Convex Duality in Singular Control – Optimal Consumption Choice with Intertemporal Substitution and Optimal Investment in Incomplete Markets

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Abstract

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In this thesis we study the problem of optimal consumption choice with investment in incomplete markets. The agent's preferences are modeled using non time-additive utilities of the type proposed by Hindy, Huang and Kreps. For such preferences the period utilities depend on the entire path of consumption up to date.

We show that a dual relationship exists between the utility optimization problem and a carefully chosen dual minimization problem. Time-inhomogeneity of the preferences and the dependence on past consumption leads to utility gradients that, in a deterministic setting, have the structure of inhomogeneously convex functions. A stochastic representation theorem is used to extend this concept to apply in the random setting. We find that the appropriate dual variables are not necessarily adapted, but that they do have adapted densities.

We illustrate the techniques by finding explicit solutions in a Wiener driven market with multiple assets. For the explicit solutions we pass to the infinite time-horizon, and show how to use the duality framework as a verification theorem. The optimal solution is to consume whenever the supremum of a certain Brownian motion with drift increases. Thus optimal consumption is singular: there is no period of time in which the agent consumes continuously.

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To my brothers Timo and Lauri

1 Introduction

The optimal behavior of agents in financial markets is an important area of research in finance and economics; in this thesis we study the question of simultaneous optimal consumption and investment. The agent, endowed with certain initial capital, is faced with the problem of choosing, at each point in time, a portion of his wealth to consume and to invest the remainder in the market. In our model, preferences are based on cumulative consumption, or, more generally, on the level of satisfaction derived from consumption up to date. In contrast to standard time-additive utilities, ours does not depend directly on the rate of consumption. Consequently, consumption can occur in a very general way: for example in gulps or in a singular way. Such preferences also satisfy important economic robustness properties.

We show here how convex duality techniques can be used to characterize the optimal consumption plan in an incomplete, possibly non-Markovian, semimartingale market. It is well known that for standard time-additive von Neumann-Morgenstern preferences, those based on the rate of consumption, the dynamic control problem that arises can be solved via the classical Hamilton-Jacobi-Bellman approach. The general duality result is also known: it has been solved by Karatzas and Žitković [34]. In order to handle the singular control problem, arising for the optimal consumption problem for non time-additive preferences, an exploration of different notions of convexity, both in the time and state variable, will be required. In this thesis we show how to carry out this program. The final result is a theorem that relates the solution of the utility optimization problem with that of a dual minimization problem.

1.1 Economic Motivation

The single-agent optimal consumption/investment problem was first considered in continuous time by Merton [39], [40] (see also Samuelson [45] for a treatment of the multi-period, discrete time case.) In his work, Merton used dynamic programming techniques to show that the optimal strategy for investors, with constant relative risk aversion, is to invest a constant proportion of wealth in the risky asset. He also showed that the rate of consumption should be proportional to current wealth. In his model the risky asset was driven by a Brownian motion. In the same year, Merton and Samuelson joined forces to prove, using utility optimization, a new pricing rule for options [41]. Under their pricing measure, obtained from utility maximizing considerations, the discounted stock process is a martingale. Their analysis pre-dates the famous Black and Scholes formula for option pricing, published in 1973.

Merton modeled the agent's preferences using time-additive von Neumann-Morgenstern utilities. These utilities are based on the rate of consumption and take the form $\mathbb{E}U(C) = \mathbb{E}\int_0^T u(t, \dot{C}_t)dt$, where C_t represents the cumulative consumption at time t. Such utilities are defined on the space of absolutely continuous consumption plans. By now, such preferences have become a standard starting point in the literature, while much important research has been conducted to address some of the limitations of Merton's model. For example, Constantinides [15] added a habit formation index; his goal was to find a model of utility that would resolve the equity premium puzzle. Applications to habit formation has also been studied by Detemple and Zapatero [23], and, more recently, Englezos and Karatzas [25], for example. Some of the other extensions include consideration of transaction costs (for instance Shreve and Soner [48]) and the introduction of convex portfolio constraints (for instance Cvitanic and Karatzas [17], Shreve and Xu [47], and Cuoco and Liu [16].)

In our model we use non-time-additive preferences: preferences that are based on the whole path of consumption up to date (c.f. Section 7.1 for the most general formulation). Such models of utility were introduced by Hindy, Huang, and Kreps [29] as a more economically appropriate alternative to the standard time-additive models. Their most important criticism focused on the concept of local substitution: consumption at near-by dates and at slightly varying rates should be close substitutes. They show that preferences continuous in the Prohorov topology exhibit this desired property and proceed to describe a class of preferences that are continuous in this topology.

For an intuitive understanding of local substitution, we can think about the act of making a large purchase such as obtaining a car. From the perspective of utility, it should not matter too much whether the car is purchased today or tomorrow. Similarly, companies making a large capital investment should not be too sensitive to the timing. Continuous consumption at slightly varying rates, but with the same overall total, should also be close. Examples to think about here are the purchasing of food items with no storage cost, or that of a company investing in preventative maintenance of infrastructure.

The intuition from these examples is valid barring any anticipated price shocks. Thought another way, however, local substitution is exactly what is needed for continuity of prices. Or rather, in the presence of randomness, we want prices to not jump at predictable times. This property comes about if there are agents in the economy who are willing to withhold consumption or to store goods in preparation of making a profit at the time of the price shock. The relationship of local subsititution and equilibrium pricing is discussed further in Hindy and Huang [27].

The early work of Hindy and Huang [28], establishing solutions in a basic Black-Scholes market model, offers interesting insights on the qualitative differences between the behavior of agents who exhibit local substitution and those whose preferences are modeled as in the classical Merton model. Key characteristics of the optimal Merton solution are that consumption occurs continuously, the rate of consumption equals a constant fraction of the current wealth, and that the agent invests a constant proportion of his wealth in the risky asset. In contrast, agents who exhibit local substitution follow what Hindy and Huang call a *ratio barrier* policy: the optimizing agent invests a constant proportion of wealth in the risky asset, and consumes just enough the keep the ratio of current wealth to the current level of satisfaction below a pre-determined level almost surely. Consequently, in a Black-Scholes market the optimizing agent consumes in a manner singular with respect to the Lebesgue measure on the time axis. In addition, it is observed that an agent with intertemporal preferences, preferences that exhibit local substitution, is less risk averse. One possible interpretation of this behavior is that with intertemporal preferences the discounted utility from past consumption allows for higher risk tolerance of future gains.

Following the work of Hindy, Huang, and Kreps [29], and Hindy and Huang [28], various authors have worked on the problem of utility optimization with intertemporal preferences. A static approach to the optimization problem, using an infinite-dimensional analogue of the Kuhn-Tucker conditions, was introduced by Bank and Riedel for a deterministic setting in [7] and for stochastic (complete) markets in [9]. For example, these authors find that if the asset prices follow geometric Brownian motion, then the agent consumes whenever the supremum of a Brownian motion with drift $(W(t) + \mu t)$ increases. Thus there is no open interval in which the agent optimally consumes all of the time; consumption is singular. A viscosity solution approach to the problem, one that can handle more general markets, was introduced by Alvarez [2]. Further solutions in the viscosity solution framework were obtained by Benth, Karlsen and Reikvam [12] and [11]. A few extensions of the model for preferences have also been explored in the literature. For example, Hindy, Huang, and Zhu [30] extend the model to include habit formation.

In this thesis we discuss how duality techniques can be applied to the problem of optimization with intertemporal preferences. Along the lines of Kramkov and Schachermayer [36], who provide a general solution to the problem of optimizing utility from terminal wealth, we show that the solution of the primal utility optimization problem is related to the solution of a dual minimization problem. In this way we obtain a general framework for treating the optimal consumption/investment problem in incomplete semimartingale markets. In order to extend the duality framework to processes, we use the process bipolar theorem developed by Žitković [51]. Karatzas and Žitković have used this bipolar theorem to apply duality methods to the case of time-additive utilities in [34]. In this thesis we show how the stochastic representation theorem due to Bank [3] can be used to extend the duality framework to non-time additive utilities. For a historical development of duality methods see, for example, [36], [32], and [34] and the references therein.

1.2 Mathematical Results

The thesis is organized as follows. We first describe the model for the financial market and the preferences used. In this section, Section 2, we also introduce the standing assumptions for the paper and the notation used.

In Sections 3–5. we cover the important analytic constructions in preparation for the proof of the main duality theorem in Section 6. Section 3 is about the space of consumption plans. We start by discussing the topology on the space of optional random measures on $[0, T]$, the most general class of consumption strategies. For this set of processes we discuss why the Prohorov metric captures the economically meaningful notion of local substitution. More details on topologies on the space of consumption plans and the relevant economic properties are provided in the article [29] by Hindy, Huang, and Kreps.

Our goal is to apply a minimax theorem and for this we will need certain continuity and compactness properties. To this end, the concept of convex compactness is introduced (see also Žitković [52]). We prove that the bounded subsets of the consumption space are convexly compact, in this sense we show that the topology has enough "compact" sets. In addition, we prove that the utility function is continuous on reasonable subsets of the consumption space, and that the natural pairing (see identity (8)) between the consumption space and our chosen dual variables is lower-semicontinious.

We then proceed to define an appropriate dual problem and a set of dual variables (processes). First, in Section 4, we discuss the deterministic preliminaries. In particular, we look at the Legendre-Fenchel transform of the utility functional $U(\cdot)$ and use this to define the appropriate conjugate (or dual) functional V. The gradients of the utility functional are established and these are shown to have the structure of inhomogeneously convex functions. This generalized notion of convexity is described in detail in Section 4.1. Once we have established the definitions, we show how to evaluate the dual functional on the space of inhomogeneously convex functions, and prove lower-semicontinuity of the dual functional. The section serves as a reference for notation on the utility functional, its dual, and the respective super and subgradients.

In the following section we extend the discussion to random processes. We show that the delicate question here is the ability (or inability) to exchange the expectation and the supremum in the definition of the conjugate functional V. Thus a careful choice of dual variables is required. Sections 5.1 and 5.2 describe the reasons behind this choice; identity (37) gives the definition of the set of dual processes. The choice begins with a second look at the budget constraint: this leads to a consideration of a larger class of processes, or deflators, than just the set of equivalent (local)martingale measures. Because the consumption plans are optional processes, it turns out that we can work interchangeably with a process and its optional projection. This fact gives us much freedom to find processes that also behave well with respect to the definition of the conjugate functional. The final step in this program is to use a stochastic representation theorem (see [3] and [4]) to establish a notion of an inhomogeneously convex envelope that can be used in the random setting (it is not the pathwise envelope, we need to go through an auxiliary process first).

This new set of dual variables that we define consists of possibly non-adapted processes and there are various important properties that we must check are true. In the final part of Section 5 we discuss these properties. Most importantly we show that the set is convex. Due to the time-inhomogeneity of the utility functional, it is not clear that the set of gradients (of utility) should be convex. In fact, we make some additional assumptions on the form of the preferences in order for this to hold. Lastly we show that the set is (Fatou) closed.

Section 6 contains the main theorem in the paper. Theorem 6.3 establishes a relationship between the primal and dual optimization problems and gives an explicit description of how these solutions are related. We also show how the theorem simplifies for a complete market, this is Theorem 6.5.

In the final part, Section 7, we discuss some examples and extensions. In particular, we show that our duality framework can also be applied to utilities based on the level of satisfaction (discounting of past consumption is accounted for) and calculate an explicit solution for time-homegeneous Wiener driven models. The examples are calculated for an infinite time-horizon where it is possible to solve the stochastic representation problem (Section 7.2). In this case we use the fact that the duality set-up can also be used as a verification theorem (and as a way to arrive at a good guess). The verification theorem is stated in Section 7.3. We are currently working on extending these calculations to more general Lévy process markets.

2 The Model

We consider an agent endowed with an initial wealth x and presented with the task of choosing an optimal consumption and investment strategy over a specified time period $[0, T]$. For simplicity, we assume that T is finite unless otherwise specified. The investor has preferences represented by an expected utility functional, to be specified later. The optimization problem then is to find a consumption plan C^* that solves

$$u(x) \triangleq \sup_{C \in \mathcal{C}(x)} \mathbb{E} U(C). \tag{1}$$

In this section we shall explain: what is the model for the financial market, what are financiable consumption plans $\mathcal{C}(x)$, and what does the utility function $U(\cdot)$ look like. This discussion is intended to further motivate the optimal consumption problem, as well as to serve as a reference for terminology and assumptions.

The financial market consists of a bond S^0 and n assets $S^1, S^2, ..., S^n$ whose dynamics are modeled by RCLL, locally bounded semimartingales. All of these instruments are modeled on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathcal{F} = (\mathbb{F}_t)_{t \in [0,T]}$ that satisfies the usual conditions; it is right-continuous and complete. We assume that the bond is constant $S^0 \equiv 1$. Observe, however, that the general setting can be recovered via a change of numéraire and appropriate discounting of consumption. In general, such markets are incomplete. In order to guarantee absence of arbitrage, we assume that there exists a measure $\mathbb{Q} \sim \mathbb{P}$ such that S is a local martingale under $\mathbb{Q}$, i.e. we assume the existence of an equivalent local martingale measure. The equivalence of no arbitrage (no free lunch with vanishing risk) and the existence of an equivalent local martingale measure was proved by Delbaen and Schachermayer [19]. In the examples section we will consider the case $T = \infty$, and such that the market is possibly only arbitrage free for every finite (deterministic) time-horizon. In general, markets with an infinite time-horizon are not arbitrage free.

In this market, we first look at pure investment strategies. Such strategies are defined by a pair (x, H). Here the constant x is the initial wealth of the investor, and $H = (H^i)_{i \leq d}$ is a predictable, S-integrable process. The interpretation is that the components H^i are the numbers of shares held of each asset S^i. The value process $X = (X_t)_{t \in [0,T]}$ associated to such a strategy is given by the stochastic integral

$$X_t = x + \int_0^t H_u \, dS_u, \quad 0 \leq t \leq T. \tag{2}$$

For any $x \in \mathbb{R}_+$, let $\mathcal{X}(x)$ denote the set of value processes with initial value $X_0 = x$ and with non-negative capital at any instant:

$$\mathcal{X}(x) \triangleq \{X \text{ is given by equation (2) and such that } X_t \geq 0 \;\; \forall t \in [0,T]\} \,. \tag{3}$$

A pure investment strategy (x, H) is admissible if the associate wealth process is an element of $\mathcal{X}(x)$.

On the other hand, a combined consumption and investment strategy is a triple (x, H, C). Here x and H are as before and the cumulative consumption $C = (C_t)_{t \in [0,t]}$ is a right-continuous, increasing, and adapted process. Thus we interpret C_t as the cumulative consumption up to time t, with $C_0 > 0$ implying a gulp of consumption at $t = 0$. We denote the set of all consumption plans by

$$\mathcal{C} \triangleq \{C : \Omega \times [0,T] \mapsto \mathbb{R}_+ \mid \; C \text{ is increasing, right-continuous, and adapted.}\} \tag{4}$$

Observe that this description of a consumption plan is very general. For instance, we have not assumed that consumption plans are absolutely continuous; consumption in gulps and in a singular way is also allowed.

Lastly, we wish to find which consumption plans can be financed with the initial wealth x available to the investor and the ability to invest in the market. In other words, we want to characterize the consumption plans C for which there exists a predictable, S-integrable process H such that the value process $V = (V_t)_{t\in[0,T]}$

$$V_t = x + \int_0^t H_u \, dS_u - C_t, \quad 0 \le t \le T \tag{5}$$

is nonnegative. Triples (x, H, C) whose associated value process is non-negative are considered admissible and the associated consumption plans are financiable with initial wealth x.

The optional decomposition theorem (see the papers by Föllmer and Kramkov [26], Kramkov [35], and El Karoui and Quenez [24]) allows us to give a dual description of this set. The important ingredient is an appropriate subset of the set of equivalent probability measures. If we restrict to positive pure investment strategies, then it turns out that the appropriate subset is the set of equivalent local martingale measures (Example 2.2 in [26] and Theorem 2.1 in [35].)

$$\mathcal{M} \triangleq \{\mathbb{Q} \sim \mathbb{P} \mid \; \text{the underlying assets } S \text{ are local martingales under } \mathbb{Q}\} \tag{6}$$

The essential statement of the theorem is that a process $V = (V_t)_{t\in[0,T]}$ is the value process of an admissible triple $(x, \bar{H}, \bar{C})$ if and only if V is a supermartingale with respect to all $\mathbb{Q} \in \mathcal{M}$. Furthermore, it can be shown (Proposition 4.3 [35]) for any $\mathbb{F}_t$-measurable random variable $Z \geq 0$, the process

$$V_t = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[Z \,|\, \mathbb{F}_t]$$

is a supermartingale for all $\mathbb{Q} \in \mathcal{M}$. In particular, if we let $Z = C_T$ then these results show that there exist an admissible strategy (V_0, H) such that

$$\int_0^t H_u \, dS_u + V_0 - C_t \;\geq\; \mathbb{E}_{\mathbb{Q}}\left[C_T \,|\, \mathbb{F}_t\right] - C_t \;\geq\; 0.$$

Here we may take any representative $\mathbb{Q} \in \mathcal{M}$. The essential result is that C is financiable with initial wealth V_0, where

$$V_0 = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[C_T].$$

In order to prepare for the dual approach to the optimization problem (1), we note that this budget constraint can also be expressed in terms of the density processes $Y^{\mathbb{Q}}$ of the equivalent local martingale measures $\mathbb{Q} \in \mathcal{M}$, namely, the set

$$\mathcal{Y}^{\mathcal{M}} \triangleq \left\{ Y^{\mathbb{Q}} \text{ s.t. } Y_t^{\mathbb{Q}} = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathbb{F}_t \right],\ 0 \leq t \leq T,\ \ \mathbb{Q} \in \mathcal{M} \right\}. \tag{7}$$

By assumption $\mathcal{M} \neq \emptyset$. For the conditional expectation we take the right continuous version, so that our processes $Y^{\mathbb{Q}}$ are RCLL. There is a natural bilinear pairing between positive processes Y, including the set $\mathcal{Y}^{\mathcal{M}}$, and the consumption plans $C \in \mathcal{C}$:

$$\mathbb{E}\langle C, Y \rangle \triangleq \mathbb{E} \int_0^T Y_t \, dC_t. \tag{8}$$

Note that the integral includes the endpoints – the same will be true of all integrals in this paper unless specifically stated otherwise. Thus $C_0 > 0$ corresponds to a gulp in consumption at time zero, i.e., a point mass at zero.

Furthermore, because all consumption plans are optional processes, Theorem 1.33 in [31] implies that

$$\mathbb{E}_{\mathbb{Q}}[C_T] = \mathbb{E}\left[\int_0^T Y_t^{\mathbb{Q}}\, dC_t\right] = \mathbb{E}\langle C, Y^{\mathbb{Q}}\rangle.$$

In other words, the set $\mathcal{C}(x)$ of consumption plans financiable with initial wealth x can be defined as

$$\mathcal{C}(x) \triangleq \{C \in \mathcal{C} \mid \mathbb{E}\langle C, Y^{\mathbb{Q}}\rangle \leq x,\ \forall Y^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}\}. \tag{9}$$

A class of utilities that makes sense for such general consumption plans was introduced by Hindy, Huang, and Kreps [29]. These authors also discuss important economic considerations and continuity properties satisfied by such utilities and we return to this discussion in the next section. There we will also discuss the choice of topology on the consumption space. For now, however, we focus on the definitions. In particular, the total utility $U(C)$ derived from a consumption plan $C \in \mathcal{C}$ is given by

$$U(C) \triangleq \int_0^T F(t, C_t)\, dt. \tag{10}$$

Of course, our optimization problem deals with expected total utilities $\mathbb{E}U(C)$. In this formulation, F is called the felicity function. In the remainder of this section we list the assumptions regarding F that we will assume throughout. We also give reasons for each assumption and references in the literature where these assumptions appear also.

The first assumption describes the appropriate concavity and boundary conditions. The second part of it is what are called the Inada conditions.

Assumption 2.1 *The felicity function $F(t, x)$ is jointly continuous in (t, x) and strictly concave, increasing in its second argument, and bounded from below.*

In addition, we assume that for each time t the derivative with respect to the second argument $F'(t,x) = \frac{\partial F(t,x)}{\partial x}$ exists, is continuous, and satisfies the Inada conditions:

$$F'(t,0) \triangleq \lim_{x \to 0} F'(t,x) = \infty, \tag{11}$$

$$F'(t,\infty) \triangleq \lim_{x \to \infty} F'(t,x) = 0 \tag{12}$$

The second assumption is required to show existence in the dual problem. Similar conditions have been imposed by Kramkov and Schachermayer [36], and Karatzas and Žitković [34], for example.

Assumption 2.2 *For each time t the asymptotic elasticity AE of the felicity function is strictly less than one:*

$$AE\, F(t,\cdot) \triangleq \limsup_{x \to \infty} \frac{F'(t,x)x}{F(t,x)} < 1.$$

This is the condition of reasonable asymptotic elasticity. The economic intuition behind this condition is that $F'(t,x)x/F(t,x)$ is the ratio (at time t) of the marginal utility $F'(t,x)$ and the average utility $F(t,x)/x$. Intuition about risk aversion would suggest that the marginal utility be much smaller than the average utility if x is large. It is related to, and in general weaker than, similar conditions imposed by Karatzas, Lehoczky, Shreve, and Xu [32]. The relationship between these conditions is described in [46].

Lastly, we need an assumption to guarantee the convexity of the set of dual variables. This assumption arises from the time-inhomogeneity of the utility function U and it guarantees that the set of utility gradients is convex.

Assumption 2.3 *The spatial derivative* $F_x = \frac{\partial F(t,x)}{\partial x}$ *of the felicity function is* C^1 *and* $F_{tx} = \frac{\partial F_x(t,x)}{\partial t}$ *is concave relative to* F_x*, i.e.* $F_{tx} \circ F_x^{-1}$ *is concave, in the second variable and for all* $t \in [0, T]$*, on the image* $F_x(t, \mathbb{R}_+)$*.*

For instance, if the felicity function is separable, $F(t, x) = \theta(t)\kappa(x)$, then F_{tx} is trivially concave relative to F_x.

3 Description of the Consumption Space

In their fundamental work, Hindy, Huang and Kreps [29] and Hindy and Huang [27], consider various topologies on the space of consumption plans $\mathcal{C}$ and their economic implications. In particular, they identify the Prohorov metric as a metric that captures the property of local substitution.

