig-polynome und unzerlegbare bimoduln "uber polynomringen. math.RT/0403496v2.

[39] W. Soergel. The combinatorics of Harish-Chandra bimodules. *Journal Reine Angew. Math.*, 429:49–74, 1992.

[40] W. Soergel. Combinatorics of Harish-Chandra modules. *Proceedings of the NATO ASI 1997, Montreal, on Representation theories and Algebraic geometry*, 1998.

[41] Catharina Stroppel. Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors. *Duke Math. J.*, 126(3):547–596, 2005.

[42] Catharina Stroppel. Parabolic category O, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology. *Compos. Math.*, 145(4):954–992, 2009.

[43] B. Webster. Kr.m2. 2005, http://katlas.math.toronto.edu/wiki/user:Ben/KRhomology.