In the first part of this section we recall the definition of the Prohorov metric and illustrate the concept of local substitution with an example (Figure 1). We then show that the appropriate extension to the case of uncertainty has good analytic properties. Most importantly, we use the notion of convex compactness introduced by Žitković [52] and show that bounded subsets of the consumption space are convexly compact. This fact will allow us to later prove a certain duality result using a new version of the minimax theorem, proved in the appendix, for convexly compact spaces. In this section, we also show that the pairing $\mathbb{E}\langle C, Y\rangle$ of consumption plans and dual processes is lower-semicontinuous in C for all Y for which there exists a lower-semicontinuous process $\tilde{Y}$ such that their optional projections agree, ${}^{\circ}\tilde{Y} = {}^{\circ}Y$.

In the second part, we show that our preferences, defined in (10), are continuous in the chosen topology. This fact is important for several reasons. For one, it means that the utility from optimal consumption can be approximated by approximating the optimal plan itself. Hindy, Huang and Kreps [29] show that preferences based on the rate of consumption are in general not continuous in this topology.

This section discusses the topological and analytic properties of the space of consumption plans, independent of the financial market. We let the terminal time T to be possibly infinite for completeness. Also, here the finite time horizon case is actually the more complicated one because of the need to impose conditions on the convergence of consumption plans at the terminal time.

3.1 Definition of the Metric and Analytic Properties

The important economic consideration in the choice of topology is local substitution: plans differing by small shifts in the time of consumption should be close. Without uncertainty, a topology that satisfies this economic robustness is that induced by the Prohorov metric

$$\begin{aligned} d(C, C') \triangleq \inf\{\varepsilon > 0 \mid \ & |C(T) - C'(T)| < \varepsilon \text{ if } T < \infty, \quad \text{and} \\ & C(t-\varepsilon) - \varepsilon \leq C'(t) \leq C((t+\varepsilon) \wedge T) + \varepsilon \ \forall t \in [0, T)\}. \end{aligned} \tag{13}$$

Figure 1 illustrates the $\varepsilon-$neighborhood of a particular consumption plan. In particular, we see that discontinuities can mean that $|C(t) - C'(t)|$ is large for $t < T$ even for plans that are close in the Prohorov metric. Behavior at the terminal time T, however, is more controlled. One consequence of requiring the total mass of consumption plans (considered as measures on $[0, T]$) to be close is that convergence in this metric is equivalent to convergence in the weak topology (see for example [29]). We will use this fact repeatedly.

In the case of uncertainty, we extend this metric as follows,

$$d_{\mathcal{C}}(C, C') \triangleq \mathbb{E}\left[d(C, C') \wedge 1\right]. \tag{14}$$

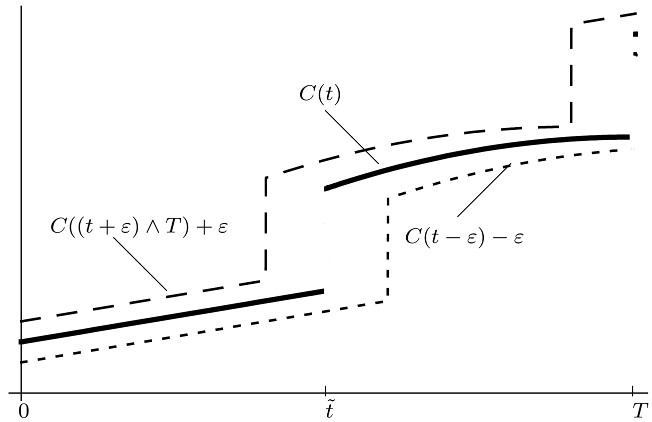

Figure 1: The ε-neighborhood of a consumption plan C is depicted above as the region bounded by the dotted and dashed lines. For $t < T$ this neighborhood is described by $C((t+\varepsilon)\wedge T)+\varepsilon$ and $C(t-\varepsilon)-\varepsilon$. For example, behavior at $\tilde{t}$ illustrates how the neighborhood behaves at points of discontinuity prior to T. The requirement that the cumulative consumption at the terminal time be close to $C(T)$ is indicated by the jump discontinuity of the dotted line at $t = T$.

We can think of convergence in this metric as equivalent to weak convergence in probability, and we will make this notion more precise in the proofs.

The topology induced by the metric $d_{\mathcal{C}}$ is equivalent to the family of norm topologies considered by Hindy and Huang [27] when restricted to the consumption set $\mathcal{C}$ (Hindy and Huang consider topologies on the commondity space, the linear span of $\mathcal{C}$.) In this topology, consumption of a known quantity at nearby predictable times are close substitutes (Proposition 2 in [27]). This topology is thus consistent with the stochastic analogue of local substitution.

In the rest of this subsection, we show that this topology also has good analytic properties. Eventually, we wish to use a minimax argument (Theorem 6.2) to prove that a desired dual relationship holds. In order to apply this theorem, however, we need enough convexly compact sets, lower-semicontinuity of the pairing $\mathbb{E}\langle\cdot,\cdot\rangle$, and (upper-semi)continuity of preferences. We treat the first two properties first, and deal with the continuity of preferences in Section 3.2.

The idea for convex compactness, a generalized notion of compactness, is due to Žitković. In [52] he shows that convex compactness can be a useful substitute for regular compactness in many problems of optimization and mathematical economics. We will show in Theorem 3.3 that sets of consumption plans that are bounded in probability are also convexly compact. Before proceeding with the proof, we mention the key concepts from Žitković [52].

Definition 3.1 (Definition 2.1 in [52]) *A convex subset E of a topological vector space X is* ***convexly compact*** *if for any non-empty set A and any family $(F_\alpha)_{\alpha\in A}$ of closed and convex subsets of E for which*

$$\bigcap_{\alpha\in D} F_\alpha \neq \emptyset \text{ for any finite } D \subset A,\, D \neq \emptyset$$

we also have that

$$\bigcap_{\alpha\in A} F_\alpha \neq \emptyset$$

In the following we let $\mathrm{Fin}(A)$ denote the set of finite subsets of the set A. By analogy with the usual notion of compactness, convex compactness can equivalently be defined in terms of the convergence of nets.

Proposition 3.2 (Proposition 2.4 in [52]) *A subset E of a topological vector space is convexly compact if and only if for every net $(x_\alpha)_{\alpha\in A}$ in the set E there exists a subnet $(y_\beta)_{\beta\in B}$ of convex combinations of $(x_\alpha)_{\alpha\in A}$ and a $y \in E$ such that $y_\beta \to y$.*

A net $(y_\beta)_{\beta\in B}$ is called a **subnet of convex combinations** of $(x_\alpha)_{\alpha\in A}$ if there exists a mapping D of B into the set of finite subsets of A, i.e. $D : B \to \mathrm{Fin}(A)$, such that

1. $y_\beta \in \mathrm{conv}\{x_\alpha \,|\, \alpha \in D(\beta)\}$ for each $\beta \in B$
2. For each $\alpha \in A$ there exists $\beta \in B$ such that $\alpha' \succeq \alpha$ for each $\alpha' \in \bigcup_{\beta'\succeq\beta} D(\beta')$

Remark: It is well known that in a metric space, sequential compactness is equivalent to compactness. However, we were unable to prove a similar statement for convex compactness. This is because for a general metric there is no relationship between the distance of points and distance between convex combinations of points. In particular, ε-neighborhoods are not necessarily convex. Thus in the following we will use the notion of a subnet of convex combinations, despite working over a metric space.

Theorem 3.3 (Convex compactness) *Let E be a convex subset of the consumption space $\mathcal{C}$ that is closed in the topology generated by the metric $d_{\mathcal{C}}$. Then E is convexly compact if and only if the set $E_T := \{C_T$ s.t. $C \in E\}$ is bounded in probability. If $T = \infty$ then we let $C_T := \lim_{t\to T} C_t$.*

Recall that a set $A \subset \mathbb{L}^0_+$ is bounded in probability if for all $\varepsilon > 0$ there exists a constant M such that $\mathbb{P}(f > M) < \varepsilon$ for all $f \in A$.

Remark: We divide the proof into two parts. First we show that sets bounded in probability are convexly compact. An important idea for this step is to use a (generic) strictly concave functional on the space of consumption plans in order to establish convergence of a net of convex combinations. We have adapted this technique from the papers of Delbaen and Schachermayer (Lemma A1.1 in [19],) and Žitković (Theorem 3.1 in [52].) The proof of the second part, that convexly compact sets are bounded in probability, is only a slight modification of what appears in [52].

PROOF : ($\Leftarrow$) Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed, convex subsets of E having the finite intersection property. The finite subsets $D \in \mathrm{Fin}(A)$ form a directed set with the order

$$D_1 \succeq D_2 \quad \text{if and only if} \quad D_1 \supseteq D_2.$$

With this directed set we associate a collection of chosen elements from the sets F_α. More specifically, for each $D \in \mathrm{Fin}(A)$, fix some arbitrary $X_D \in \bigcap_{\alpha\in D} F_\alpha$ (if $D = \emptyset$, then let $X_D \in E$). Observe that if $C_D \in \mathrm{conv}(X_{D'} \mid D' \succeq D)$, then, because each set F_α is convex, $C_D \in \bigcap_{\alpha\in D} F_\alpha$. We will show that we can pick these convex combinations C_D so that the resulting net converges to a consumption plan $C \in \bigcap_{\alpha\in A} F_\alpha$.

The first step is to consider, for any $D \in \mathrm{Fin}(A)$ the supremum

$$s_D \triangleq \sup\left\{ \mathbb{E}\int_0^T \phi(C_t)\, d\mu(t) \,\middle|\, C \in \mathrm{conv}(X_{D'} \mid D' \succeq D) \right\},$$

where ϕ is the strictly concave, bounded function $\phi(x) := 1 - e^{-x}$, and μ is a probability measure with full support on the interval $[0, T]$ and with a point mass at T if $T < \infty$. These choices ensure that the supremum is uniformly bounded by some constant $W \in \mathbb{R}_+$. In particular, because $s_{D_1} \geq s_{D_2}$ if $D_2 \succeq D_1$, the net $(s_D)_{D\in\mathrm{Fin}(A)}$ is bounded, monotone decreasing, and hence convergent to some $s_\infty \in [0, W]$.

In order to construct the appropriate net of consumption plans, we pick, for each $D \in \mathrm{Fin}(A)$, some $C_D \in \mathrm{conv}(X_{D'} \,|\, D' \succeq D)$ such that

$$\mathbb{E}\int_0^T \phi(C_D)\,dt \geq s_D - \frac{1}{\#D}$$

holds.

At this point we also make a further comment about the function ϕ: for any given constant $K > 0$, there exists a constant $\varepsilon > 0$ such that

$$\phi\left(\frac{x_1+x_2}{2}\right) \geq \frac{1}{2}\phi(x_1) + \frac{1}{2}\phi(x_2) + \varepsilon$$

holds for all x_1, x_2 such that $\min(x_1, x_2) \leq K$ and $|x_1 - x_2| > 1/K$. Indeed, for such $x_1 < x_2$

$$\begin{aligned}\phi\left(\frac{x_1+x_2}{2}\right) - \frac{1}{2}\left[\phi(x_1)+\phi(x_2)\right] &= \frac{1}{2}e^{-x_1}\left[\left(1-e^{-\frac{x_2-x_1}{2}}\right)\right.\\ &\quad \left.-e^{-\frac{x_2-x_1}{2}}\left(1-e^{-x_2-x_1 2}\right)\right]\\ &\geq \frac{1}{2}e^{-x_1}\left(1-e^{-\frac{x_2-x_1}{2}}\right)\left(1-e^{-\frac{x_2-x_1}{2}}\right)\\ &\geq \frac{1}{2}e^{-K}\left(1-e^{-\frac{1}{2K}}\right)^2.\end{aligned}$$

We can integrate this result with respect to the product measure $\bar{\mathbb{P}} := \mathbb{P}\otimes\mu$ in order to obtain the estimate

$$\begin{aligned}\mathbb{E}\int_0^T \phi\left(\frac{C_{D_1}(t)+C_{D_2}(t)}{2}\right)d\mu(t) \geq\ & \frac{1}{2}\mathbb{E}\int_0^T \phi(C_{D_1})\,d\mu(t) + \frac{1}{2}\mathbb{E}\int_0^T \phi(C_{D_2})\,d\mu(t)\\ &+ \varepsilon\bar{\mathbb{P}}\left[|C_{D_1}-C_{D_2}| > 1/K, \min(C_{D_1}, C_{D_2}) \leq K\right]\end{aligned}$$

Observe that because $\frac{1}{2}(C_{D_1}+C_{D_2}) \in \mathrm{conv}(X_{D'} \,|\, D' \succeq D_1 \cap D_2)$ we can obtain an estimate for the probability

$$\begin{aligned}\varepsilon\bar{\mathbb{P}}\left[\,|C_{D_1}-C_{D_2}| > 1/K, \min(C_{D_1}, C_{D_2}) \leq K\,\right] \leq & s_{D_1\cap D_2} - \frac{1}{2}(s_{D_1}+s_{D_2})\\ &+\frac{1}{2}\left(\frac{1}{\#D_1}+\frac{1}{\#D_2}\right).\end{aligned}$$

Furthermore, because the net $(s_D)_{D\in \mathrm{Fin}(A)}$ is convergent, for any given $\kappa > 0$ there exists a set $D(\kappa) \in \mathrm{Fin}(A)$ such that

$$s_\infty + \kappa \geq s_D \geq s_\infty \quad \forall D \succeq D(\kappa).$$

If we choose D_1 and D_2 such that $D_i \succeq D(\kappa)$ for $i = 1, 2$ and such that $\#D_1, \#D_2 \geq \kappa^{-1}$ then we have that,

$$s_{D_1 \cap D_2} - \frac{1}{2}(s_{D_1} + s_{D_2}) + \frac{1}{2}\left(\frac{1}{\#D_1} + \frac{1}{\#D_2}\right) \leq 2\kappa.$$

In addition, because the set E_T is bounded in probability, for any given $\kappa > 0$ there exists a constant M such that $\mathbb{P}(C_T \geq M) < \kappa$ for all $C \in E$. Because consumption plans are increasing, this implies that

$$\bar{\mathbb{P}}\left[\min(C_{D_1}, C_{D_2}) \geq M\right] < \kappa.$$

If we choose $K > M$, these two estimates together show that

$$\begin{aligned}\bar{\mathbb{P}}\left[|C_{D_1} - C_{D_2}| > 1/K\right] \leq & \bar{\mathbb{P}}\left[|C_{D_1} - C_{D_2}| > 1/K, \min(C_{D_1}, C_{D_2}) \leq K\right] \\ & + \bar{\mathbb{P}}\left[\min(C_{D_1}, C_{D_2}) \geq K\right] \\ \leq & 3\kappa\end{aligned}$$

In particular, our sequence is Cauchy in probability:

$$\bar{\mathbb{P}}\left[|C_{D_1} - C_{D_2}| > \lambda\right] \to 0 \ \ \forall \lambda > 0$$

The next step is to show that convergence in (product) probability implies convergence in the topology generated by the metric: $d_{\mathcal{C}}(\cdot,\cdot) = \mathbb{E}(1 \wedge d(\cdot,\cdot))$.

The first step is to calculate,

$$\begin{aligned}\mathbb{E}[1 \wedge d(C_{D_1}, C_{D_2})] =& \int_{\{d(C_{D_1},C_{D_2})>\lambda\}} 1 \wedge d(C_{D_1}, C_{D_2})\, d\mathbb{P} \\ &+ \int_{\{d(C_{D_1},C_{D_2})\leq\lambda\}} 1 \wedge d(C_{D_1}, C_{D_2})\, d\mathbb{P} \\ \leq& \mathbb{P}[d(C_{D_1}, C_{D_2}) > \lambda] + \lambda\end{aligned}$$

We want to show that the probability on the right hand side converges to zero with respect to the net indexed by $D \in \mathrm{Fin}(A)$. This fact will show that $(C_D)_{D\in\mathrm{Fin}(A)}$ is a Cauchy net with respect to $d_{\mathcal{C}}$.

For a finite time horizon, we need to consider two possible sources of non-convergence: failure to converge at the terminal time T in probability, and failure to converge for some $t \in [0, T)$ with respect to the metric $d_{\mathcal{C}}$.

Convergence at terminal time points is a direct consequence of convergence in the product topology. In fact, for a finite time-horizon (this case can be ignored for $T = \infty$) the measure μ has a point mass at $t = T$, thus

$$\mathbb{P}\left[|C_{D_1}(T) - C_{D_2}(T)| > \lambda\right] \to 0 \quad \forall\, \lambda > 0.$$

In particular, for all $\varepsilon > 0$ there exists a set D such that if $D_1 \supseteq D$ and $D_2 \supseteq D$ then $\mathbb{P}[|C_{D_1}(T) - C_{D_2}(T)| > \lambda] < \varepsilon$.

For $t \in [0, T)$ we proceed by contradiction. To simplify notation, we assume for the remainder, without loss of generality, that $|C_{D_1}(T) - C_{D_2}(T)| \leq \lambda$. Thus, if we assume, to reach a contradiction, that for a fixed $\lambda > 0$ and $\omega \in \Omega$ the estimate $d(C_{D_1}(\omega), C_{D_2}(\omega)) > 2\lambda$ is true, then there exists $t \in [0, T)$ such that at least one of the following holds:

$$C_{D_1}(t) > C_{D_2}((t + 2\lambda) \wedge T) + 2\lambda,$$

or

$$C_{D_1}(t) < C_{D_2}((t - 2\lambda) \vee 0) - 2\lambda.$$

Without loss of generality, we may assume that the first statement holds. In particular, because consumption plans are increasing, we have the stronger statement:

$$C_{D_1}(t) > C_{D_2}((t + \varepsilon) \wedge T) + 2\lambda \ \ \forall\, \varepsilon \in (0, 2\lambda).$$

Combined with the fact that C_{D_1} is right-continuous, we deduce from the triangle inequality that there exists a $\delta(\omega) > 0$ such that

$$\text{if } s \in [t, t + \delta(\omega)] \ \text{ then } \ |C_{D_1}(s)(\omega) - C_{D_2}(s)(\omega)| > \lambda.$$

In particular,

$$\int_0^T 1_{\{|C_{D_1}(\omega) - C_{D_2}(\omega)| > \lambda\}}\, d\mu(t) \geq \mu([t, t + \delta(\omega)]) > 0.$$

The Fubini-Tonelli theorems imply that

$$\bar{\mathbb{P}}[|C_{D_1} - C_{D_2}| > \lambda] > \mathbb{E}[\mu([t, t + \delta])].$$

Taking limits on the left hand side (there is not limit to take on the right), shows that $\mathbb{E}[\mu([t, t + \delta]] = 0$. In particular, the result $\mu[t, t + \delta(\omega)] > 0$ holds only on a set of $\mathbb{P}$-measure 0. In particular,

$$\mathbb{P}(d(C_{D_1}, C_{D_2}) > 2\lambda) \to 0 \ \forall\ 2\lambda > 0.$$

Consequently, there exists $D(\lambda) \in \text{Fin}(A)$ such that ,

$$\mathbb{E}(1 \wedge d(C_{D_1}, C_{D_2})) \leq 2\lambda \ \ \forall D_1, D_2 \succeq D(\lambda).$$

We thus have that $(C_D)_{D\in \text{Fin}(A)}$ is a Cauchy net in the space of consumption plans $\mathcal{C}$ endowed with the metric $d_{\mathcal{C}}$. In addition, we have that the space $\mathcal{C}$ is complete. To show this, first observe that for a fixed $\omega \in \Omega$ the space $\mathcal{C}(\omega) = \{C(\omega)\,|\, C\in\mathcal{C}\}$ is complete with respect to the Prohorov metric $d(\cdot,\cdot)$. In addition, we can use the techniques in the proof of continuity of preferences (Lemma 3.5) to show that if C_n is a Cauchy sequence of consumption plans, then there exists a subsequence C_{n_k} such that $C_{n_k}(\omega)$ are Cauchy for almost all $\omega\in\Omega$. Lastly we note that to show completeness, it is sufficient to consider sequences because $\mathcal{C}$ is a metric space.

In particular, there exists a consumption plan $C_\infty \in E$ such that $C_D \longrightarrow C_\infty$ in the metric $d_{\mathcal{C}}$. Furthermore, since

$$C_{D'} \in \bigcap_{\alpha\in D} F_\alpha \quad \forall D' \supseteq D$$

and because each set F_α is closed, we must have that the limit $C_\infty \in \bigcap_{\alpha\in D} F_\alpha$ for all $D\,\in$Fin(A). In fact, $C_\infty \in \bigcap_{\alpha\in A} F_\alpha$ and so this intersection is not empty.

($\Rightarrow$) It remains to show that if the set E is convexly compact, then the end values E_T are necessarily bounded in probability. Suppose that $E_T \subset \mathbb{L}^0_+$ is not bounded in probability. Then there exists an $\varepsilon \in (0,1)$ and a sequence $\{C^n\}_{n\in\mathbb{N}} \subset E$ such that

$$\mathbb{P}(C^n_T \geq n) > \varepsilon \;\; \forall n\in\mathbb{N}.$$

Because the set E is convexly compact, there exists a subnet $\{C^\beta\}_{\beta\in B}$ of convex combinations of $\{C^n\}_{n\in\mathbb{N}}$ that converges to some $C\in E$. In particular,

$$\forall n\in\mathbb{N}\; \exists \beta_n \;\; \text{s.t.} \;\; C^{\beta'} \in \text{conv}(C^m,\, m\geq n) \quad \forall\, \beta' \succeq \beta_n.$$

Lemma 9.8.6 (p. 205) in [20] allows to construct an estimate for these convex combinations. In particular, we may write $C_T^\beta = \sum_{j=n}^m \lambda_j C_T^j$ where

$$\mathbb{P}\left(\lambda_j C_T^j \geq \lambda_j n\right) > \varepsilon$$

based on the construction of the sequence $(C_T^n)_{n\in\mathbb{N}}$. The lemma then shows that for any $0 < \eta < 1$, $\mathbb{P}\left(C_T^\beta \geq n\eta\varepsilon\right) \geq \frac{\varepsilon(1-\eta)}{1-\eta\varepsilon}$. Letting $\eta = 1/2$ we obtain,

$$\mathbb{P}\left(C_T^\beta \geq \frac{n\varepsilon}{2}\right) \geq \frac{\varepsilon}{2}.$$

Because convergence in the topology generated by the metric $d_{\mathcal{C}}(\cdot,\cdot)$ implies convergence of the terminal values, then for large enough β

$$\mathbb{P}\left(|C_T - C_T^\beta| > \frac{n\varepsilon}{4}\right) \leq \frac{\varepsilon}{4}.$$

Combining these two estimates, we can show that

$$\mathbb{P}\left(C_T \geq \frac{n\varepsilon}{4}\right) \geq \mathbb{P}\left(C_T^\beta \geq \frac{n\varepsilon}{4}\right) - \mathbb{P}\left(|C_T - C_T^\beta| > \frac{n\varepsilon}{4}\right) \geq \frac{\varepsilon}{4}.$$

In particular, $\mathbb{P}(C_T = +\infty) > 0$ and $C \notin E$, a contradiction. □

We also have the following regularity result.

Lemma 3.4 *Let Y be a non-negative and (pathwise) lower-semicontinuous stochastic process. Then the mapping*

$$C \mapsto \mathbb{E}\langle C, Y\rangle$$

is lower-semicontinuous with respect to $d_{\mathcal{C}}$. In fact, it is enough to assume that there exists a process (not necessarily adapted) $\tilde{Y} \geq 0$ with lower-semicontinuous paths such that the optional projections agree: ${}^\circ Y = {}^\circ\tilde{Y}$. The result also holds for all non-negative supermartingales Y.

Proof : The proof has three main parts. In the first we show that the mapping is continuous if the paths of Y are continuous. We then extend the result for lower-semicontinuous Y using an approximating family of functions. In the second part we prove that if Y is a nonnegative, RCLL, supermartingale of class (D) then it can be written as the optional projection of a lower-semicontinuous process. In the third part we use a localization procedure to extend the result for all nonnegative supermartingales Y.

(i) Because $\mathcal{C}$ is a metric space, it suffices to show that if $(C_n)_{n\in\mathbb{N}}$ is a sequence of consumption plans converging to some $C \in \mathcal{C}$, then

$$\mathbb{E}\int_0^T Y_t\, dC(t) \leq \liminf_{n\to\infty} \mathbb{E}\int_0^T Y_t\, dC_n(t). \tag{15}$$

The first step in the proof is to pass to the subsequence $(C'_n)_{n\in\mathbb{N}}$ that achieves this limit inferior. In other words, such that

$$\liminf_{n\to\infty} \mathbb{E}\int_0^T Y_t\, dC_n(t) = \lim_{n\to\infty} \mathbb{E}\int_0^T Y_t\, dC'_n(t)$$

holds.

Because this subsequence is also convergent to C in the topology induced by the metric $d_{\mathcal{C}}$, we can use the arguments in Lemma 3.5 to extract a subsequence $(C'_{n_k})_{k\in\mathbb{N}}$ such that C'_{n_k} converges to C in the weak topology on $[0,T]$, almost surely. If Y is pathwise continuous, then Fatou's lemma and the definition of weak convergence imply

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\int_0^T Y_t\, dC'_n(t) &= \lim_{k\to\infty} \mathbb{E}\int_0^T Y_t\, dC'_{n_k}(t)\\ &\geq \mathbb{E}\liminf_{k\to\infty}\int_0^T Y_t\, dC'_{n_k}(t)\\ &= \mathbb{E}\int_0^T Y_t\, dC(t).\end{aligned}$$

Thus the result is true for pathwise continuous processes Y. For the more general case, we first make the remark that, because each consumption plan C_n is optional,

$$\mathbb{E}\int_0^T Y_t\, dC_n(t) = \mathbb{E}\int_0^T {}^{\circ}Y_t\, dC_n(t).$$

The assumptions of the lemma imply that it is sufficient to prove relation (15) for pathwise lower-semicontinuous processes. Lower-semicountinuous functions that are bounded from below, however, can be written as pointwise limits of increasing families of Lipschitz continuous functions (for example theorem 3.13 in [1]). For each $\omega \in \Omega$ denote this approximating sequence by $(Y_\alpha(\omega))_{\alpha\in\mathbb{N}}$. Alternatively, we may consider $(Y_\alpha)_{\alpha\in\mathbb{N}}$ as a family of Lipschitz continuous processes, converging pointwise to the process Y. These processes Y_α are $\mathbb{F}_T \otimes \mathcal{B}[0,T]$-measurable, but not necessarily adapted.

We make some observations about this approximating family. First, because $Y_\alpha \leq Y$,

$$\mathbb{E}\int_0^T Y_\alpha(t)\, dC_n(t) \leq \mathbb{E}\int_0^T Y(t)\, dC_n(t) \quad \forall n \in \mathbb{N}.$$

Also, because each Y_α is continuous,

$$\mathbb{E}\int_0^T Y_\alpha(t) dC(t) \leq \liminf_{n\to\infty} \mathbb{E}\int_0^T Y_\alpha(t) dC_n(t).$$

Furthermore, monotone convergence implies that

$$\lim_{\alpha\to\infty} \mathbb{E}\int_0^T Y_\alpha(t)\, dC(t) = \mathbb{E}\int_0^T Y(t) dC(t).$$

Combining these observations finishes the proof:

$$\begin{aligned}\mathbb{E}\int_0^T Y(t)\, dC(t) = \lim_{\alpha\to\infty} \mathbb{E}\int_0^T Y_\alpha(t)\, dC(t) &\leq \lim_{\alpha\to\infty} \liminf_{n\to\infty} \mathbb{E}\int_0^T Y_\alpha(t)\, dC_n(t) \\ &\leq \liminf_{n\to\infty} \mathbb{E}\int_0^T Y(t)\, dC_n(t).\end{aligned}$$

Because the pairing is stable with respect to optional projections, $\mathbb{E}\langle C, Y\rangle = \mathbb{E}\langle C, {}^{\circ}Y\rangle$, we only need to require that Y is the optional projection of something lower-semicontinuous in order for the conclusion of the lemma to hold.

(ii) Next we show that if Z is a nonnegative RCLL supermartingale of class (D), then there exists a lower-semicontinuous process $(\zeta_t)_{t\in[0,T]}$ (not necessarily adapted) such that

$$Z_t = \mathbb{E}[\zeta_t \,|\, \mathbb{F}_t] \text{ a.s. for } t \in [0, T]$$

Furthermore, because Z is right-continuous, we may write $Z = {}^{\circ}\zeta$.

Theorem 8, chapter VII [22] states that if Z is a RCLL, nonnegative, supermartingale of class (D), then there exists a predictable, integrable, increasing (so right continuous by definition) process A indexed by $[0, \infty]$ which is zero at $t = 0$ but may have a jump at infinity, such that

$$Z_t = \mathbb{E}[A_\infty - A_t \,|\, \mathbb{F}_t] \text{ a.s. for } t \in [0, \infty].$$

For a finite time-horizon, a similar statement holds with $Z_T = 0$. We can always add in Z_T without affecting the continuity properties. Letting $\zeta_t = A_\infty - A_t$ proves the first statement. Because ζ is right-continuous and decreasing, it must be lower-semicontinuous. Because ζ is right-continuous, ${}^{\circ}\zeta$ is right-continuous also (Theorem 47 VI in [22]). In particular, the identity $Z_S = \mathbb{E}[\zeta_S \,|\, \mathbb{F}_S]$ holds almost surely for all finite stopping times S, i.e. $Z = {}^{\circ}\zeta$.

Consequently, the mapping $C \mapsto \mathbb{E}\langle C, Y\rangle$ is lower-semicontinuous for all nonnegative RCLL supermartingales of class (D).

(iii) As a final step, we prove that the mapping is lower-semicontinuous for all nonnegative, RCLL supermartingales Y (for a finite time horizon). We start by observing that there exists a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times such that $\lim_{n\to\infty}\tau_n = T$-a.e. and such that the stopped process Y^{τ_n} is of class (D) for each n. For this step, we argue as in the proof of Corollary 5.5 (by letting $\tau_n \triangleq n \wedge \inf\{t > 0 \,|\, Y_t \geq n\}$).

Now suppose that $(C_n)_{n\in\mathbb{N}}$ is a sequence of consumption plans converging to some $C \in \mathcal{C}$, then

$$\liminf_{n\to\infty} \mathbb{E}\langle C^n, Y\rangle \geq \liminf_{n\to\infty} \mathbb{E}\langle (C^n)^{\tau_m}, Y\rangle \geq \mathbb{E}\langle C^{\tau_m}, Y^{\tau_m}\rangle.$$

The last inequality follows from the results proven in parts (i) and (ii). Taking limits as m tends to infinity, we obtain

$$\lim_{m\to\infty} \mathbb{E}\langle C^{\tau_m}, Y^{\tau_m}\rangle = \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_m} Y_t dC(t) = \mathbb{E}\int_0^T Y_t dC_t = \mathbb{E}\langle C, Y\rangle.$$

In the above we have used the monotone convergence theorem to justify the operations with limits. Putting these two parts together, we obtain the desired result:

$$\liminf_{n\to\infty} \mathbb{E}\langle C^n, Y\rangle \geq \mathbb{E}\langle C, Y\rangle.$$

□

In the remainder of this subsection we discuss the relationship of these results regarding the mapping $C \mapsto \mathbb{E}\langle C, Y\rangle$ with the literature on equilibrium pricing. Intuitively, prices should not jump in the absence of a surprise. This is because if, for example, a price jump upwards is known in advance, there will be some agents willing to withhold consumption today, and to sell later at the higher price. Thus in an equilibrium the price today should be very close to the price tomorrow. This intuition relies on the assumption that there are agents who are willing to delay consumption. The idea is thus related to the notion that consumption at nearby dates should be close substitutes.

Hindy and Huang [27] define an information surprise as either a nonpredictable stopping time, or a predictable stopping time τ such that the filtration $\mathbb{F}$ is not quasi-leftcontinuous at τ. A filtration $\mathbb{F}$ is **quasi-leftcontinuous** at a predictable time τ if $\mathbb{F}_\tau = \bigvee_n \mathbb{F}_{\tau_n} (= \mathbb{F}_{\tau-})$ where (τ_n) is an announcing sequence for τ. If the filtration is quasi-leftcontinuous, then martingales do not jump at predictable times.

To see what the implications are for supermartingales, we look at Theorem 14, chapter VI in [22]: if X is a right-continuous supermartingale that is closed on the right by X_∞ (if X is nonnegative then it is closed by 0) and if S and T are two predictable stopping times such that $S \leq T$ then X_{S^-} and X_{T^-} are integrable and

$$X_{S^-} \geq \mathbb{E}[X_{T^-} \,|\, \mathbb{F}_{S^-}] \geq \mathbb{E}[X_T \,|\, \mathbb{F}_{S^-}] \text{ a.s.}$$

(with equality if X_∞ closes X as a martingale.) In particular, if we let $S = T$ then the statement becomes

$$X_{S^-} \geq \mathbb{E}[X_S \,|\, \mathbb{F}_{S^-}] \text{ a.s.}$$

Thus, if a filtration is quasi-leftcontinuous, then supermartingales have only downward jumps at predictable times. In general, supermartingales jump only downward, except possibly at information surprises. Thus in our setting, where the dual processes Y are supermartingales, lower-semicontinuity of the pairing $C \mapsto \mathbb{E}\langle C, Y\rangle$ is the best that we can hope for. The pairing is not in general continuous.

However, if we know that the process Y defines an equilibrium price, then more can be said. In particular, returning to the first step in the proof of Lemma 3.4 we note that if Y is continuous and bounded then the dominated convergence theorem can be used to show that the pairing is continuous. Viewed as a price, this makes sense. Consumption plans that are close in the agents' preferences should have similar prices.

The existence of continuous equilibrium prices for Hindy-Huang-Kreps utilities have been established for stochastic pure exchange economies by Bank and Riedel [8], Martins-da-Rocha and Riedel [37] and Martins-da-Rocha and Riedel [38]. In these papers, the construction of equilibrium prices rests on economic considerations which help prove continuity. In particular, Theorem 2 in [8] states that, under the assumption of quasi-leftcontinuity, every equilibrium price functional is continuous in the topology generated by $d_{\mathcal{C}}$. The continuity results do not apply in our setting, because, for example, the processes $Y^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}$ are not necessarily bounded. Note, however, that in our setting Y cannot be directly interpreted as a price either. Also, in our result, quasi-leftcontinuity is not needed.

3.2 Preferences

We have made a point of using a topology with desired economic robustness properties, choosing the topology first and then proving that the consumption space has good analytic properties. In this section we show that preferences, as specified by our expected utility functional (10), are continuous as well.

In their paper, Hindy, Huang, and Kreps [29] show that standard time-additive utilities of the von Neumann-Morgenstern form

$$\mathbb{E}U(C) = \mathbb{E}\int_0^T F(t, \dot{C}(t))\, dt$$

are continuous with respect to $d_{\mathcal{C}}$ if and only if the felicity function $F(t, \cdot)$ is linear. In that case the utility is not strictly concave, or, in other words, the agent is risk neutral. Thus these utilities are not compatible with our framework. In particular, while consumption in gulps can be approximated by consumption in rates, the same is not necessarily true of the utilities that the consumption plans generate.

Note, however, that lack of continuity does not in general prevent one from solving the utility optimization problem. For instance, Karatzas and Žitković [34] use the duality technique to solve the utility optimization problem for the above von Neumann-Morgenstern preferences. Their method of proof is an extension of the work of Kramkov and Schachermayer [36] to processes, and uses a product topology on the space $\Omega \times [0, T]$. For the purposes of applying a minimax theorem, they endow bounded subsets of this space with the L^∞ topology. In that case, the preferences are, however, upper-semicontinuous. This fact is enough to prove that a duality relationship exists.

Lemma 3.5 *In the topology generated by the metric $d_{\mathcal{C}}$, the expected utility functional*

$$C \mapsto \mathbb{E}U(C) = \mathbb{E}\int_0^T F(t, C_t) dt$$

is continuous on all subsets of consumption plans $E \subseteq \mathcal{C}$ such that the set $\{U(C)\}_{C \in E}$ is uniformly integrable.

Remark: If the set E is bounded in the sense that there exists a constant M such that $C_T \leq M$ a.s. $\omega \in \Omega$ for all $C \in E$ and if $T < \infty$, then $\{U(C)\}_{C \in E}$ is uniformly integrable. These bounded sets will be important later, when we apply the minimax theorem. Recall from Theorem 3.3 that they are also convexly compact.

Proof : First, we prove the deterministic result, and then extend the results via a diagonalization argument. Let $(C_k)_{k \in \mathbb{N}} \in \mathcal{C}_M$ be a sequence of (deterministic) consumption plans converging to C in the Prohorov metric, or equivalently, in the weak topology. Recall that weak convergence also implies that C_k converges pointwise at all continuity points of $C(\cdot)$ and at the terminal time T. Furthermore, for large enough k we may assume that $C_k(t) \leq C_T + 1$, and hence also that $-F(t, C_k(t))$ is bounded from below. We can thus apply Fatou's lemma;

$$\begin{aligned}
\liminf_{k \to \infty}(-U(C_k)) &= \liminf_{k \to \infty} \int_0^T -F(t, C_k(t))dt \\
&\geq \int_0^T -F(t, \limsup_{k \to \infty} C_k(t))dt \\
&= \int_0^T -F(t, C(t))\, dt = -U(C).
\end{aligned}$$

The second to last equality follows because the set of discontinuities of C has Lebesgue measure zero. We thus obtain the desired result:

$$\limsup_{k \to \infty} U(C_k) \leq U(C)$$

Applying the same reasoning to $U(C_k)$, which is nonnegative and hence bounded from below, gives $\liminf_{k \to \infty} U(C_k) \geq U(C)$. Combining results, we have that the limit exists and that it is equal to $U(C)$.

We are now ready to treat the case with uncertainty.

Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of consumption plans that is convergent, with respect to the metric $d_{\mathcal{C}}$, to some consumption plan C. Because convergence in $\mathbb{L}^1$ implies convergence in probability, we have that

$$\lim_{n\to\infty} \mathbb{P}(d(C_n, C) > \varepsilon) = 0 \quad \forall \varepsilon > 0.$$

In particular, the sequence is Cauchy in probability with respect to the Prohorov metric $d(\cdot,\cdot)$,

$$\lim_{n,m\to\infty} \mathbb{P}(d(C_n, C_m) > \varepsilon) = 0 \quad \forall \varepsilon > 0.$$

Thus there exists a $\mathbb{P}$-a.e. convergent subsequence. In fact, we may choose $n_k \in \mathbb{N}$ such that the sets $E_{n_k} := \{d(C_{n_{k+1}}, C_{n_k}) > \frac{1}{2^k}\}$ have measure $\mathbb{P}(E_{n_k}) < \frac{1}{2^k}$. Then the set $F_m := \bigcup_{k=m}^{\infty} E_{n_k}$ has measure $\mathbb{P}(F_m) \leq \frac{1}{2^{m-1}}$. In particular, for all $\omega \in F_m^c$ and for all $j \geq l \geq N \geq m$ we have

$$d(C_{n_j}, C_{n_l}) \leq \sum_{r=l}^{j-1} d(C_{n_{r+1}}, C_{n_r}) \leq \sum_{r=N}^{\infty} d(C_{n_{r+1}}, C_{n_r}) \leq \frac{1}{2^{N-1}}.$$

Hence the subsequence $(C_{n_k})_{k\in\mathbb{N}}$ is Cauchy sequence for all $\omega \in \cup_{m=1}^{\infty} F_m^c = (\cap_{m=1}^{\infty} F_m)^c$. This set has measure 1, so in particular, the subsequence is convergent a.s. to a consumption plan C. This convergence is with respect to the Prohorov metric. For each such ω continuity of the deterministic function $U(\cdot)$ implies that

$$\lim_{k\to\infty} U(C_{n_k}) = U(C).$$

Furthermore, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(C, C') < \delta$ then $|U(C) - U(C')| < \varepsilon$. Because the sequence $(C_n)_{n\in\mathbb{N}}$ is Cauchy in probability, for all such ε and $\delta(\varepsilon)$ there exists an N_δ such that for all $n, m > N_\delta$

$$\mathbb{P}(d(C_n, C_m) \geq \delta) < \varepsilon.$$

We may also pick K_ε such that for all $k \geq K_\varepsilon$,

$$|U(C_{n_k}) - U(C)| < \varepsilon.$$

Now fixing m, n_k large enough, we have that for $\omega \in \{d(C_m, C_{n_k}) < \delta\}$,

$$|U(C_m) - U(C)| \leq |U(C_m) - U(C_{n_k})| + |U(C_{n_k}) - U(C)| \leq 2\varepsilon$$

Thus

$$\mathbb{P}(|U(C_m) - U(C)| > 2\varepsilon) < \varepsilon \quad \forall m \geq N_\varepsilon$$

and we have that the sequence $\{U(C_n)\}_{n\in\mathbb{N}}$ converges in probability to $U(C)$. By assumption, the family $(U(C_n))_{n\in\mathbb{N}}$ is uniformly integrable. Convergence in probability thus implies convergence in L^1. $\square$

4 Duality Theory: Deterministic Preliminaries

At the heart of the duality approach is a family of upper bounds to the primal problem, in our case the indirect utility $u(x)$, that allow us to turn the maximization problem into a question of minimization over a different set of variables, the dual variables. In this section we begin the discussion of the appropriate dual problem and dual variables in a deterministic setting.

In the deterministic setting, a candidate upper bound can be found using traditional Lagrangian reasoning. To do this, we look at our question as a constrained optimization problem over the set $\mathcal{C}$ of right-continuous, increasing functions:

$$u(x) = \sup_{C \in \mathcal{C}} U(C) \quad \text{s.t.} \quad \langle C, Y \rangle \leq x \text{ for all } Y \in \mathcal{Y}(1),$$

for some set of constraints, represented by $\mathcal{Y}(1)$, which may be infinite. For each constraint, or pricing function, Y we can consider the Legendre-Fenchel transform

$$\begin{aligned} L(Y, y) &= \sup_{C \in \mathcal{C}} [U(C) - \langle C, yY \rangle + yx] \\ &\triangleq V(yY) + yx \end{aligned}$$

Most importantly, this transform gives an upper bound to our optimization problem: $u(x) \leq L(Y, y)$ for all $Y \in \mathcal{Y}(1)$ and $y \in \mathbb{R}_+$.

The ultimate goal is to show that there exists a dual feasible pair (Y, y) such that we in fact have equality above, also in the case of uncertainty. In other words, we wish to relate the question of optimal consumption to a minimization problem over the dual pairs (y, Y). In this section we focus on the first step: the relationship between our objective function $U(\cdot)$ and its conjugate $V(\cdot)$. In the next section we discuss the problems that arise with the above argument in the case of uncertainty and how to handle these. It turns out that the structure of the problem is maintained, but for a different choice of dual variables.

4.1 Inhomogeneously Convex Functions

In this section we present the definition of an inhomogeneously convex function. Later we will see that the (super)gradients of U are of this type. The reader wishing to skim this section on a first reading should note that the statement of Lemma 4.3 is most important for what follows.

In order to define inhomogeneous convexity, we begin by presenting the results for functions $g : [0,T] \times \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\}$ that are continuous and strictly increasing from $-\infty$ to ∞ in the second argument. In some sense, this is the most natural class of functions to consider. All of the proofs and details can be found in Bank [3], but, given the importance of this structure on our further analysis, we include the important results here for completeness. At the end of this section we provide an argument for when these results can be applied to functions g defined only on the half line. As a specific example, the reader is encouraged to think about how these results apply to $-F'(t,x)$.

Definition 4.1 *A function $x : [0,T] \mapsto \mathbb{R}$ is **inhomogeneously convex**, or g-convex for short, if for all $0 \leq s < t < u \leq T$ we have that*

$$x(t) \leq x(s) + \int_s^t g(r, \ell_{s,u}),$$

where $\ell_{s,u}$ *is the unique constant satisfying*

$$x(u) = x(s) + \int_s^u g(r, \ell_{s,u}).$$

Remark: If the function does not depend on the time variable, i.e. $g(t,x) = g(x)$, then we recover the class of convex functions. In general, many of the important properties of convexity remain true for g-convexity. The first such property, the existence of a density, is described in the lemma below.

Lemma 4.2 (Proposition 3.8 in [3]) *The following are equivalent:*

(i) x *is* g*-convex*

(ii) For all $0 \leq s < t < u \leq T$ *we have that*

$$\ell_{s,t} \leq \ell_{t,u}$$

(iii) There exists an increasing function $l : [0,T] \mapsto \mathbb{R}$ *such that*

$$x(t) - x(s) = \int_s^t g(r, \ell(r))\, dr$$

we will call $g(r, \ell(r))$ *the "density" of* x*.*

In addition, for each function x, there exists a maximal g-convex function $\check{x}$ such that $x \geq \check{x}$ (proposition 3.9 in [3].) Such a function is called the **g-convex envelope** of x. The next lemma describes the key properties of this envelope.

Lemma 4.3 (Proposition 3.13 in [3]) *Let* $\check{x}$ *be the* g*-convex envelope of a lower-semicontinuous function* x*, then the following statements hold.*

(i) $\check{x}(0) = x(0)$*, and* $\check{x}(T) = x(T)$

(ii) Let $\check{\ell}$ *be the unique increasing, right-continuous function* $\check{\ell} : [0,T] \to \mathbb{R}$ *such that* $g(\cdot, \check{\ell}(\cdot))$ *is a density for* $\check{x}$ *on* $(0,T)$*. Define*

$$\check{\ell}(0) = \lim_{t\downarrow 0} \check{\ell}(t) \quad \text{and} \quad \check{\ell}(T) = \lim_{t\uparrow T} \check{\ell}(t)$$

Then $d\check{\ell}$ *induces a Borel measure on* $[0,T]$ *such that*

$$supp(d\check{\ell}) \subseteq \{\check{x} = x\}$$

(iii) The function $\check{x}$ *is absolutely continuous on the closed interval* $[0,T]$

We are now ready to make a connection with the derivative of the felicity function $F'(t,x)$. Based on the previous discussion, if x is lower-semicontinuous and $\check{x}(T) = 0$ then we may write

$$\check{x}(s) = -(\check{x}(T) - \check{x}(s)) = -\int_s^T g(r, \check{\ell}(r))dr \tag{16}$$

Assume further that x is nonnegative. Then it is possible to apply these results to functions defined only on the half-line. In particular, we can apply these results to $F' : [0,T] \times \mathbb{R}_+ \to [0,\infty]$. First we extend F' to the whole real line by defining

$$g(t,\ell) = \begin{cases} -F'\left(t, -\frac{1}{\ell}\right) & \ell < 0 \\ 0 & \ell = 0 \\ \ell & \ell > 0 \end{cases}$$

With this definition we observe that the function g is strictly increasing from $-\infty$ to ∞, and it is continuous if the Inada conditions (11) and (12) are satisfied. The theory developed above thus applies. In particular, there exists a g-convex envelope of the function x, and this envelope has the properties described in lemma (4.3). To show that only nonpositive values of ℓ are relevant, it suffices to find one g-convex function $\check{k}$ such that $\check{k}(t) \leq x(t)$ and such that the increasing process ℓ describing the density of $\check{k}$ is nonpositive. This is because we can represent the g-convex function $\check{k}$ as in equation (16)

$$\check{k}(s) = -\int_s^T g(r, \ell(r))dr$$

and so it is a decreasing function of ℓ. Thus, because $\check{x} \geq \check{k}$, we also have that the density of $\check{x}$ is described by a nonpositive, increasing process $\check{\ell} \leq \ell$.

In order to construct such a $\check{k}$ we start by defining $\ell_{s,t}$ to be the unique constant such that

$$x(t) = x(s) + \int_s^t g(v, \ell_{s,t})dv$$

From the definition of g we note that if $x(t) \leq x(s)$ then $\ell_{s,t} \leq 0$. Because $0 = x(T) \leq x(0)$ we have that

$$\ell_0 \triangleq \inf_{s,t \in [0,T]} \ell_{s,t} \leq 0$$

Now define

$$\check{k}(t) = x(0) + \int_0^t g(v, \ell_0)dv$$

It is easy to check that it is g-convex and dominated by x. This establishes the result. In particular,

$$\check{x}(t) = \int_t^T F'(s, \tilde{\ell}(s))ds, \quad \tilde{\ell}(s) = \frac{-1}{\check{\ell}(s)} \tag{17}$$

and the process $\tilde{\ell}$ is increasing and nonegative! In this sense we may talk about $(-F')$-convex functions and $(-F')$-convex envelopes of nonegative functions x (c.f. representation in equation (16).)

Remark: Because we only require that x is lower-semicontinuous, we can obtain the above representation by re-defining, if necessary, $x(T) = 0$. In many applications, this will not be a problem. For instance, in our problem there is no gain from consumption at the terminal time, and thus, the terminal values are not of concern.

4.2 The Conjugate Pair

At the beginning of this section, we informally defined the conjugate function V of the utility U. We now make this definition precise and show how inhomogeneously convex functions come into play. The end of this subsection is devoted to regularity properties of V such as lower-semicontinuity.

First, observe that in the deterministic setting, the natural domain $\mathcal{H}$ of U is the set of right-continuous, increasing, non-negative functions on $[0, T]$, i.e.

$$\mathcal{H} \triangleq \{h | h \text{ is the distribution function of a nonnegative measure on } [0, T]\}. \tag{18}$$

With this notation, we define, for every k in the set

$$\mathcal{K} \triangleq \{k : [0, T] \to \mathbb{R}_+ \,|\, k \text{ Borel measurable}\}$$

the functional

$$V(k) \triangleq \sup_{h \in \mathcal{H}} [U(h) - \langle h, k \rangle]. \tag{19}$$

The bilinear pairing is defined, by analogy with the pairing (8), as follows,

$$\langle h, k \rangle \triangleq \int_0^T k(t) dh(t). \tag{20}$$

Observe that the domain of definition $\mathcal{K}$ contains the paths of processes $Y^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}$, and is thus a good candidate for the set of dual functions. The set $\mathcal{K}$, while convenient for its generality, contains many poorly behaving functions for which V is difficult to calculate explicitly. It turns out that for many of the concrete results of this section, we need to make the additional assumption that k is lower-semicontinuous.

Remark: We will show that if k is lower-semicontinuous, then $V(k)$ is uniquely determined by its action on a strict subset of $\mathcal{K}$, the subset of inhomogeneously convex functions. Recall that for lower-semicontinuous functions, the inhomogeneously convex envelope has special properties. These are detailed in Lemma 4.3, most important of which is the statement about the support of the measure induced by the right-continuous increasing function $\check{\ell}$ associated with $\check{k}$. The support of this measure is contained in the set where the function k and its inhomogeneously convex envelope $\check{k}$ are equal. In the next section (Proposition 5.7 and Corollary 5.10) we show how to extend this result to the stochastic processes $Y^{\mathbb{Q}}$, which are only lower-semicontinuous in expectation.

The first result is closely related to the classical Legendre-Fenchel transform for functions on $\mathbb{R}^n$.

Lemma 4.4 *(i) The functionals U and V (defined in (10) and (19) respectively) are mutually conjugate. In other words, if V is defined by equation (19), then the following reciprocal relation is also true:*

$$U(h) = \inf_{k \in \mathcal{K}} [V(k) + \langle h, k \rangle]. \tag{21}$$

(ii) The function

$$\nabla_t U(h) \triangleq \int_t^T F'(s, h(s))\, ds \ \ (0 \le t \le T) \tag{22}$$

is a supergradient of the concave functional U *at the point* h*, in the sense that*

$$U(\tilde{h}) - U(h) \le \langle \tilde{h} - h, \nabla U(h) \rangle, \qquad \forall\ \tilde{h} \in \mathcal{H} \tag{23}$$

Furthermore, this supergradient is unique.

Remark: The supergradients of U are $(-F')$-convex functions with $\nabla_T U(C) = 0$.

Proof : We start with the proof of (ii); it is a straightforward application of the theorem of Fubini-Tonelli. Begin with

$$\begin{aligned} \langle \tilde{h} - h, \nabla U(h) \rangle &= \int_0^T \int_t^T F'(s, h(s))\, ds\, d(\tilde{h} - h) \\ &= \int_0^T F'(s, h(s))(\tilde{h} - h)(s)\, ds, \end{aligned}$$

and compare this expansion, term by term, with

$$U(\tilde{h}) - U(h) = \int_0^T F(s, \tilde{h}(s)) - F(s, h(s)) ds.$$

The supergradient property now follows from the fact that the felicity function $F(t, \cdot)$ is concave and has (super)gradient $F'(t, \cdot)$.

The uniqueness of supergradients follows as a special case of Proposition 1.3 in [3]. The proof involves showing that if we write $g(\varepsilon) = U(h + \varepsilon 1_{[t,T]})$, then $\partial^+ g(0) = \nabla_t U(h)$.

To prove (i) we note that the definition (19) implies that

$$U(h) \le V(k) + \langle h, k \rangle$$

for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$. Thus

$$U(h) \leq \inf_{k \in \mathcal{K}} [V(k) + \langle h, k \rangle].$$

For the reverse inequality, it suffices to show that there exists a $k^* \in \mathcal{K}$ that achieves this infimum. In fact, setting $k^* = \nabla U(h)$ and using definition (19) along with (23)we have

$$\begin{aligned} V(\nabla U(h)) + \langle h, \nabla U(h) \rangle &= \sup_{\tilde{h} \in \mathcal{H}} [U(\tilde{h}) - \langle \tilde{h} - h, \nabla U(h) \rangle] \\ &= U(h) \end{aligned}$$

□

Furthermore, the structure of the super-gradients gives the following corollary.

Corollary 4.5 *The utility function U is also a solution to a minimization problem $\inf_{\check{k} \in \check{\mathcal{K}}} [V(\check{k}) + \langle h, \check{k} \rangle]$ over the smaller set ,*

$$\check{\mathcal{K}} \triangleq \left\{ \check{k} \in \mathcal{K} \;\middle|\; \check{k}(s) = \int_s^T F'(t, \check{\ell}(t)) dt, \text{ for a function } \check{\ell} : [0,T] \to [0,\infty] \text{ that is right-continuous, increasing and } \check{\ell}(T) = \lim_{t \to T} \check{\ell}(t) \right\},$$

i.e.,

$$U(h) = \inf_{\check{k} \in \check{\mathcal{K}}} [V(\check{k}) + \langle h, \check{k} \rangle].$$

Remark: The set $\check{\mathcal{K}}$ consists of inhomogeneously convex functions $\check{k}$ such that $\check{k}(T) = 0$. Because the value of $\check{k}$ is not influenced by $\check{\ell}(T)$, we make the additional assumption of left-continuity at the terminal time. This way we can show (Corollary 4.8) that V is strictly convex when restricted to $\check{\mathcal{K}}$. Similarly, it is easy to see that U is strictly concave on the subset of $\mathcal{H}$ that consists of distribution functions that are left-continuous at $t = T$. In the remainder of this section we study the properties of the conjugate functional V. In particular, we find a characterization of its subgradients, show how to evaluate this functional on the set $\check{\mathcal{K}}$ (and in fact for all lower-semicontinuous k), and prove that it is lower-semicontinuous and strictly convex on $\check{\mathcal{K}}$. This discussion will also highlight the special role played by inhomogeneously convex functions.

Lemma 4.6 *If $\check{k}$ is $(-F')$-convex, with $\check{k}(t) = \int_t^T F'(s, \check{\ell}(s))\, ds$ then*

$$V(\check{k}) = U(\check{\ell}) - \langle \check{\ell}, \check{k} \rangle. \tag{24}$$

In particular, if $\check{k} > 0$ on $[0, T)$, then

$$V(\check{k}) = \int_0^T [F(t, \check{\ell}_t) - F'(t, \check{\ell}_t)\check{\ell}_t]\, dt. \tag{25}$$

Furthermore, if $\check{k}$ is the $(-F')$-convex envelope of $k \in \mathcal{K}$ and if k is lower-semicontinuous with $k(T) = 0$, then

$$V(k) = V(\check{k}).$$

Proof : As long as the function $\check{k} > 0$ on $[0, T)$, we have that $\check{\ell} < \infty$ and that the supremum is achieved at the point h^* such that $\nabla U(h^*) = \check{k}$. In other words, $h^* = \check{\ell}$. The characterization in equation (25) follows from the Fubini-Tonelli theorem.

Now let $t^* = \inf\{t \geq 0 \,|\check{k}(t) = 0\}$. If $t^* < T$ then we have that

$$\langle h, \check{k}_t \rangle = \int_0^{t^*} \check{k}\, dh(t).$$

In other words, there is no constraint on consumption after time t^* and we may let it go to infinity. However, we also have that $\check{\ell}(t) = \infty$ for all $t \geq t^*$. Thus equation (24) still holds, provided that we define $F(t,\infty) = \lim_{x\to\infty} F(t,x)$. If this limit is infinite for some $t \geq t^*$ then $V(\check{k}) = \infty$ also.

For the second part, we note that if k is not $(-F')$-convex, then it dominates its $(-F')$-convex envelope $\check{k}$. Thus, for each $h \in \mathcal{H}$ we have the inequality

$$U(h) - \langle h, k\rangle \;\leq\; U(h) - \langle h, \check{k}\rangle$$

Consequently, $V(\check{k}) \geq V(k)$. However, letting h^* be as above, we have that

$$supp(dh^*) = supp(d\check{l}) \subset \{k = \check{k}\}$$

according to lemma (4.3). This inclusion implies that $\langle h^*, k\rangle = \langle h^*, \check{k}\rangle$. In particular,

$$U(h^*) - \langle h^*, k\rangle = U(h^*) - \langle h^*, \check{k}\rangle$$

Thus $V(\check{k}) \leq V(k)$ and we must in fact have equality,

$$V(k) = V(\check{k})$$

□

Remark: The result does not extend to functions k that are not lower-semicontinuous. For example, if $k(t) = 2$ at all rational points and $k(t) = 1$ at all irrational points, then $k_*(t) = 1$ and $\check{\ell}$ is constant, representing a point mass at $t = 0$. But because $k_*(0) \neq k(0)$, we have that $\langle \check{\ell}, k\rangle \neq \langle \check{\ell}, k_*\rangle$.

Another important result is that for $\check{k} \in \check{\mathcal{K}}$ the subgradient of V evaluated at $\check{k}$ is unique. This result is expanded on in the following lemma.

Lemma 4.7 *(i) $V(\check{k}) < \infty$ if and only if either the felicity function $F(t,\cdot)$ is bounded from above for all $t \in [0,T]$, or $\check{k}(t) > 0$ for all $t \in [0,T)$.*

(ii) Suppose that $V(\check{k}) < \infty$ and let k^* be lower-semicontinuous and such that its $(-F')$-convex envelope is $\check{k}^*(t) = \int_t^T F'(s, \check{\ell}^*(s))ds$, then $-\check{\ell}^*$ is a subgradient of V at k^*. This means that

$$V(k) - V(k^*) \geq \langle -\check{\ell}^*, k - k^* \rangle \quad \forall k \in \mathcal{K}$$

Furthermore, the subgradient is unique if we assume that it is left-continuous at $t = T$. We call this subgradient $\nabla V(k^*)$.

Proof : The statement (i) follows from the characterization (24) and the observation that if $\check{k}(t^*) = 0$ then $\check{k}(t) = 0$ for all $t > t^*$.

To prove (ii) we first note that the subgradient property follows easily from the definition of V as the conjugate functional to U, at least when restricted to $\check{\mathcal{K}}$. In fact, let $\check{l}^* = h^*$ and for any k, $k^* \in \mathcal{K}$ we have that

$$V(\check{k}) - V(\check{k}^*) \geq \langle -h^*, \check{k} - \check{k}^* \rangle$$

Then because $V(k) = V(\check{k})$, $\langle h^*, k^* \rangle = \langle h^*, \check{k}^* \rangle$, and $k \geq \check{k}$, we have in addition that

$$V(k) - V(k^*) \geq \langle -h^*, k - k^* \rangle.$$

In particular, $-h^* = -\check{\ell}^*$ is a subgradient at k^*. Observe that equation (24) guarantees that this result holds even if $\check{\ell}^*(t) = \infty$ for some $t < T$. In this case, however, the subgradient is not a genuine consumption plan, but a limit of a sequence of plans.

For uniqueness, we observe first that in the dual description of U

$$U(h) = \inf_{\check{k} \in \check{\mathcal{K}}} [V(\check{k}) + \langle h, \check{k} \rangle]$$

the infimum is attained at any point $\check{k}$ such that $-h \in \{\nabla V(\check{k})\}$, the (convex) set of subgradients of V at $\check{k}$. Suppose that this set is not a singleton, i.e. that there exists $h_1, h_2 \in \mathcal{H}$ such that $-h_1, -h_2 \in \{\nabla V(\check{k})\}$ and $h_1 \neq h_2$. Then from the definition of a subgradient, the set $\{\nabla V(\check{k})\}$ must contain the line between h_1 and h_2. This, however, is a contradiction to the strict concavity of the utility functional U. Thus, as an element of $-\mathcal{H}$, the subgradient must be unique.

If $\check{k}(t) > 0$ and the felicity function is not bounded, then the associated $\check{\ell}$ is finite. The above considerations thus show that $-\check{l}$ is the unique subgradient. If the felicity function is bounded, however, it still makes sense to talk about gradients of V for functions $\check{k}$ that possibly hit zero. In this case, however, we can also extend the definition of U to act on possibly infinite functions $h : [0, T] \to [0, \infty]$. We can do this by a limiting procedure and by applying the bounded convergence theorem. Observe that for such h, the dual description (21) still holds and we again have uniqueness of subgradients. □

This lemma has an important corollary

Corollary 4.8 *Restricted to the set* $\{\check{k} \in \check{\mathcal{K}}$ *s.t.* $V(\check{k}) < \infty\}$, *the functional* V *is strictly convex.*

Proof : Let $\check{k}_1, \check{k}_2 \in \check{\mathcal{K}}$ and such that $V(\check{k}_i) < \infty$, then for $\lambda \in (0, 1)$ we have

$$\begin{aligned} V(\lambda\check{k}_1 + (1-\lambda)\check{k}_2) &= \sup_{h \in \mathcal{H}} [\lambda(U(h) - \langle h, \check{k}_1 \rangle) + (1-\lambda)(U(h) - \langle h, \check{k}_2 \rangle)] \\ &\leq \lambda \sup_{h \in \mathcal{H}} [U(h) - \langle h, \check{k}_1 \rangle] + (1-\lambda) \sup_{h \in \mathcal{H}} [U(h) - \langle h, \check{k}_2 \rangle] \\ &= \lambda V(\check{k}_1) + (1-\lambda) V(\check{k}_2) \end{aligned}$$

with equality if and only if the supremum is achieved at the same h^* in each of the three supremums above (possibly allowing for infinite values of h^*). From the characterization (24) of V, however, we know that this can only happen if $\check{k}_1 = \check{k}_2$. Hence V is strictly convex. □

Finally, we describe in what sense V is lower-semicontinuous on $\check{\mathcal{K}}$. This result is important for showing the existence of a solution to the dual problem.

Lemma 4.9 *Assume that the asymptotic elasticity of the felicity function $F(t,\cdot)$ is less than one for each $t \in [0,T]$ and let $(\check{k}_n)_{n\in\mathbb{N}} \in \check{\mathcal{K}}$ be a sequence such that the corresponding processes $\check{\ell}_n$ converge to a right-continuous, increasing function $\check{\ell} : [0,T] \to [0,\infty]$ at all points of continuity of $\check{\ell}$. Let $\check{k} \in \check{\mathcal{K}}$ be such that it has density $-F'(t,\check{\ell}(t))$. Then we have*

$$\liminf_{n\to\infty} V(\check{k}_n) \geq V(\check{k}).$$

Proof : If $\check{\ell}(T) < \infty$ then we may assume that for n large enough $\check{\ell}_n(T) < M < \infty$. Also, because $F(\cdot, M)$ and $F'(t,M)$ are continuous, they are bounded on the interval $[0,T]$. We can apply the dominated convergence theorem to obtain

$$\begin{aligned}
\lim_{n\to\infty} V(\check{k}_n) &= \lim_{n\to\infty} \int_0^T F(t,\check{\ell}_n(t)) - F'(t,\check{\ell}_n(t))\check{\ell}_n(t)\,dt \\
&= \int_0^T \lim_{n\to\infty} \left[F(t,\check{\ell}_n(t)) - F'(t,\check{\ell}_n(t))\check{\ell}_n(t)\right]\,dt \\
&= V(\check{k}).
\end{aligned}$$

Now let $t^* = \inf\{t \in [0,T] \,|\, \check{\ell}(t) = \infty\}$. For the case $\check{\ell}(t^*) = \infty$, first assume that the felicity function is bounded for each $t \in [0,T]$. In this case, we first show that the asymptotic elasticity is zero. To prove this fact, observe that for any $\varepsilon > 0$ we may pick x_0 large enough such that $F(t, x_0) > M(t) - \varepsilon$. Then using the fundamental theorem of calculus, we obtain

$$\varepsilon > F(t, x_0 + x) - F(t, x_0) = \int_{x_0}^{x_0+x} F'(t, y)dy \geq F'(t, x_0 + x)x$$

Because the derivative is decreasing. This calculation gives the estimate

$$\frac{F'(t, x_0 + x)(x_0 + x)}{F(t, x_0 + x)} \leq \frac{\varepsilon}{M(t) - \varepsilon} \frac{x_0 + x}{x}.$$

Letting x tend to infinity, we have that $AE\, F(t, \cdot) \leq \frac{\varepsilon}{M(t)-\varepsilon}$. Letting $\varepsilon \to 0$ we obtain the desired result. $AE\, F(t, \cdot) = 0$. In addition we may write it as a genuine limit;

$$AEF(t, \cdot) = \lim_{x\to\infty} \frac{F'(t, x)x}{F(t, x)}.$$

We are now ready to show lower-semicontinuity. In fact, the definition of V gives,

$$V(k) = \int_0^{t^*} F(t, \check{\ell}(t)) - F'(t, \check{\ell}(t))\check{\ell}(t)\, dt + \int_{t^*}^{T} F(t, \infty)dt$$

and applying the dominated convergence theorem to the limit and simplifying we get,

$$\begin{aligned}
\lim_{n\to\infty} V(\check{k}_n) &= \lim_{n\to\infty} \int_0^T F(t, \check{\ell}_n(t)) - F'(t, \check{\ell}_n(t))\check{\ell}_n(t)\, dt \\
&= \int_0^T \lim_{n\to\infty} \left(1 - \frac{F'(t, \check{\ell}_n(t))\check{\ell}_n(t)}{F(t, \check{\ell}_n(t))}\right) F(t, \check{\ell}_n(t))\, dt \\
&= \int_0^{t^*} F(t, \check{\ell}(t)) - F'(t, \check{\ell}(t))\check{\ell}(t)\, dt + \int_{t^*}^{T} (1 - AE\, F(t, \cdot))\, F(t, \infty)\, dt \\
&= V(\check{k})
\end{aligned}$$

The last equality follows because the asymptotic elasticity is zero.

If the felicity function F is unbounded and we have $\check{\ell}(t^*) = \infty$ for some $t^* < T$ as above, then we may use Fatou's lemma to show that both $V(\check{k})$ and $\liminf_{n\to\infty} V(\check{k}_n)$ are infinite. For this we use an expansion similar to above. If $\check{\ell}(T) = \infty$ but is finite for $t < T$, then we may only apply Fatou to get the inequality

$$\liminf_{n\to\infty} V(\check{k}_n) \geq V(\check{k})$$

In this sense the convex function V is lower-semicontinuous at the boundary with respect to convergence is distribution of the right-continuous, increasing functions $\check{\ell}$. □

5 Duality Theory: Fully Random

In this section we lay the foundations for duality theory in a market with uncertainty. To start, observe that if we follow the logic of Section 4, we are led to the upper-bound

$$u(x) \leq \sup_{C \in \mathcal{C}} \mathbb{E}\left[U(C) - \langle C, yY \rangle + yx\right], \quad \text{for all } Y \in \mathcal{Y}^{\mathcal{M}}(1).$$

Unfortunately, it is not true that we can exchange the supremum with the expectation operator, and recover the function V (which acts on functions and not on stochastic processes). This is because there is no guarantee that for a general process Y an ω by ω optimization will lead to an *adapted* optimal consumption plan. However, we will show in Corollary 5.10 that for a slightly different class of processes $\check{\mathcal{Y}}$, this pathwise approach does work and also provides an upper bound to $u(x)$. In fact, in the next section we prove that this bound is tight, i.e., that there exists a dual process for which we obtain equality above. In a sense this new set of processes $\check{\mathcal{Y}}$ is chosen to enforce the non-anticipativity constraint; this property is illustrated in the first part of Corollary 5.10. For an additional example see Davis and Karatzas [18], in addition to budget requirements; we give a more precise definition later in this section.

Skipping ahead, in Section 6 we will formulate the main duality result (Theorem 6.3) in terms of these processes $\check{\mathcal{Y}}$. First, however, we discuss the relationship between the sets $\mathcal{Y}^{\mathcal{M}}$ and $\check{\mathcal{Y}}$ in more analytic terms. One key idea that we use is the stochastic representation theorem in [4].

The first two parts of this section are devoted to explaining the choice of dual variables $\check{\mathcal{Y}}$ and formulating the dual problem (40). To begin, we extend the budget constraint to a more tractable set of variables. In particular, we use the process bipolar theorem of Žitković [51], to show that a financiable consumption plan must satisfy an integrability constraint with respect to the whole set of deflator processes, i.e., with respect to processes Y such that XY is a supermartingale for all $X \in \mathcal{X}$. We call this class of processes $\mathcal{Y}$.

Finally, we note that in fact any set of processes whose optional projections coincide with the set $\mathcal{Y}$ could be used for the budget constraint. The collection of all such processes is too large, however, to be analytically tractable. But we are free to choose a convenient subset, as long as it, or optional projections of its elements, contain the set $\mathcal{Y}^{\mathcal{M}}$. Interchanging a process with its optional projection will be important for the definition of $\check{\mathcal{Y}}$, the set that we ultimately plan to use for the dual problem.

The last part of this section is devoted to proving important analytic properties of the set $\check{\mathcal{Y}}$. In particular, we find conditions under which the set is convex, and show that it is Fatou closed. For this part we will need the extra assumption, Assumption 2.3, on the felicity function; this assumption is necessary and sufficient for the set $\check{\mathcal{Y}}$ to be convex. Finally, we also prove existence in the dual problem.

5.1 The Budget Constraint Revisited

We now prove an important extension of the budget constraint (9) introduced in the model description. In Section 2 we showed that a consumption plan is financiable with initial capital x if and only if

$$\sup_{Y^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}} \mathbb{E} \int_0^T Y_t^{\mathbb{Q}} \, dC_t \leq x.$$

This result expresses the budget constraint in terms of the economically meaningful pricing measures $\mathbb{Q} \in \mathcal{M}$ and their densities. Recall that $\mathcal{Y}^{\mathcal{M}}$ is the set of density processes of these pricing measures (the set is defined in (7.) However, it is also possible to express the constraint in terms of the set $\mathcal{Y}(1)$ instead, where we define, for each $y \in \mathbb{R}_+$,

$$\mathcal{Y}(y) \triangleq \{Y \text{ adapted, } \text{RCLL, } Y_0 \leq y, \, (YX)_t \text{ is a supermartingale for all } X \in \mathcal{X}\}. \tag{26}$$

The significance of the sets $\mathcal{Y}(1)$ for the duality approach, and also the reason that makes it work for the budget constraint, is that it is the process-bipolar of the set $\mathcal{Y}^{\mathcal{M}}$. The process bipolar theorem and the related concepts are developed in [51]. For an application in utility optimization via duality, we refer the reader to [34]. These ideas are an extension of the bipolar theorem for subsets of $\mathbb{L}^0_+$ (a non-locally-convex space) proved in [13] and the application of this theorem in [36] to portfolio optimization.

We will use these bipolar results to prove the following theorem.

Lemma 5.1 *The consumption plan $C \in \mathcal{C}$ is financiable with initial wealth x if and only if*

$$\sup_{Y \in \mathcal{Y}(1)} \mathbb{E} \int_0^T Y_t \, dC_t \leq x \tag{27}$$

In fact, if $\mathcal{B}$ is any set of processes such that $\mathcal{Y}^{\mathcal{M}} \subseteq \mathcal{B} \subseteq \mathcal{Y}(1)$, then the budget constraint can be written in terms of $\mathcal{B}$.

PROOF : Theorem 4 in [51] shows that $\mathcal{Y}$ is the process bipolar of $\mathcal{Y}^{\mathcal{M}}$. What this means in our case is that, letting $\mathcal{D}$ denote the set of non-increasing, adapted RCLL processes, for all $Y \in \mathcal{Y}$ there exists sequences of processes $Y_n^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}$ and $D_n \in \mathcal{D}$ such that $Y_n^{\mathbb{Q}} D_n$ is Fatou-convergent to Y. We recall the definition of Fatou-convergence for stochastic processes in Section 5.3.2, where we also use the concept more extensively, see also [36]. Most importanly, what we get is that

$$Y_t = \liminf_{s \downarrow t} \liminf_{n \to \infty} (Y^{\mathbb{Q}} D_n)(s) \leq \liminf_{s \downarrow t} \liminf_{n \to \infty} Y_n^{\mathbb{Q}}(s)$$

Furthermore, from taking conditional expectations and applying Fatou's lemma, we have that

$$Y_t = \mathbb{E}[Y_t|\mathbb{F}_t] = \mathbb{E}\left[\liminf_{s \downarrow t} \liminf_{n \to \infty} Y_n^{\mathbb{Q}}(s) \middle| \mathbb{F}_t \right] \leq \liminf_{n \to \infty} Y_n^{\mathbb{Q}}(t)$$

The last inequality follows because each $Y_n^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}$ is a non-negative local martingale and hence also a supermartingale. We may thus write (applying Fatou again)

$$\mathbb{E} \int_0^T Y_t dC_t \leq \liminf_{n \to \infty} \mathbb{E} \int_0^T Y_n^{\mathbb{Q}}(t) dC_t \leq x$$

The last part of the theorem follows from the original statement of the budget constraint in terms of $\mathcal{Y}^{\mathcal{M}}$. □

5.2 Choice of Dual Variables

In Section 4 we defined inhomogeneously convex functions and showed how these are the right choice of dual variables in the deterministic setting. We are now ready to discuss the implications of these results in the stochastic setting. Recall that the objective is to identify a suitable class of dual variables (processes) and to find a candidate for the dual of the indirect utility. We begin with a representation result (Theorem 3 in [4], and Lemma 3.1 in [3]):

Theorem 5.2 (Stochastic representation) *Suppose that the function* $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is continuous and, for any* $t \in [0,T]$, $g(t,\cdot) : \mathbb{R} \to \mathbb{R}$ *is strictly decreasing from* $+\infty$ *to* $-\infty$*. Let* X *be a non-negative optional process of class (D). Assume in addition that* X *is lower-semicontinous in expectation with* $X(T) = 0$*. Then there exists an optional process* L *such that*

$$X(\tau) = \mathbb{E}\left[\int_{\tau}^{T} g(t, \sup_{\tau \le v \le t} L_v)\, dt \middle| \mathbb{F}_\tau\right] \tag{28}$$

holds almost surely, for every stopping time τ*. Furthermore, If L is a solution to the representation problem 30, then so is its upper-rightcontinuous modification*

$$\tilde{L} \triangleq \limsup_{s \searrow t} L(s) = \lim_{\varepsilon \downarrow 0} \sup_{s \in [t,(t+\varepsilon)\wedge T]} L(s)$$

This solution is unique up to optional sections.

Definition 5.3 *An optional process* X *of class* (D) *is called* ***lower-semicontinous in expectation*** *if, for every stopping time* τ*, we have*

$$\liminf_{n\to\infty} \mathbb{E}X_{\tau_n} \ge \mathbb{E}X_\tau$$

whenever $(\tau_n,\, n = 1,2,3,\ldots)$ *is a monotone sequence of stopping times converging to* τ *almost surely.*

Observe that if X is a non-negative, right-continuous supermartingale of class (D), then X is lower-semicontinuous in expectation. In fact, the optional sampling theorem shows that if $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times converging to τ from below, then $\liminf_n \mathbb{E}X_{\tau_n} \geq \mathbb{E}X_\tau$. Because X is of class (D) and right continuous, this result holds also for $\tau_n \downarrow \tau$.

Furthermore, when X is non-negative, the representation theorem holds also when the function g is replaced with the derivative of the felicity function $F'(t,x) : [0,T] \times [0,\infty) \to [0,\infty)$. Given such a representation result, we can easily find inhomogeneously convex envelopes, in an appropriate pathwise sense, of the processes $Y \in \mathcal{Y}$ that are of class (D). Corollary 5.5 below shows that we can dispense with the class (D) requirement. It is then possible to apply duality theory (strictly speaking, the representation theorem is applied to $Y1_{[0,T)}(t)$.)

Corollary 5.4 *Let X be as in Theorem 5.2 and let F be a function that satisfies Assumption 2.1, then the conclusions of Theorem 5.2 also apply. In particular, there exists an optional process $L(\omega,t)$ taking values in $[0,\infty]$ such that*

$$X(\tau) = \mathbb{E}\left[\int_\tau^T F'(t, \sup_{\tau \leq v \leq t} L_v)\, dt \,\middle|\, \mathbb{F}_\tau\right] \tag{29}$$

almost surely, and for every stopping time τ.

Proof : By analogy with the deterministic argument, use F' to define a new function

$$g(t,\ell) = \begin{cases} F'\left(t, -\frac{1}{\ell}\right) & \ell < 0 \\ 0 & \ell = 0 \\ -\ell & \ell > 0 \end{cases}$$

that satisfies the conditions of the representation result (Theorem 5.2.) In particular, there exists an optional process $\tilde{L}$ such that

$$X(\tau) = \mathbb{E}\left[\int_\tau^T g(t, \sup_{\tau \leq v \leq t} \tilde{L}_v)\, dt \,\middle|\, \mathbb{F}_\tau\right]$$

The key step of the proof of theorem (5.2) is a level process argument and we modify this only slightly in order to show that the process $\tilde{L}$ takes only nonpositive values. The key constructions are taken from the proof of Theorem 3 in [4] (see also [3]). In particular, these authors show that for each $\ell \in \mathbb{R}$ there exists an optional process

$$Z^{\ell}(\sigma) = \operatorname*{ess\,inf}_{\tau \in \mathcal{S}(\sigma)} \mathbb{E}\left[X(\tau) + \int_{\sigma}^{\tau} F(t,\ell)\, dt \,\middle|\, \mathbb{F}_{\sigma} \right]$$

such that the mapping $\ell \mapsto Z^{l}(t,\omega)$ is continuously decreasing from

$$Z^{-\infty}(t,\omega) \triangleq \lim_{\ell \downarrow -\infty} Z^{\ell}(t,\omega) = X(t,\omega)$$

Furthermore, the level process

$$\tilde{L}(t,\omega) \triangleq \sup \left\{ \ell \in \mathbb{R} \,|\, Z^{\ell}(t,\omega) = X(t,\omega) \right\}$$

solves the representation problem (30). Also, because Lebesgue measure has no atoms, we can let $\tilde{L}(T,\omega) = 0$ for all $\omega \in \Omega$. For the special case that X is nonnegative, we need to show that the process $\tilde{L}$ is nonpositive. To this end, suppose that $\ell > 0$, and observe that because $X_T = 0$,

$$Z^{\ell}(t) = \operatorname*{ess\,inf}_{\tau \in \mathcal{S}(t)} \mathbb{E}\left[X(\tau) + \int_{t}^{\tau} F(t,\ell)\, dt \,\middle|\, \mathbb{F}_{t} \right]$$

$$\leq \mathbb{E}[X_T \,|\, \mathbb{F}_t] - \ell(T-t) < 0 \leq X_t$$

Thus $\tilde{L}(t,\omega) \leq 0$ for all $t \in [0,T]$, and $\omega \in \Omega$. As in the deterministic setting, as a last step we take,

$$L(t,\omega) = -\frac{1}{\tilde{L}(t,\omega)}$$

$\square$

We next show that for supermartingales, a localization argument can be used to dispense with the class (D) requirement.

Corollary 5.5 *Suppose that the function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and, for any $t \in [0,T]$, $g(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is strictly decreasing from $+\infty$ to $-\infty$. Let X be a non-negative, right-continuous supermartingale such that $X(T) = 0$. Then there exists an optional process L such that for every stopping time τ, we have*

$$X(\tau) = \mathbb{E}\left[\int_\tau^T g(t, \sup_{\tau \le v \le t} L_v)\, dt \,\middle|\, \mathbb{F}_\tau\right], \quad a.s. \tag{30}$$

Furthermore, if L is any progressively measurable solution to (30), then so is its upper-rightcontinuous modification.

Remark: This corollary, just like the original representation theorem, also applies to supermartingales on an infinite time horizon whenever the limit at infinity is defined and equal to 0.

Proof : The key contribution of this corollary is a localization argument that allows us to extend the results of the representation theorem to supermartingales that are not necessarily of class (D). The last statement, that the upper-rightcontinuous modification is also a solution, is Lemma 3.1 in [3].

The key fact is that non-negative supermartingales are locally of class (D). In particular, if τ_n is the stopping time $\tau_n \triangleq n \wedge \inf\{t > 0 \mid X_t \ge n\}$, then

$$X_{\tau_n \wedge S} \le n + X_{\tau_n} \quad \text{for all stopping times } S.$$

Because τ_n is bounded, we may apply the optional sampling theorem to conclude that X_{τ_n} is integrable: $\mathbb{E}[X_{\tau_n}] \le \mathbb{E}[X_0]$. In particular, the family of random variables $\{X_{\tau_n \wedge S}\}_{S \in \mathcal{S}}$ ($\mathcal{S}$ is the set of all stopping times) is bounded by an integrable random variable and hence it is uniformly integrable. Thus the stopped process $X^{\tau_n}(t) = X_{\tau_n \wedge t}$ is of class (D). We also have that $\lim_{n\to\infty} \tau_n = T$, so that $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence.

The stochastic representation theorem, Theorem 5.2, can be applied to show that for each n, there exists an optional process $(\xi_t^n)_{t\in[0,T]}$ such that for all stopping times σ

$$X_\sigma^{\tau_n} = \mathbb{E}\left[\int_\sigma^T g(s, \sup_{v\in[\sigma,s]} \xi_v^n) ds) \;\middle|\; \mathbb{F}_\sigma \right]$$

almost surely. Section 4.3 in Bank and ElKaroui [4] shows how to characterize this optional solution in terms of a family of Snell envelopes. In particular, we have

$$\xi^n(\omega, t) \triangleq \sup \left\{ \ell \in \mathbb{R} \mid Z^{\ell,n}(\omega, t) = X^{\tau_n}(\omega, t) \right\} \tag{31}$$

where

$$Z^{\ell,n}(\sigma) = \operatorname*{ess\,inf}_{\kappa\geq\sigma} \mathbb{E}\left[X^{\tau_n}(\kappa) + \int_\sigma^\kappa g(t,\ell)dt \;\middle|\; \mathbb{F}_\sigma \right] \tag{32}$$

(κ is a stopping time).

We can view the process $\bar{Z}^{\ell,n}(t) := Z^{\ell,n}(t) - \int_0^t g(s,\ell)ds$ as the Snell envelope (from below) of the process $X^{\ell,n}(t) := X^{\tau_n}(t) + \int_0^t g(s,\ell)ds$. In particular, $\bar{Z}^{\ell,n}$ is the largest submartingale dominated by $X^{\ell,n}$. This fact can be used to show that $Z^{\ell,n}$ (or equivalently $\bar{Z}^{\ell,n}$) is right continuous. In fact, because $\bar{Z}^{\ell,n}$ is a submartingale, its right- and left-hand limits exists a.s. along any countable set, and the limit does not depend on the choice of countable set (see for example Theorem VI.1.2 in [22]). Let $\bar{Z}_{t^+}^{\ell,n}$ denote this limit. It is also a submartingale. Then the right-continuity of the filtration implies that $\bar{Z}_t^{\ell,n} \leq \bar{Z}_{t^+}^{\ell,n}$ (also Theorem

VI.1.2 in [22]). Furthermore, because $X^{\ell,n}$ is right-continuous, we have

$$\bar{Z}_t^{\ell,n} \le \bar{Z}_{t^+}^{\ell,n} \le \limsup_{s_m \downarrow t} X_{s_m}^{\ell,n} = X_t^{\ell,n}.$$

By the maximality of $\bar{Z}^{\ell,n}$ we must have that $\bar{Z}_t^{\ell,n} = \bar{Z}_{t^+}^{\ell,n}$.

Observe also that the processes $Z^{\ell,n}$ satisfy (Lemma 4.12 in [4]),

$$Z^{\ell,n}(s) \le X^{\tau_n}(s) \quad \text{for all } s \in [0,T] \tag{33}$$

and

$$\ell \mapsto Z_s^{\ell,n}(\omega) \text{ is continuous and strictly decreasing in } \ell. \tag{34}$$

The next step is to show that this characterization leads to a solution that is upper-semicontinuous from the right when the process to be represented X is right-continuous. For this part, we adapt arguments found in Bank and Küchler [6] Theorem 1.

According to Proposition 2 in Dellacherie and Lenglart [21] it is sufficient to show that $\lim_{m\to\infty} \xi_{S_n}^n \le \xi_S^n$ for all sequences of stopping times $S_m \downarrow S$ and such that the limit $\lim_{m\to\infty} \xi_{S_m}^n := \zeta^n$ exists almost surely.

Combining the right continuity of $Z^{\ell,n}$, the properties (34) and (33), and the definition of ξ^n we conclude that for all $\varepsilon > 0$

$$Z^{\zeta-\varepsilon,n}(S) = \lim_{S_m \downarrow S} Z^{\zeta-\varepsilon,n}(S_m) = \lim_{S_m \downarrow S} X_{S_m}^{\tau_n} = X_S^{\tau_n}.$$

Letting ε tend to zero, we conclude that $\zeta^n \le \xi_S^n$, as desired. Hence the optional solution ξ^n defined in terms of the Snell envelopes, is also upper-rightcontinuous.

According to Theorem 1 in Bank and ElKaroui [4], any upper-rightcontinuous solution ξ^n to the representation problem is given by

$$\xi_S^n = \operatorname*{ess\,inf}_{\kappa > S} \ell_{S,\kappa}^n \quad \text{a.s. and for all stopping times } S \tag{35}$$

where κ is a stopping time and $\ell^n_{S,\kappa}$ is the unique $\mathcal{F}_S$ measurable solution of

$$\mathbb{E}[X^{\tau_n}_\kappa - X^{\tau_n}_S \mid \mathbb{F}_S] = \mathbb{E}\left[\int_S^\kappa g(t, \ell^n_{S,\kappa})dt \,\middle|\, \mathbb{F}_s\right].$$

Without loss of generality, we may assume that $\tau_n \geq \sigma$, because in the limit as n tends to infinity this is true almost surely: $\mathbb{P}(\lim_{n\to\infty}\{\tau_n \geq S\}) = 1$. For the family $(X^{\tau_n})_{n\in\mathbb{N}}$ of supermatingales, it turns out that $\ell^n_{S,\kappa}$ is increasing in n in the sense that $\ell^m_{S,\kappa} \geq \ell^n_{S,\kappa}$-a.e. for $m \geq n$ (on the set $\{\tau_n \geq S\}$). In fact, we can use the optional sampling theorem for bounded stopping times to show that

$$\mathbb{E}[X^{\tau_m}_\kappa - X^{\tau_m}_S \mid \mathbb{F}_S] \leq \mathbb{E}[X^{\tau_n}_\kappa - X^{\tau_n}_S \mid \mathbb{F}_S]$$

for all $m \geq n$. Because $g(t,\cdot)$ is decreasing, this proves that $\ell^m_{S,\kappa} \geq \ell^n_{S,\kappa}$-a.e.

Recall that the essential infimum I of a family G of $\mathcal{F}$ measurable random variables is the greatest-lower-bound of G in the sense of a.e.-equivalence. More precisely, $I = \operatorname{ess\,inf}_{g\in G} g$ if

(i) $I \leq g$ a.e. for all $g \in G$.

(ii) If h is any $\mathcal{F}$ measurable random variable such that $h \leq g$ a.e. for all $g \in G$ then $h \leq I$ a.e.

In particular, we have that $\xi^n_S \leq \ell^n_{S,\kappa} \leq \ell^m_{S,\kappa}$ a.e. and hence also that $\xi^n_S \leq \xi^m_S$ a.e. for each stopping time S.

The optional section theorem (see for example Bass [10] Corollary 2.4) can then be used to show that

$$\mathbb{P}(\xi^n_t \leq \xi^m_t \text{ for all } t \in [0,T]) = 1.$$

In particular, for a fixed S, the process $\sup_{v\in[S,t]} \xi^n_v$ is increasing in n point-wise. We can thus apply the monotone convergence theorem to conclude

$$X_S = \lim_{n\to\infty} X_S^{\tau_n} = \mathbb{E}\left[\int_S^T g(t, \lim_{n\to\infty} \sup_{v\in[S,t]} \xi^n_v) dt \;\middle|\; \mathbb{F}_S\right]$$
$$= \mathbb{E}\left[\int_S^T g(t, \sup_{v\in[S,t]} \lim_{n\to\infty} \xi^n_v) dt \;\middle|\; \mathbb{F}_S\right]$$

almost surely for a fixed stopping time S. Because each ξ^n is optional, the limit is also. In particular, we obtain a representation of X with respect to the optional process

$$L \triangleq \lim_{n\to\infty} \xi^n.$$

□

The representation theorem Theorem 5.2 and Corollary 5.5 show that each $Y1_{[0,T)}$ such that $Y \in \mathcal{Y}$ can be represented as an optional projection

$$Y_t 1_{[0,T)}(t) = \mathbb{E}\left[\int_t^T F'(s, \sup_{t\le v\le s} L(v))\, ds \;\middle|\; \mathbb{F}_t\right]$$

for some progressively measurable process L that is upper-rightcontinuous. In order to make a connection with inhomogeneously convex functions, we undo the optional projection and instead look at the possibly nonadapted process

$$\tilde{Y}_t = \int_t^T F'(s, \sup_{t\le v\le s} L(v))\, ds$$

whose optional projection is an element $Y1_{[0,T)} \in \mathcal{Y}$, ${}^{\circ}\tilde{Y} = Y$. The pathwise inhomogeneously convex envelope of this process is easy to describe.

Corollary 5.6 *With the process L defined as above, set*

$$\check{L}(t) \triangleq \sup_{0\le v\le t} L(v) \quad \textit{for all } t \in [0,T) \quad \textit{and } \check{L}_T \triangleq \lim_{t\uparrow T} \check{L}_t.$$

Then we have that the pathwise $(-F')$-convex envelope $\check{Y}$ of $\tilde{Y}$ is given as,

$$\check{Y}_t \triangleq \int_t^T F'(s, \check{L}(s))\, ds \tag{36}$$

and it satisfies

$$supp(d\check{L}) \subset \{\check{Y}(t) = \tilde{Y}_*(t)\}$$

Here Y_* denotes the lower-semicontinuous envelope of $\tilde{Y}$. Moreover, the process $\check{L}$ is increasing and adapted. Furthermore, if the process L is pathwise upper-rightcontinuous then $\check{L}$ is right-continuous.

REMARK: The left-continuity of $\check{L}$ at the terminal time is a convention, convenient for the duality framework. Observe that the value of $\check{Y}$ is not influenced by the value of $\check{L}(T)$. In this corollary we take as given a specific realization of the process L and hence the support $supp(d\check{L})$ can be defined pathwise. The result of the corollary then follows directly from the arguments in the deterministic case. We also have the important

Proposition 5.7 *Let $\tilde{Y}$, $\check{Y}$ and $\check{L}$ be as in the above corollary and let $C^* = \check{L}$, then*

$$\langle C^*, \tilde{Y} \rangle = \langle C^*, \check{Y} \rangle.$$

PROOF : The first step is to show that the process $\tilde{Y}$ is lower-semicontinuous from the right. Observe that we may write

$$\tilde{Y}(s) = \int_0^T 1_{(s,T]}(t)\, F'(t, \sup_{v \in [s,t]} L(v))\, dt.$$

Because the integrand is positive, we my apply Fatou's lemma to obtain

$$\begin{aligned}\liminf_{s\downarrow u} \tilde{Y}(s) &\geq \int_0^T \liminf_{s\downarrow u} 1_{(s,T]}(t)\, F'(t, \sup_{v\in[s,t]} L(v))\, dt \\ &= \int_0^T 1_{(u,T]}(t)\, F'(t, \sup_{v\in(u,t]} L(v))\, dt \\ &\geq \int_0^T 1_{(u,T]}(t)\, F'(t, \sup_{v\in[u,t]} L(v))\, dt = \tilde{Y}(u).\end{aligned}$$

Approaching from the left, we have only a partial result: if $\check{L}(\cdot)$ has a discontinuity at t, i.e. if the measure $d\check{L}$ has an atom at t, then $\tilde{Y}(\cdot)$ is lower-semicontinuous at t. In order to prove this, we first observe that a discontinuity at t implies the strict inequality

$$L(t) = \sup_{v\in[0,t]} L(v) > \lim_{\varepsilon\downarrow 0} \sup_{v\in[0,t-\varepsilon]} L(v).$$

In otherwords, L is upper-semicontinuous from the left (and the right) at the point t: $L(t) > \limsup_{v\uparrow t} L_v$. We can now proceed as above to obtain

$$\begin{aligned}\liminf_{s\uparrow u} \tilde{Y}(s) &\geq \int_0^T \liminf_{s\uparrow u} 1_{(s,T]}(t)\, F'(t, \sup_{v\in[s,t]} L(v))\, dt \\ &= \int_0^T 1_{[u,T]}(t)\, F'(t, \lim_{\varepsilon\downarrow 0} \sup_{v\in[u-\varepsilon,t]} L(v))\, dt \\ &= \tilde{Y}(u).\end{aligned}$$

This suggests that it is helpful to write the measure $d\check{L}$ as a sum of a part $d\check{L}_\lambda$ that is absolutely continuous with respect to Lebesgue measure and a part $d\check{L}_s$ that is singular with respect to Lebesgue measure. In this notation, what we have shown so far is that

$$\int_0^T \check{Y}(t) d\check{L}_s(t) = \int_0^T \tilde{Y}_*(t) d\check{L}_s(t) = \int_0^T \tilde{Y}(t) d\check{L}_s(t).$$

The equality of the integrals with respect to $d\check{L}_\lambda$ follows from a theorem due to Young (Theorem 2 in [50]): the left and right limit inferior of a function of one real variable can differ at most countably many points. Because $\tilde{Y}$ is already lower-semicontinuous from the right, this shows that $\tilde{Y}(t) \neq \tilde{Y}_*(t)$ in at most countably many points. Hence

$$\int_0^T \check{Y}(t) d\check{L}_\lambda(t) = \int_0^T \tilde{Y}_*(t) d\check{L}_\lambda(t) = \int_0^T \tilde{Y}(t) d\check{L}_\lambda(t).$$

□

In the remainder of this section we discuss the importance of making a distinction between the deflator processes $\mathcal{Y}(y)$ defined by (26), and the set

$$\check{\mathcal{Y}}(y) \triangleq \Big\{ \check{Y}_t = \int_t^T F'(s, \check{L}(s)) ds \;\Big|\; \check{L} \text{ is adapted, increasing, and right-continuous, and } \exists Y \in \mathcal{Y}(y) \text{ s.t. } {}^\circ\check{Y}_t \leq Y_t \Big\} \tag{37}$$

Throughout we shall let $\mathcal{Y}$ denote the union $\bigcup_{y \in \mathbb{R}_+} \mathcal{Y}(y)$, and similarly for $\check{\mathcal{Y}}$. Also, we make one comment about notation. Although we have used the notation $\check{Y}$, it is not necessarily true that $\check{Y} \in \check{\mathcal{Y}}$ is the pathwise inhomogeneously convex envelope of some $Y \in \mathcal{Y}$. It is however, the pathwise inhomogeneously convex envelope of some $\tilde{Y}$ such that ${}^\circ\tilde{Y} \in \mathcal{Y}$.

In the Corollary 5.10 we will make clear how these sets relate to the original optimization problem (1). First, however, we mention a few theorems regarding integration with respect to optional random measures, an exposition of which is presented in notes by Bass [10].

The first step is to notice that every right-continuous, increasing process A defines a measure μ_A on $(\Omega \times [0,T], \mathbb{F}_T \otimes \mathcal{B}([0,T]))$ via $\mu_A(B) := \mathbb{E} \int_0^T 1_B(t,\omega) dA_t(\omega)$.

Also, define $\mu_A(X) := \mathbb{E}\int_0^T X_t dA_t$ for any measurable process X. Then we have the following existence result.

Theorem 5.8 (Theorem 5.1 in [10]) *If μ is a finite measure on $(\Omega\times[0,T], \mathbb{F}\otimes\mathcal{B}([0,T]))$ such that $\mu(X) = 0$ when $X \equiv 0$. Then there exists a (unique) increasing and right-continuous process A such that $\mu = \mu_A$.*

This construction does not require that the process A be adapted. However, if A is an optional process, then integration with respect to this measure has some nice properties:

Theorem 5.9 (Jacod [31], Theorem 1.33) *Let μ_A be a finite measure on $(\Omega\times[0,T], \mathbb{F}_T\otimes\mathcal{B}([0,T]))$ such that $\mu_A(X) = 0$ if $X \equiv 0$. Then for every bounded measurable process X and its optional projection ${}^\circ X$ we have that $\mu_A(X) = \mu_A({}^\circ X)$ if and only if A is optional.*

Observe that this measure is well defined, and that these results are valid, for any finite non-negative process X as well. In particular, if C is a consumption plan, then $\mathbb{E}\langle C, X\rangle = \mathbb{E}\langle C, {}^\circ X\rangle$ for an arbitrary process $X \geq 0$. In particular,

Corollary 5.10 *Fix $Y \in \mathcal{Y}$ and apply the representation theorem to obtain a (possibly non-adapted) process $\tilde{Y}$ such that ${}^\circ\tilde{Y} = Y$. Then following the construction in Corollary 5.6, we obtain a process $\check{Y} \in \check{\mathcal{Y}}$ such that*

$$\mathbb{E}V(\check{Y}) = \sup_{C\in\mathcal{C}} \mathbb{E}\left[U(C) - \langle C, \check{Y}\rangle\right] = \mathbb{E}\left[\sup_{h\in\mathcal{H}}[U(h) - \langle h, \check{Y}\rangle]\right], \tag{38}$$

and such that

$$\sup_{C\in\mathcal{C}} \mathbb{E}[U(C) - \langle C, \check{Y}\rangle] = \sup_{C\in\mathcal{C}} \mathbb{E}[U(C) - \langle C, Y\rangle] \tag{39}$$

Proof : Proposition 5.7 gives the pathwise equality

$$\arg\max_{h\in\mathcal{H}}\{U(h)-\langle h,\tilde{Y}\rangle\}=\arg\max_{h\in\mathcal{H}}\{U(h)-\langle h,\check{Y}\rangle\}$$

The supremum on the left hand side is achieved by the associated process $h^*(t) = \check{L}(t)$. Moreover, because $\check{L}$ is adapted, the functions $h^*(t)(\omega)$ that achieve the supremum on the right hand side in fact form an acceptable consumption plan $C \in \mathcal{C}$. This proves the first identity,

$$\mathbb{E}V(\check{Y})=\mathbb{E}\sup_{h\in\mathcal{H}}\left[U(h)-\langle h,\check{Y}\rangle\right]=\sup_{C\in\mathcal{C}}\mathbb{E}\left[U(C)-\langle C,\check{Y}\rangle\right].$$

We also have that

$$\mathbb{E}\left[\sup_{h\in\mathcal{H}}[U(h)-\langle h,\tilde{Y}\rangle]\right]=\mathbb{E}\left[\sup_{h\in\mathcal{H}}\left[U(h)-\langle h,\check{Y}\rangle\right]\right]=\sup_{C\in\mathcal{C}}\mathbb{E}\left[U(C)-\langle C,\tilde{Y}\rangle\right]$$

Combining this result with the fact that C is optional, gives us

$$\sup_{C\in\mathcal{C}}\mathbb{E}\left[U(C)-\langle C,\check{Y}\rangle\right]=\sup_{C\in\mathcal{C}}\mathbb{E}\left[U(C)-\langle C,Y\rangle\right].$$

□

Remark: In the above proof we have glossed over the technicality that the representation theorem applies to processes X such that $X(T) = 0$. Notice, however, that because $\mathbb{E}\langle C, Y\rangle \geq \mathbb{E}\langle C, Y1_{[0,T)}\rangle$ and because

$$\arg\max\left(\sup_{C\in\mathcal{C}}\mathbb{E}\left[U(C)-\langle C,Y1_{[0,T)}\rangle\right]\right)$$

includes (the maximizer is not unique) a consumption plan C^* that is left-continuous at T, we also have that

$$\sup_{C\in\mathcal{C}}\mathbb{E}\left[U(C)-\langle C,Y\rangle\right]=\sup_{C\in\mathcal{C}}\mathbb{E}\left[U(C)-\langle C,Y1_{[0,T)}\rangle\right].$$

Combining this corollary with the results in a deterministic setting, we have the upper bound

$$\sup_{C \in \mathcal{C}(x)} \mathbb{E}U(C) \leq \mathbb{E}V(\check{Y}) + yx, \quad \forall \check{Y} \in \check{\mathcal{Y}}(y).$$

We are thus led to consider the function

$$v(y) \triangleq \inf_{\check{Y} \in \check{\mathcal{Y}}(y)} \mathbb{E}V(\check{Y}) \tag{40}$$

The final goal will be to show that u and v satisfy a dual relationship (Theorem 6.3). In other words, we wish to show that there is no duality gap:

$$\sup_{C \in \mathcal{C}(x)} \mathbb{E}U(C) = \inf_{y \in \mathbb{R}_+} \inf_{\check{Y} \in \check{\mathcal{Y}}(y)} \mathbb{E}\{V(\check{Y}) + yx\}.$$

5.3 Dual Variables in Detail

In this section we try to collect some of the important properties of the dual variables that are needed for the application of minimax methods. As a first step, we note that the set $\mathcal{Y}(y)$ is convex. To prove the corresponding property for the sets $\check{\mathcal{Y}}(y)$, however, requires some additional work. Throughout we assume that Assumptions 2.1, 2.2 and 2.3 hold. They will be needed to prove that $\check{\mathcal{Y}}(y)$ is convex and Fatou closed.

5.3.1 Convexity

In this section we find necessary and sufficient conditions for the set $\check{\mathcal{Y}}(y)$ to be convex for all $y \in \mathbb{R}_+$. We finish the section with a discussion of examples and a derivative test for the necessary and sufficient conditions. First we prove an important

Lemma 5.11 *Let ψ be a C^1-function of two variables $\psi : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ that is strictly decreasing in the second variable. Then the function L defined implicitly by*

$$\psi(t, L_t) = \lambda\, \psi(t, L_t^1) + (1-\lambda)\, \psi(t, L_t^2). \tag{41}$$

is increasing and right continuous for all choices of increasing, right continuous functions L^1, L^2 and any $\lambda \in [0,1]$ if and only if $\psi_t(t,\cdot)$ is concave relative to $\psi(t,\cdot)$ (if $\psi_t \circ \psi^{-1}$ is concave in the usual sense on $\psi(t, \mathbb{R}_+)$.) In other words, if we let $u = \psi(t,b)$ and $v = \psi(t,c)$ then we require that for all $b, c \in \mathbb{R}_+$, $t \in [0,T]$, and $\lambda \in [0,1]$,

$$\lambda\, \psi_t(\psi^{-1}(t,u)) + (1-\lambda)\, \psi_t(\psi^{-1}(t,v)) \leq \psi_t\left(\psi^{-1}(t, \lambda u + (1-\lambda) v)\right), \tag{42}$$

where we have used the notation $\partial_t \psi = \psi_t$.

Remark: The concept of a relatively convex/concave function is standard, equivalent formulations and applications are discussed, for example, in [14], [43], and [42]. Observe, however, that because ψ is decreasing in the second variable, the image $\psi(t, \mathbb{R}_+)$ has an inverted order. Thus we find (see Proposition 5.13) that an equivalent condition for $\psi_t(t,\cdot)$ relatively concave to $\psi(t,\cdot)$ is that the relative difference quotients are increasing. If the second partial derivative ψ_{tx} exists, then this is equivalent to the fact that $\psi_{tx}(t,x)/\psi_x(t,x)$ is increasing in x for all $t \in [0,T]$.

Remark: For simplicity of notation, we let $\phi(t,\cdot) := \psi(t,\cdot)^{-1}$. The assumption that ψ is C^1 is needed to guarantee that the inverse ϕ and its time ϕ_t and spatial ϕ_x derivatives exist. In addition, with this assumption, we can apply the implicit function theorem to calculate these derivatives:

$$\phi_x(t,y) = \frac{1}{\psi_x(t,\phi(t,y))} \quad \text{and} \quad \phi_t(t,y) = -\frac{\psi_t(t,\phi(t,y))}{\psi_x(t,\phi(t,y))}.$$

Proof : Denote by $\mathcal{L}$ the set of all nondecreasing and right-continuous functions $L : [0,T] \to \mathbb{R}_+$. We need to show that if $L_1, L_2 \in \mathcal{L}$ then $L \in \mathcal{L}$ if and only if relation (42) holds.

First assume that the functions L^1 and L^2 are differentiable. With this assumption, the implicitly defined function L is also differentiable. At the end we show how to use approximations to obtain the proof in the general case.

($\Leftarrow$) To simplify notation, define y_t such that $L_t = \phi(t, y_t)$. Next, assuming that each of the functions L^1 and L^2 is differentiable, we calculate the derivative

$$\begin{aligned} dL_t \;&= \left[\phi_t(t,y_t) + \lambda\phi_x(t,y_t)\psi_t(t,L_t^1) + (1-\lambda)\phi_x(t,y_t)\psi_t(t,L_t^2)\right] dt \\ &\quad + \lambda\phi_x(t,y_t)\psi_x(t,L_t^1)dL_t^1 + (1-\lambda)\phi_x(t,y_t)\psi_x(t,L_t^2)dL_t^2. \end{aligned} \tag{43}$$

Observe that, because the function ψ is strictly decreasing,

$$\lambda\phi_x(t,y_t)\psi_x(t,L_t^1)dL_t^1 + (1-\lambda)\phi_x(t,y_t)\psi_x(t,L_t^2)dL_t^2 \geq 0. \tag{44}$$

In particular, we have that

$$dL_t \;\geq\; \left[\phi_t(t,y_t) + \lambda\phi_x(t,y_t)\psi_t(t,L_t^1) + (1-\lambda)\phi_x(t,y_t)\psi_t(t,L_t^2)\right] dt.$$

As a final step, we express all of the derivatives in terms of the derivatives of ψ to obtain,

$$dL_t \geq \left[-\frac{\psi_t(t,\phi(t,y_t))}{\psi_x(t,\phi(t,y_t))} + \lambda\frac{\psi_t(t,L_t^1)}{\psi_x(t,\phi(t,y_t))} + (1-\lambda)\frac{\psi_t(t,L_t^2)}{\psi_x(t,\phi(t,y_t))}\right] dt.$$

Recall that $\psi_x(t,\phi(t,y_t))$ is negative. Thus, if relation (42) holds, then $dL_t \geq 0$.

($\Rightarrow$) Returning to the calculation of the derivative of L in equation (43), we observe that if the functions L^1 and L^2 are constant, then the identity (44) is in fact an equality. In particular, for $L_t^1 = b$ and $L_t^2 = c$, we have that

$$\psi_x(t,\phi(t,y_t))dL_t = [-\psi_t(t,\phi(t,y_t)) + \lambda\psi_t(t,b) + (1-\lambda)\psi_t(t,c)]\,dt.$$

Assuming that $dL_t \geq 0$, we have the identity (42).

It remains to show that we can extend the arguments (for both directions of the implication) to the case when the functions L^1 and L^2 are not necessarily differentiable. To this end, let $\tilde{L} \in \mathcal{L}$, and define for each $n \in \mathbb{N}$,

$$\tilde{L}_n(s) \triangleq n \int_s^{s+\frac{1}{n}} \tilde{L}(t)\,dt.$$

Note that, because $\tilde{L}$ is increasing, both its left and right hand limits exist at each point. This is enough to show that the left and right hand derivatives of $\tilde{L}_n$ also exist for each $n \in \mathbb{N}$. Similarly, it is easy to see that right continuity implies

$$\lim_{n\to\infty} \tilde{L}_n(t) = \tilde{L}(t) \ , \ \ \forall t \in [0,T]$$

Furthermore, observe that equation (43) remains valid if the derivatives are replaced with left and right hand derivatives $D^{\pm}L(s)$ and $D^{\pm}L^i(s), i = 1,2$. We now have an approximating sequence of processes L_n corresponding to the approximations L_n^1 and L_n^2, for which the theorem holds. Because the pointwise limit of increasing functions is increasing, we have that the theorem holds for general processes $L^1, L^2 \in \mathcal{L}$ $\square$

An important consequence on this lemma is

Corollary 5.12 *If the felicity function F satisfies assumption (2.3), then the set $\check{\mathcal{Y}}(y)$ is convex for all y.*

Proof : Observe that the assumptions imply that the previous lemma holds for $\psi(t,x) = F'(t,x)$. What remains to show is that the relation (42) can be extended to a statement about processes. To this end, let $\check{Y}^1, \check{Y}^2 \in \check{\mathcal{Y}}(y)$ for a fixed $y \in \mathbb{R}_+$. Denote by L^1 and L^2 the adapted, increasing, right-continuous processes such that

$$\check{Y}^1_t = \int_t^T F'(s, L^1_s) ds, \qquad \check{Y}^2_t = \int_t^T F'(s, L^2_s) ds.$$

Then we must show that the process $\check{Y}_t := \lambda \check{Y}^1_t + (1-\lambda)\check{Y}^2_t$ is also an element of $\check{\mathcal{Y}}(y)$ for all $\lambda \in [0,1]$. Because optional projection is linear, ${}^\circ\check{Y}$ is dominated by some $Y \in \mathcal{Y}(y)$. Next we show that the process $\check{Y}$ is pathwise inhomogeneously convex. In particular, if we define the process L implicitly via

$$F'(s, L_s) = \lambda F'(s, L^1_s) + (1-\lambda) F'(s, L^2_s), \tag{45}$$

then

$$\check{Y}_t = \int_t^T F'(s, L_s) ds.$$

It remains to show that the process L is adapted, right-continuous, and increasing. Adaptivity and right-continuity are inherited from the processes L^1 and L^2 because F' is continuous. Finally, the previous lemma proves that, given the assumptions on F', the process L is also increasing.

Note that the result also holds if $L^1(t) = \infty$ or $L^2(t) = \infty$ for some $t \in [0,T]$. This follows from the Inada conditions by taking limits in Equation 41. □

The following proposition describes a relationship between the relation (42) and the usual notion of convexity. In particular, if ψ_t is differentiable in the second variable, it gives an easy derivative test for when the identity is true.

Proposition 5.13 *The function ψ, satisfying the assumptions of lemma (5.11), satisfies the inequality (42) if and only if the difference quotients are increasing (for $x < y < z$):*

$$\frac{\psi_t(t,x) - \psi_t(t,y)}{\psi(t,x) - \psi(t,y)} \leq \frac{\psi_t(t,x) - \psi_t(t,z)}{\psi(t,x) - \psi(t,z)} \leq \frac{\psi_t(t,y) - \psi_t(t,z)}{\psi(t,y) - \psi(t,z)} \qquad (46)$$

If the second partial derivative ψ_{tx} exists, then the above condition is equivalent to

$$\beta(t,x) \triangleq \frac{\psi_{tx}(t,x)}{\psi_x(t,x)} \text{ is increasing in } x \ \ \forall\ t \in [0,T].$$

Proof : (⇒) The proof of (46) is almost identical to the standard arguments for convex functions. In particular, if we let

$$\lambda = \frac{\psi(t,x) - \psi(t,y)}{\psi(t,x) - \psi(t,z)}$$

then

$$\psi(t,y) = (1-\lambda)\,\psi(t,x) + \lambda\,\psi(t,z).$$

The equivalence of the identities (42) and (46) follows from substituting for λ and rearranging terms. It remains to prove the second statement in the proposition.

We make a few calculations to prove the second statement.

(⇒) Observe that existence of the second partial derivative allows us to calculate β as the limit of difference quotients as follows,

$$\lim_{y \downarrow x} \frac{\dot\psi(t,x) - \dot\psi(t,y)}{\psi(t,x) - \psi(t,y)} = \lim_{y \downarrow x} \frac{\frac{\dot\psi(t,x) - \dot\psi(t,y)}{x-y}}{\frac{\psi(t,x) - \psi(t,y)}{x-y}}$$
$$= \frac{\psi_{tx}}{\psi_x} = \beta(t,x).$$

It is now easy to see that the inequalities (46) imply that $\beta(t,x)$ is increasing in x.

(⇐) An application of Cauchy's (extended) mean value theorem shows that if $\beta(t,\cdot)$ is increasing, then for $x < y < z$,

$$\frac{\psi_t(t,x) - \psi_t(t,y)}{\psi(t,x) - \psi(t,y)} \leq \frac{\psi_t(t,y) - \psi_t(t,z)}{\psi(t,y) - \psi(t,z)}.$$

Defining λ as before and rearranging terms, gives the identity in (42).

□

In the remainder of this section we discuss examples. In order to gain some intuition about the identity (42), we start by looking at a function for which it does not hold.

Example 1. Suppose that the felicity function is given as $F(t,x) := 1 - e^{-tx}$, then the set $\check{\mathcal{Y}}(y)$ is not convex. For this choice of felicity function, we have the following partial derivatives

$$F_x(t,x) = t\,e^{-tx} \qquad F_{xx}(t,x) = -t^2\,e^{-tx}$$
$$F_{xt}(t,x) = -tx\,e^{-tx} + e^{-tx} \quad F_{xtx}(t,x) = -2t\,e^{-tx} + t^2\,e^{-tx}$$

Hence the ratio

$$\frac{F_{xtx}}{F_{xx}} = \frac{2}{t} - x$$

is strictly decreasing for all $t \in [0, T]$. Figure 2 gives one example of what can go wrong.

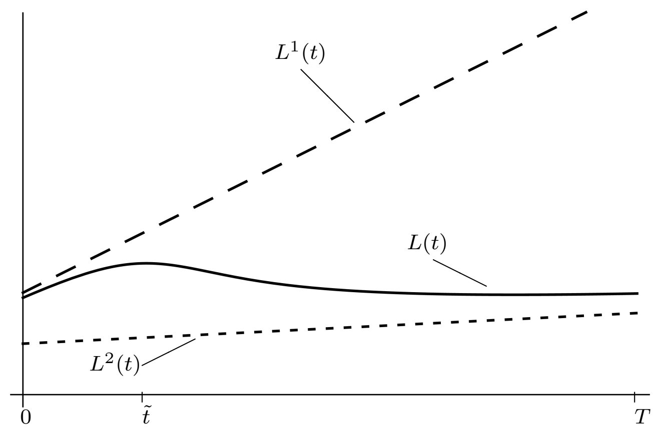

Figure 2: An example where the set of dual variables $\check{\mathcal{Y}}(y)$ is not convex. If the felicity function is given by $F(t, x) = 1 - e^{-tx}$, then $\check{\mathcal{Y}}(y)$ is not necessarily convex. In particular, if we let let $L^1(t) = 2 + t$ and $L^2(t) = 1 + 0.1t$, then the implicitly defined function L is increasing only up to a time $\tilde{t}$ and is decreasing from then onwards. Recall that L is defined by $F'(t, L(t)) = \lambda F'(t, L^1(t)) + (1 - \lambda)F'(t, L^2(t)))$, where for this image we have used $\lambda = 0.9$.

Remark: The derivative test allows us to construct counter examples as solutions to a simple PDE. We used this method to arrive at the example discussed above.

Example 2. If the felicity function is separable, in other words $F(t, x) = \theta(t)\tilde{u}(x)$, then the set $\check{\mathcal{Y}}(y)$ is convex. This is because the time dependency drops out in equation (41).

Example 3. In this example we allow for some time inhomogeneity. Suppose that $F(t,x) = e^{-t\alpha}\tilde{u}(e^{-t\gamma}x)$ for a utility function $\tilde{u}$ such that F satisfies the assumptions (2.1), (2.2) and (2.3) and for $\alpha, \gamma > 0$. If we assume further that $\tilde{u}$ is three times differentiable then,

$$\frac{F_{xtx}}{F_{xx}} = -(\alpha + 2\gamma) - \frac{\gamma x\, e^{-\gamma t}\tilde{u}'''(x\, e^{-\gamma t})}{\tilde{u}''(x\, e^{-\gamma t})}.$$

Thus, according to Corollary 5.12 and Proposition 5.13, the set $\check{\mathcal{Y}}(y)$ is convex if and only if the ratio $\frac{x\tilde{u}'''(x)}{\tilde{u}''(x)}$ is not increasing. For instance, if $\tilde{u}(x) = 1 - e^{-cx}$ for some $c > 0$, then

$$\frac{x\tilde{u}'''(x)}{\tilde{u}''(x)} = -cx.$$

This ratio is strictly decreasing and we have the desired convexity. Observe also that this is a more reasonable choice of a felicity function than the (similar) choice in example 1. In this example we also have exponential utility, but with a time-invariant risk aversion.

5.3.2 Fatou Closure

We next show that the set $\check{\mathcal{Y}}(y)$ is Fatou closed if the felicity function satisfies the Inada conditions (2.1) and the assumption of reasonable asymptotic elasticity (2.2). The result will follow from a series of lemmas identifying the structure of the Fatou limit. These lemmas will also be used later to prove existence in the dual problem.

First, we recall the definition of Fatou convergence in the setting of stochastic processes (see for example [36]). If $\{X^n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative, RCLL, adapted processes, then $\{X^n\}_{n\in\mathbb{N}}$ is **Fatou convergent** to an adapted RCLL process X if

$$X_t = \limsup_{s \downarrow t} \limsup_{n \to \infty} X_s^n$$
$$= \liminf_{s \downarrow t} \liminf_{n \to \infty} X_s^n$$

almost surely for all $t \in [0, T]$. A set A of RCLL, non-negative, adapted processes is said to be **Fatou closed** if whenever $(X_n)_{n \in \mathbb{N}}$ is a sequence in A Fatou convergent to some process X, then $X \in A$ also.

An important result that we will need is that the set $\mathcal{Y}(y)$ is Fatou closed. This theorem is proved by Žitković [51], Theorem 4. Our task in this section, is to show that the set $\check{\mathcal{Y}}(y)$ is also Fatou closed. Because processes in this set are decreasing, we can make use of a result which states that, for a dense subset, the Fatou limit is equivalent to an ordinary limit.

Lemma 5.14 (Lemma 4.2 [35]) *Let $(A^n)_{n \in \mathbb{N}}$ be a sequence of positive, decreasing processes on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Assume also that the collection of random variables $\operatorname{conv}((A_0^n)_{n \in \mathbb{N}})$ is bounded in probability. Then there exists a sequence $B^n \in conv(A^n, A^{n+1}, \ldots)$ and a $[0, \infty)$ valued decreasing process B such that*

$$B_t = \lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} B_{t+\varepsilon}^n = \lim_{\varepsilon \downarrow 0} \liminf_{n \to \infty} B_{t+\varepsilon}^n \tag{47}$$

for almost every $\omega \in \Omega$. Furthermore, there exists a countable dense subset $\mathcal{T} \subset [0, T]$ containing the set $\{0, T\}$ such that

$$B_t = \lim_{n \to \infty} B_t^n \ \forall t \in \mathcal{T}$$

Remark: Strictly speaking, Lemma 4.2 in [35] is about convergence of increasing processes. However, Lemma 5.14 is a straightforward modification for decreasing processes.

The characterization of the limit (47) will be used to show that the limit of pathwise inhomogeneously convex processes is also pathwise inhomogeneously convex ($\mathbb{P}$-a.s.). We can now state the main result of this section.

Theorem 5.15 *Let $(Z^n)_{n\in\mathbb{N}} \in \check{\mathcal{Y}}(y)$ be a sequence that is Fatou convergent to some process Z. Then $Z \in \check{\mathcal{Y}}(y)$ also.*

Remark: The proof of this theorem will proceed in several steps the goal of which is to construct a density $-F'(t, L_t)$ for Z and to show that L is increasing, right-continuous, and adapted. First, we restrict our attention to the individual paths $Z(\omega)$ for $\omega \in \Omega$ such that equation (47) holds. In what follows we suppress the ω from the notation and look at Z, L, Z^n, L^n as functions of time only.

We also use that the limit inferior and the limit superior of a sequence of increasing functions is increasing. We will use this fact repeatedly.

Lemma 5.16 *In the above notation, let Z denote the Fatou limit of a sequence $(Z^n)_{n\in\mathbb{N}} \in \check{\mathcal{Y}}(y)$, and let $L_{s,t}$ be the unique constant such that*

$$Z(s) - Z(t) = \int_s^t F'(u, L_{s,t})\, du$$

Then we have that

$$\lim_{r\downarrow s} \liminf_{n\to\infty} L^n_r \leq L_{s,t} \leq \limsup_{n\to\infty} L^n_t \tag{48}$$

PROOF : The definition of limit-superior and equation (47) imply

$$Z(s) - Z(t) = \lim_{\varepsilon \downarrow 0} \left(\limsup_{n\to\infty} Z^n_{s+\varepsilon} - \limsup_{n\to\infty} Z^n_{t+\varepsilon} \right) \leq \lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \left(Z^n_{s+\varepsilon} - Z^n_{t+\varepsilon} \right).$$

Similar calculations can be made to show the opposite inequality for limit inferiors. Now supposing that we may apply Fatou's lemma to both functions F' and $(-F')$, then

$$\lim_{\varepsilon\to 0} \int_{s+\varepsilon}^{t+\varepsilon} \liminf_{n\to\infty} F'(u, L^n_u)\, du \leq \int_s^t F'(u, L_{s,t})\, du \leq \lim_{\varepsilon\to 0} \int_{s+\varepsilon}^{t+\varepsilon} \limsup_{n\to\infty} F'(u, L^n_u)\, du$$

Continuing these calculations, we obtain the further relation

$$\begin{aligned} \int_s^t F'(u, L_{s,t})\, du &\geq \lim_{\varepsilon\to 0} \int_{s+\varepsilon}^{t+\varepsilon} \liminf_{n\to\infty} F'(u, L^n_u)\, du \\ &= \int_s^t \liminf_{n\to\infty} F'(u, L^n_u)\, du \\ &\geq \int_s^t F'(u, \limsup_{n\to\infty} L^n_t)\, du. \end{aligned}$$

In these calculations, the first equality follows from the fact that the Lebesgue measure is atomless. In the last inequality we have removed the time dependence from the second variable. We can do this because $F(t, \cdot)$ is decreasing while each process $L^n(t)$ is increasing in t. From these calculations, we can conclude that

$$L_{s,t} \leq \limsup_{n\to\infty} L^n_t.$$

The proof of the second inequality is similar. In this case, however, we note that the stronger identity

$$\int_s^t F'(u, \liminf_{n\to\infty} L^n_u)\, du \leq \int_s^t F'(u, \lim_{r\downarrow u} \liminf_{n\to\infty} L^n_r)\, du$$

also holds. The last step of the proof is to justify the use of Fatou's lemma. Because the felicity function is increasing, $F'(t, x)$ is non-negative and hence bounded from below. It remains to show that it is also bounded from above.

Because we are applying Fatou's lemma to the integral over $[s+\varepsilon, t+\varepsilon]$ it is enough to prove that $(F'(t, L_t^n))_{t\in[t_1,t_2]}$ is bounded for $t_1 \neq 0$.

We may assume, without loss of generality, that at $t = 0$ the Fatou limit of processes is in fact a genuine limit: $\lim_{n\to\infty} Z_0^n = Z_0$. Thus, for a given $\varepsilon > 0$ and large enough n, we have that $Z_0 + \varepsilon > Z_0^n$. Because each of these processes is decreasing, we obtain the bound

$$M := Z_0 + \varepsilon \geq Z_0^n - Z_s^n = \int_0^s F'(t, L_t)\, dt \geq s\, F'(\bar{t}, L_s)$$

The special time $\bar{t} \in [0, s]$ comes from an application of the mean value theorem for integrals. We have also used the fact that function $L(\cdot)$ is increasing.

Because the felicity function is strictly concave and continuously differentiable, the inverse of the derivative exists, it is decreasing, and continuous. We denote this inverse by $\phi(t, \cdot) = (F'(t, \cdot))^{-1}$. Because we have assumed that F' is C^1, the inverse ϕ is continuous in both variables. It is also strictly greater than zero on the compact interval $[0, s]$. These facts imply the bounds

$$L_s \geq \phi\left(\bar{t}, \frac{M}{s}\right) \geq \min_{t\in[0,s]} \phi\left(t, \frac{M}{s}\right) = \varepsilon(s) > 0.$$

For $t \in [t_1, t_2]$, the bound above gives

$$F'(t, L_t^n) \leq F'(t,^n L_{t_1}) \leq F'(t, \varepsilon(t_1)) \quad \text{for all } t \in [t_1, t_2].$$

The result now follows from the continuity of the function $F'(\cdot, x)$ and the compactness of the interval $[t_1, t_2]$. □

The next step in the proof of the theorem is a characterization of the density of the limiting process Z.

Lemma 5.17 *In the setting of lemma (5.16) we also have that,*

$$\lim_{t \downarrow t_0} \liminf_{n \to \infty} L_t^n \geq \limsup_{n \to \infty} L_{t_0}^n. \tag{49}$$

Furthermore, $L_{s,t} \leq L_{t,u}$ *if* $s < t < u$ *and the limit* Z *is* $(-F')$*-convex. Define*

$$\check{L}_t \triangleq \lim_{s \downarrow t} L_{t,s}. \tag{50}$$

Then $(-F'(t, \check{L}(t)))$ *is a version of the density of* Z *and* $\check{L}$ *is right-continuous and increasing.*

Proof : Combining the results of equations (48) and (49) shows that $L_{s,t} \leq L_{t,u}$ if $s < t < u$. In particular,

$$L_{s,t} \leq \limsup_{n \to \infty} L_t^n \leq \lim_{r \downarrow t} \liminf_{n \to \infty} L_r^n \leq L_{t,u}.$$

According to lemma (4.2), this is equivalent to inhomogeneous convexity. Lemma (4.3) shows that the inhomogeneously convex function $Z(\omega, \cdot)$ has a density, and that a version of this density is given by the equation (50). Right-continuity follows from the definition.

It remains to prove the first statement (49). We first treat the case when $s, t \in \mathcal{T}$. For these time points the limit, $Z_s - Z_t = \lim_{n \to \infty} \int_s^t F'(u, L_u^n)\, du$ exists. Furthermore, the mean value theorem for integrals combined with the fact that the functions $L^n(\cdot)$ are increasing, gives the following bounds

$$(t-s)F'(t_*, L_t^n) \leq \int_s^t F'(u, L_u^n)\, du \tag{51}$$

$$(t-s)F'(t^*, L_s^n) \geq \int_s^t F'(u, L_u^n)\, du \tag{52}$$

for some $t_*, t^* \in [s,t]$ which may depend on n. As the next step, we take limits in equations (51) and (52) (limit-superior and limit-inferior respectively);

$$(t-s)\limsup_{n\to\infty} F'(t_*(n), L_t^n) \leq \lim_{n\to\infty}\int_s^t F'(u, L_u^n)\,du,$$
$$(t-s)\liminf_{n\to\infty} F'(t^*(n), L_t^n) \geq \lim_{n\to\infty}\int_s^t F'(u, L_u^n)\,du.$$

The next estimate allows us to remove the dependence on n of the time points $t^*(n)$, and $t_*(n)$:

$$\limsup_{n\to\infty} F'(t^*(n), L_t^n) \geq \inf_{u\in[s,t]} F'(u, \liminf_{n\to\infty} L_t^n),$$
$$\liminf_{n\to\infty} F'(t_*(n), L_s^n) \leq \sup_{u\in[s,t]} F'(u, \limsup_{n\to\infty} L_s^n).$$

Combining these results, we obtain

$$\inf_{u\in[s,t]} F'(u, \liminf_{n\to\infty} L_t^n) \leq \sup_{u\in[s,t]} F'(u, \limsup_{n\to\infty} L_s^n).$$

As a last step, we observe that, because $F'(\cdot,\cdot)$ is continuous in both the time and space variable, if we take the limit as t approaches s (along $t_n \in \mathcal{T}$ if you like), then we obtain

$$F'(s, \limsup_{n\to\infty} L_s^n) \geq F'(s, \lim_{t\downarrow s}\liminf_{n\to\infty} L_t^n).$$

The proof for arbitrary $s,t \in [0,T]$ follows the same reasoning, but is notationally more complex. We omit the details, but note that repeatedly applying equation (47) shows that

$$Z(s) - Z(t) = \lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\int_{s+\varepsilon}^{t+\varepsilon} F'(u, L_u^n)\,du = \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{s+\varepsilon}^{t+\varepsilon} F'(u, L_u^n)\,du.$$

We use this identity, instead of the existence of the limit, $\lim_{n\to\infty}\int_s^t F'(u, L_u^n)\,du$, to relate the estimates on $\liminf$ and $\limsup$. $\square$

Proof of Theorem 5.15 Returning for a moment to consider $Z(\omega, t)$ as a process, note that we have shown that the pathwise limit $Z(\omega, t)$ is inhomogeneously convex for almost every $\omega \in \Omega$ (when the sequence $(Z^n)_{n\in\mathbb{N}}$ converges as processes in the sense of Fatou). Furthermore, because the filtration is assumed right-continuous, the definition of $\check{L}$ in Lemma 5.17 shows that it is adapted if for each n the processes L^n are adapted.

The remaining step in the proof of the Theorem 5.15 is to show that $Z \in \check{\mathcal{Y}}(y)$ for the particular y. From the definition, we easily see that for each $\check{Z}^n$ there exists a non-negative supermartingale $\tilde{Y}^n \in \mathcal{Y}(y)$ such that $\tilde{Y}^n \geq {}^\circ(\check{Z}^n)$. Lemma 4.2 in [36] shows that there exists a sequence of convex combinations $Y^n \in \mathrm{conv}(\tilde{Y}^n, \tilde{Y}^{n+1}, \dots)$ that is Fatou convergent to a supermartingale Y. In fact, $Y \in \mathcal{Y}(y)$ because the set $\mathcal{Y}(y)$ is Fatou closed (Theorem 4 [51]). Observe also that if we take further convex combinations of the processes $\check{Z}^n$ then the limit is not altered, abusing notation call this sequence also $(\check{Z}^n)_{n\in\mathbb{N}}$.

The optional projection ${}^\circ(\check{Z}^n_t)$ is the unique optional process such that ${}^\circ(\check{Z}^n_T) = \mathbb{E}[(\check{Z}^n_T)\,1_{\{T<\infty\}}\,|\,\mathbb{F}_T]$, for all stopping times T. Combined with Fatou's lemma,

$$\begin{aligned} Y_t = \liminf_{s\downarrow t} \liminf_{n\to\infty} Y^n_s &\geq \liminf_{s\downarrow t} \liminf_{n\to\infty} \mathbb{E}[\check{Z}^n_s\,|\,\mathbb{F}_t] \\ &\geq \liminf_{s\downarrow t} \mathbb{E}\left[\liminf_{n\to\infty} \check{Z}^n_s \,\middle|\, \mathbb{F}_t\right] \end{aligned}$$

Next, let $\tilde{Z}_t := \liminf_{n\to\infty} \check{Z}^n_t$. The choice of our subsequence and Lemma 5.14 imply that $\liminf_{s\downarrow t} \tilde{Z}_s = \lim_{s\downarrow t} \tilde{Z}_s = Z_t$. In particular, the limit exists and we can use Lévy's convergence result;

$$\lim_{s\downarrow t} \mathbb{E}[\tilde{Z}_s\,|\,\mathbb{F}_t] = \mathbb{E}[Z_t\,|\,\mathbb{F}_t]$$

Combining this with the previous results, we obtain $Y \geq {}^\circ Z$. In particular, the limit $Z \in \check{\mathcal{Y}}(y)$. □

Remark: In the above argument it suffices to show that $\mathbb{P}(Y_t \geq (^\circ Z)_t) = 1$ for all $t \in [0,T]$ because both processes are right-continuous. Right-continuity of $Z \in \check{\mathcal{Y}}$ follows from Theorem 47 in [22], which states that the optional projection of a right-continuous process is also right-continuous.

An important corollary of this discussion is a characterization of the (stochastic) density of the limiting process $\check{Y}$.

Lemma 5.18 *Let $(\check{Y}^n)_{n\in\mathbb{N}} \in \check{\mathcal{Y}}(y)$ beFatou convergent to some $\check{Y} \in \check{\mathcal{Y}}(y)$ and denote by $(L^n)_{n\in\mathbb{N}}$ and L the right-continuous, adapted processes describing their densities. Then the limit $\lim_{n\to\infty} L^n(\omega,t)$ exists for almost every $\omega \in \Omega$ at every continuity point t of $\check{L}(\omega,\cdot)$. In particular,*

$$\liminf_{n\to\infty} L^n(\omega,t) = \limsup_{n\to\infty} L^n(\omega,t) = \check{L}(\omega,t)$$

at all but at countably many times $t \in [0,T]$. This countable set may depend on $\omega \in \Omega$

Proof : Fix $\omega \in \Omega$ such that (47) holds (this is true a.s). For this choice of ω the results of Lemmas 48 and 49 hold for the functions $\{\check{L}^n(\omega,\cdot)\}_{n\in\mathbb{N}}$ and $\check{L}(\omega,\cdot)$. Now let t be a point of continuity of $\check{L}(\omega,\cdot)$ and pick $s < t$ (we may assume that $t \neq 0, T$). Equation (48) shows that for all $r > s$, $L_{s,r} \leq \limsup_n L^n_r$. Let $r \downarrow s$ to show (we suppress ω from the notation from now on),

$$\check{L}_s \leq \lim_{r\downarrow s} \limsup_n L^n_r \leq \liminf_{n\to\infty} L^n_t.$$

The last inequality is obtained from (49). From these same equations, we can also conclude that for all $v > t$, $\limsup_{n\to\infty} L^n_t \leq L_{t,v}$. Taking the limit we obtain,

$$\limsup_{n\to\infty} L_t^n \le \lim_{v\downarrow t} L_{t,v} = \check{L}_t.$$

This gives the sequence of inequalities

$$\check{L}_s \le \liminf_{n\to\infty} L_t^n \le \limsup_{n\to\infty} L_t^n \le \check{L}_t.$$

Now let $s \uparrow t$ and observe that because t is a point of continuity, we must in fact have equality: $\liminf_{n\to\infty} L_t^n = \limsup_{n\to\infty} L_t^n = \check{L}_t$. □

5.3.3 Existence in the Dual Problem

An important consequence of the characterization of the Fatou limit is existence in the dual problem. This fact is a direct corollary of the following theorem.

Theorem 5.19 *Let $\{\check{Y}^n\}_{n\in\mathbb{N}} \in \check{\mathcal{Y}}(y)$ be a sequence that is Fatou convergent to a process $\check{Y} \in \check{\mathcal{Y}}(y)$. Then we have*

$$\liminf_{n\to\infty} \mathbb{E}V(\check{Y}^n) \ge \mathbb{E}V(\check{Y})$$

PROOF : Now assume that the processes $(L^n)_{n\in\mathbb{N}}$ and $\check{L}$ are finite valued. Then we may apply Fatou's lemma to show

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}V(\check{Y}^n) &= \liminf_{n\to\infty} \mathbb{E}\int_0^T (F(t, L_t^n) - F'(t, L_t^n)L_t^n)\, dt \\ &\ge \mathbb{E}\int_0^T \left(F(t, \liminf_{n\to\infty} L_t^n) - F'(t, \liminf_{n\to\infty} L_t^n)\liminf_{n\to\infty} L_t^n\right) dt.\end{aligned}$$

We have also used the fact that the function $G(t,x) := F(t,x) - F'(t,x)x$ is increasing in its second variable. In fact, $G'(t,x) = -F''(t,x)x$ which is positive because the felicity function is concave. The key fact used is Lemma 5.18. It shows that the Lebesgue integral is not changed if the limit-inferior is replaced with $\check{L}$. In particular,

$$\liminf_{n\to\infty} \mathbb{E}V(\check{Y}^n) \geq \mathbb{E}\int_0^T \left(F(t,\check{L}_t) - F'(t,\check{L}_t)\check{L}_t\right)\, dt = \mathbb{E}V(\check{Y}).$$

The argument for infinite valued processes requires only a few extra steps. This is because the definition of the conjugate functional $V(\cdot)$ now splits into two parts. In order to describe it, we define the stopping time $\tau_* := \inf\{t > 0 \,|\, \check{L}_t = \infty\}$ and observe that because $\check{L}$ is right-continuous, $\check{L}_{\tau_*} = \infty$. Thus,

$$\mathbb{E}V(\check{Y}) = \mathbb{E}\int_0^T \left(F(t,\check{L}_t) - F'(t,\check{L}_t)\check{L}_t\right) 1_{\{t<\tau_*\}} + F(t,\infty)1_{\{t\geq\tau_*\}}\, dt. \tag{53}$$

We first consider the case when $\check{L}_n(T) < \infty$ for all n large enough. In this part we will need the assumption of reasonable asymptotic elasticity (2.2): the asymptotic elasticity $AEF(t,\cdot)$ is less than one for $t \in [0,T]$. Furthermore, observe that if the felicity function is bounded, $F(t,\infty) = M(t) < \infty$, then the asymptotic elasticity is zero. This fact was proven in the proof of Lemma 4.9. With this assumption, Fatou's lemma gives,

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{E}V(\check{Y}^n) &= \liminf_{n\to\infty} \mathbb{E}\int_0^T F(t,L_t^n) - F'(t,L_t^n)L_t^n\, dt \\
&\geq \mathbb{E}\int_0^T \liminf_{n\to\infty}\left(1 - \frac{F'(t,L_t^n)L_t^n}{F(t,L_t^n)}\right) F(t,L_t^n)\, dt \\
&= \int_0^T \left(F(t,\check{L}_t) - F'(t,\check{L}_t)\check{L}_t\right) 1_{\{t<\tau_*\}} \\
&\qquad + (1 - AE\, F(t,\cdot))F(t,\infty)1_{\{t\geq\tau_*\}}\, dt.
\end{aligned}$$

The second to last equality follows because the limit-inferior is equivalent to $\check{L}$ at all but countably many points. In particular, because $AE < 1$, the last expression is infinite if the felicity function is unbounded. Comparing with equation (53), we observe that the same is true of $\mathbb{E}V(\check{Y})$. If, on the other hand, the felicity function is bounded, then also $AE = 0$ and the second term is just $1 \cdot F(t, \infty) = M(t)$. Again the result follows.

From this proof, we observe that the result is in fact true for arbitrary $\check{L}_n$, finite or possibly taking infinite values, because

$$\liminf_{x\to\infty} \left(1 - \frac{F'(t,x)x}{F(t,x)}\right) F(t,x) = F(t,\infty).$$

To calculate the limit inferior of the product, we have used the fact that $\lim_{x\to\infty} F(t,x)$ exists. □

REMARK: If the increasing processes $\check{L}_n$ are uniformly bounded, then we have equality:

$$\lim_{n\to\infty} \mathbb{E}V(\check{Y}^n) = \mathbb{E}V(\check{Y}).$$

This theorem has an important

Corollary 5.20 (Existence in the dual problem) *Let the dual* v *to the value function be defined as*

$$v(y) \triangleq \inf_{\check{Y}\in\check{\mathcal{Y}}(y)} \mathbb{E}V(\check{Y})$$

Then there exists a process $\check{Y}^* \in \check{\mathcal{Y}}(y)$ *that achieves this infimum. In addition, this solution is unique if* $v(y) < \infty$.

PROOF : Let $(\check{Y}^n)_{n\in\mathbb{N}} \in \check{\mathcal{Y}}(y)$ be a sequence of processes such that $v(y) = \lim_n \mathbb{E}V(\check{Y}^n)$. According to lemmas (5.14) and theorem (5.15) there exists a sequence of convex combinations $Z^n \in \mathrm{conv}(\check{Y}^n, \check{Y}^{n+1}, \ldots)$ that converges to some process $Z \in \check{\mathcal{Y}}(y)$. Because the functional V is convex, $\lim_n \mathbb{E}V(\check{Y}^n) \geq \liminf_{n\to\infty} \mathbb{E}V(Z^n)$. Furthermore, we have just shown that $\liminf \mathbb{E}V(Z^n) \geq \mathbb{E}V(Z)$. Because the sequence $\mathbb{E}V(\check{Y}^n)_{n\in\mathbb{N}}$ approaches the infimum $v(y)$, we must have that $v(y) = \mathbb{E}V(Z)$, in particular, the infimum is attained. This solution is unique because V is strictly convex for $\check{Y} \in \check{\mathcal{Y}}$ and $v(y) < \infty$. $\square$

6 Proof of the Duality Theorem

We are now ready to prove a dual characterization of the optimal consumption problem. In particular, we establish a relationship between the optimal solutions C^* and $\check{Y}^*$ of the primal and dual problems respectively. This result is Theorem 6.3. Furthermore, in Theorem 6.5 we will show how things simplify in a complete market.

The first step is to apply a minimax theorem to prove a relationship between the primal and dual functions ($u(x)$ and $v(y)$ respectively.) The general structure of this proof is inspired by the proof of Lemma 3.4 in [36]. The use of these techniques is possible due to our choice of dual variables and the analytic properties satisfied by the space of consumption plans.

Theorem 6.1 *The value functions u and v (defined in (1) and (40) respectively) are conjugate in the sense that*

$$v(y) = \sup_{x>0}[u(x) - xy] \tag{54}$$

for all $y \in \mathbb{R}_+$.

We apply the minimax theorem to show that

$$\sup_{C\in\mathcal{C}_n} \inf_{Y\in\mathcal{Y}(y)} \mathbb{E}[U(C) - \langle C, Y\rangle] = \inf_{Y\in\mathcal{Y}(y)} \sup_{C\in\mathcal{C}_n} \mathbb{E}[U(C) - \langle C, Y\rangle], \tag{55}$$

where $\mathcal{C}_n = \{C \in \mathcal{C} \mid \ C(T) \leq n\}$. The version of the theorem we use is proved in the appendix.

Theorem 6.2 (Minimax) *Let A be a nonempty convex subset of a topological vector space, and B a nonempty, closed, convex, and convexly compact subset of a topological vector space. Let $h : A \times B \to \mathbb{R}$ be convex on A, and concave and upper-semicontinuous on B. Then*

$$\inf_A \sup_B h = \sup_B \inf_A h.$$

Proof of Theorem 6.1 In our application, we take A to be the set of dual variables $\mathcal{Y}(y)$ which is convex. Also, let B be the set of bounded consumption plans $\mathcal{C}_n$ for some $n > 0$. It is clear that $\mathcal{C}_n$ is closed and convex. According to Theorem 3.3 it is also convexly compact. Finally, for the function h we take the mapping $(C, Y) \mapsto \mathbb{E}[U(C) - \langle C, Y \rangle]$. Because the bracket is bilinear, the map is clearly convex in the Y variable. It is also concave in C because the utility function is concave. Furthermore, combining the results of Lemmas 3.5 and 3.4 we see that it is also upper-semicontinuous. This result follows because $Y \in \mathcal{Y}(y)$ is the optional projection of something with lower-semicontinuous paths. Thus we know that the relation (55) indeed holds for all $n \in \mathbb{N}$.

As a next step, we look at the limit as n tends to infinity. On the left hand side we calculate

$$\begin{aligned}
\lim_{n\to\infty} \sup_{C\in\mathcal{C}_n} \inf_{Y\in\mathcal{Y}(y)} \mathbb{E}[U(C) - \langle C, Y \rangle] &= \lim_{n\to\infty} \sup_{C\in\mathcal{C}_n} [\mathbb{E}U(C) - \sup_{Y\in\mathcal{Y}(y)} \mathbb{E}\langle C, Y \rangle] \\
&= \sup_{C\,\text{bounded}} [\mathbb{E}U(C) - \sup_{Y\in\mathcal{Y}(y)} \mathbb{E}\langle C, Y \rangle] \\
&= \sup_{x>0} \sup_{C,\pi_y(C)=xy} [\mathbb{E}U(C) - \sup_{Y\in\mathcal{Y}(y)} \mathbb{E}\langle C, Y \rangle] \\
&= \sup_{x>0} [u(x) - xy].
\end{aligned}$$

Some explanation of the validity of these statements is in order. First, observe that in the third equality, we have removed the assumption that the consumption plan is bounded. However, because all admissible plans are almost surely finite, each plan can be approximated by a monotonically increasing sequence of truncated processes. A more critical step is the classification of the consumption plans by their price

$$\pi_y(C) = \sup_{Y \in \mathcal{Y}(y)} \mathbb{E}\langle C, Y \rangle.$$

This step is justified because the constraint set $\mathcal{Y}(y)$ scales linearly in y. In other words, because $\mathcal{Y}(y) = y\mathcal{Y}(1)$, the budget constraint can be written in terms of any of the sets $\mathcal{Y}(y)$:

$$C \in \mathcal{C}(x) \text{ iff } \mathbb{E}\langle C, Y \rangle \leq xy \;\; \forall Y \in \mathcal{Y}(y).$$

We next treat the right hand side. For this, we will need our earlier observations about integrating with respect to an optional random measure. In particular, Corollary 5.10 justifies the equality

$$\lim_{n\to\infty} \inf_{Y \in \mathcal{Y}(y)} \sup_{C \in \mathcal{C}_n} \mathbb{E}[U(C) - \langle C, Y \rangle] = \lim_{n\to\infty} \inf_{\check{Y} \in \check{\mathcal{Y}}(y)} \sup_{C \in \mathcal{C}_n} \mathbb{E}[U(C) - \langle C, \check{Y} \rangle].$$

This equality shows that we can do the rest of our analysis with the variables $\check{Y} \in \check{\mathcal{Y}}(y)$ instead.

As preparation for the next step, we make a few simplifying definitions. First, we define an approximation to the conjugate functional V:

$$V_n(k) \triangleq \sup_{h \in \mathcal{H},\, h \leq n} [U(h) - \langle h, k \rangle].$$

In Lemma 4.4 we showed that $\arg\max(\sup_{h\in\mathcal{H}} U(h) - \langle h, \check{k} \rangle) = \check{l}$ from the concavity of $F(t, \cdot)$ at each point in time. Similarly we can show that the supremum for the truncated problem, when $h \in \mathcal{H}, h \leq n$, is achieved at $\check{l} \wedge n$. In particular, the optimum is adapted and we can conclude that

$$\mathbb{E}V_n(\check{Y}) = \mathbb{E}\left[\sup_{h \in \mathcal{H},\, h \leq n} [U(h) - \langle h, \check{Y} \rangle]\right] = \sup_{C \in \mathcal{C}_n} \mathbb{E}[U(C) - \langle C, \check{Y} \rangle].$$

Similarly we define the dual function for this truncated problem,

$$v_n(y) \triangleq \inf_{\check{Y} \in \check{\mathcal{Y}}(y)} \mathbb{E}V_n(\check{Y}).$$

Thus it remains to show that

$$\lim_{n \to \infty} v_n(y) = v(y).$$

Because $\mathcal{C}_n \subset \mathcal{C}$, we know that $\mathbb{E}V_n(\check{Y}) \leq \mathbb{E}V(\check{Y})$. Consequently, $v_n(y) \leq v(y)$ also. Furthermore, because the sequence $v_n(y)$ is increasing in n, we know that it converges. In order to relate this limit to the dual function v, we first choose a subsequence $\{\check{Y}^n\}_{n \in \mathbb{N}} \in \check{\mathcal{Y}}(y)$ such that

$$\lim_{n \to \infty} \mathbb{E}V_n(\check{Y}^n) = \lim_{n \to \infty} v_n(y).$$

From Lemma 5.14 we know that there exists a sequence

$$\check{Z}^n \in \operatorname{conv}(\check{Y}^n, \check{Y}^{n+1}, \check{Y}^{n+2}, \ldots)$$

that converges to some process $\check{Z}$. Furthermore, Theorem 5.15 shows that $\check{Z} \in \check{\mathcal{Y}}(y)$.

Observe that because the functionals V_n are convex and increasing in n,

$$\mathbb{E}V_n(\check{Z}^n) = \mathbb{E}V_n\left(\sum_{i=n}^{N} \lambda_i \check{Y}^i\right) \leq \sum_{i=n}^{N} \lambda_i \mathbb{E}V_n(\check{Y}^i) \leq \max_{i=n,\ldots N} \mathbb{E}V_n(\check{Y}^i) \leq \mathbb{E}V_{i^*}(\check{Y}^{i^*}),$$

where i^* is some index greater than or equal to n. Thus, renaming indices and taking limit inferiors (on the right hand side, this is a genuine limit by the choice of the sequence $(\check{Y}^n)_{n\in\mathbb{N}}$), we obtain

$$\liminf_{n\to\infty} \mathbb{E}V_n(\check{Z}^n) \le \liminf_{n\to\infty} \mathbb{E}V_n(\check{Y}^n).$$

The last step is to show that

$$\liminf_{n\to\infty} \mathbb{E}V_n(\check{Z}_n) \ge \mathbb{E}V(\check{Z}). \tag{56}$$

Because $\check{Z} \in \check{\mathcal{Y}}(y)$, we also have that $\mathbb{E}V(\check{Z}) \ge v(y)$ and hence in fact must have equality. The proof of the inequality (56) follows ideas similar to those used to prove that $\mathbb{E}V(\cdot)$ is lower-semicontinuous (Theorem 5.19). The key part in this proof is the characterization of the density of the limit $\check{Z}$ in Lemma 5.18. In particular, this description can be used to show that the approximate functionals $\mathbb{E}V_n(\cdot)$ are also lower-semicontinuous. Given this fact, we know that for any fixed N,

$$\liminf_{n\to\infty} \mathbb{E}V_n(\check{Z}^n) \ge \liminf_{n\to\infty} \mathbb{E}V_N(\check{Z}^n) \ge \mathbb{E}V_N(\check{Z}).$$

Furthermore, because V_N is increasing in N, the limit $\lim_{N\to\infty} \mathbb{E}V_N(\check{Z})$ exists. In the setting of the proof of Theorem 5.19, we can use monotone convergence to show

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}V_N(\check{Z}) &= \mathbb{E}\int_0^T \lim_{N\to\infty} \big(F(t,\check{L}_t \wedge N) - F'(t,\check{L}_t)(\check{L}_t \wedge N)\big)\, 1_{\{t<\tau_*\}} \\ &\qquad + F(t,\infty) 1_{\{t\ge\tau_*\}}\, dt \\ &= \mathbb{E}V(\check{Z}),\end{aligned}$$

where, as before, we define $\tau_* := \inf\{t > 0 \,|\, \check{L}_t = \infty\}$ and $\check{L}$ is such that $\check{Z}_t = \int_t^T F'(s,\check{L}_s)ds$. □

The important consequence of this theorem is

Theorem 6.3 *(i) The indirect utility $u(x)$ is continuously differentiable on $(0, \infty)$.*

(ii)Let x, $y \in \mathbb{R}_+$ be such that $u'(x) = y$. Then the solutions to the primal and dual problems are related in the following way: if $C^*(x)$ is such that

$$u(x) = \sup_{C \in \mathcal{C}(x)} \mathbb{E}U(C) = \mathbb{E}U(C^*(x))$$

and $\check{Y}^*(y) \in \check{\mathcal{Y}}(y)$ satisfies

$$v(y) = \inf_{\check{Y} \in \check{\mathcal{Y}}(y)} \mathbb{E}V(\check{Y}) = \mathbb{E}V(\check{Y}^*(y))$$

then, almost surely, we have that,

$$\nabla V(\check{Y}^*(y)) = -C^*(x)$$

and

$$\nabla U(C^*(x)) = \check{Y}^*(y).$$

In particular, the representation $\check{Y}^*_t = \int_t^T F'(s, \check{L}^*(s))ds$ proves that

$$\check{L}^*(y) = C^*(x).$$

Remark: Because the process $\check{L}$ is left continuous at the terminal time T, the theorem shows that the optimal consumption plan has no gulp at the terminal time (as expected).

Proof : The basic idea of the proof of (ii) is to notice that for each $\check{Y} \in \check{Y}(y)$, the function $V(\check{Y}) + \langle C^*(x), \check{Y} \rangle$ provides an upperbound to $U(C^*(x))$ (for all $\omega \in \Omega$). This fact and the budget constraint prove

$$\begin{aligned}\mathbb{E}[|V(\check{Y}^*(y)) + \langle C^*(x), \check{Y}^*(y)\rangle - U(C^*(x))|] &= \mathbb{E}[V(\check{Y}^*(y)) \\ &\quad + \langle C^*(x), \check{Y}^*(y)\rangle - U(C^*(x))] \\ &\leq v(y) + xy - u(x) \\ &= 0.\end{aligned}$$

Thus for $\mathbb{P}$-almost surely $\omega \in \Omega$ the relation

$$V(\check{Y}^*(y)(\omega)) + \langle C^*(x)(\omega), \check{Y}^*(y)(\omega)\rangle = U(C^*(x))(\omega)$$

holds. In other words, $\check{Y}^*(y)(\omega)$ achieves the unique (see Lemma 4.7) infimum in

$$U(C^*(x))(\omega) = \inf_{k \in \mathcal{K}} [V(k) + \langle C^*(x)(\omega), k\rangle].$$

This proves the relation

$$\nabla V(\check{Y}^*(y)) = -C^*(x).$$

Similar considerations prove the second statement

$$\nabla U(C^*(x)) = \check{Y}^*(y).$$

To prove part (i) we first show that $v(y)$ is strictly convex for $y \in \{v < \infty\}$. From Corollary 5.20 we know that there exist a $\check{Y} \in \check{\mathcal{Y}}(y)$ such that $v(y) = \mathbb{E}V(\check{Y})$ and that this solution is unique. Now fix $y_1 < y_2$ such that $v(y_1) < \infty$ with optimal solutions $\check{Y}(y_1)$ and $\check{Y}(y_2)$ respectively. Then in particular,

$$\frac{\check{Y}(y_1) + \check{Y}(y_2)}{2} \in \check{\mathcal{Y}}\left(\frac{y_1 + y_2}{2}\right).$$

This implies that

$$
\begin{aligned}
v\left(\frac{y_1+y_2}{2}\right) &\leq \mathbb{E}\left[V\left(\frac{\check{Y}(y_1)+\check{Y}(y_2)}{2}\right)\right] \\
&< \frac{1}{2}\mathbb{E}V(\check{Y}(y_1)) + \frac{1}{2}\mathbb{E}V(\check{Y}(y_2)) \\
&= \frac{1}{2}v(y_1) + \frac{1}{2}v(y_2).
\end{aligned}
$$

Because v is strictly convex, and because u and v satisfy the dual relation (54), we know that u is continuously differentiable on $(0,\infty)$ (see for example Rockafellar [44]. □

Corollary 5.20 gives existence in the dual problem. Below we prove the corresponding result for the primal problem.

Corollary 6.4 *Suppose that the utility functional U is uniformly integrable on the set of financiable plans $\mathcal{C}(x)$, and if the felicity function satisfies assumptions (2.1) and (2.2) then there exists a consumtion plan $C^* \in \mathcal{C}(x)$ that achieves the supremum in*

$$u(x) = \sup_{C\in\mathcal{C}(x)} \mathbb{E}U(C)$$

REMARK: The uniform integrability assumption ensures that the value of the optimization problem (1) is finite. For sufficient conditions for when uniform integrability holds, see Lemma 2.1 in Bank and Riedel [9].

PROOF : Let $\{C^n\}_{n\in\mathbb{N}}$ be a sequence of consumption plans in $\mathcal{C}(x)$ such that

$$\lim_{n\to\infty} \mathbb{E}U(C^n) = \sup_{C\in\mathcal{C}(x)} \mathbb{E}U(C).$$

Then Lemma 4.2 in [35] shows that there exists a sequence of convex combinations $B^n \in \mathrm{conv}(C^n, C^{n+1}, \ldots)$ and a $C \in \mathcal{C}$ such that $B^n \to C$ in the sense of Fatou. Without loss of generality, we may also assume that $B^n(T) \to C(T)$ if $T < \infty$. Because $\mathcal{C}(x)$ is convex, we have that $B^n \in \mathcal{C}(x)$ for all $n \in \mathbb{N}$. Observe also that Fatou convergence implies convergence in the metric $d_{\mathcal{C}}$. In particular, we can use lower-semicontinuity of the pairing (Lemma 3.4) to show that the limit $C \in \mathcal{C}(x)$. Additionally, continuity of preferences with respect to convergence in $d_{\mathcal{C}}$ implies that

$$\mathbb{E}(U(C)) = \lim_{n\to\infty} \mathbb{E}U(B^n).$$

Concavity of U gives the result. □

We finish with a theorem specializing these results for a complete market.

Theorem 6.5 *In a complete market, the unique dual optimizer is given by $\check{Z}$ where Z is the density of the unique equivalent martingale measure $\mathbb{Q}$. In particular, for x and y such that $u'(x) = y$, if the representation result gives*

$$yZ_t = \mathbb{E}\left[\int_t^T F'(t, \sup_{v\in[\tau,t]} L(y)(v))\, dt \,\middle|\, \mathbb{F}_t \right],$$

then the optimal solution $C^*(x)$ satisfies

$$C^*(x)(t) = \sup_{v\in[0,t]} L(y)(v).$$

Proof : The definition (19) of the dual functional V guarantees that it is a decreasing function in the sense that if k_1 and k_2 are two deterministic functions such that $k_1(t) \le k_2(t)$ for all t, then $V(k_1) \ge V(k_2)$. To arrive at a similar statement for random processes, observe first that identity (39) shows that for all $Y \in \mathcal{Y}(y)$ there exists a corresponding $\check{Y} \in \check{\mathcal{Y}}(y)$ such that

$$\mathbb{E}V(\check{Y}) = \sup_{C \in \mathcal{C}} \mathbb{E}[U(C) - \langle C, Y \rangle].$$

Conversely, for each $\check{Y} \in \check{\mathcal{Y}}(y)$ there exists a $Y^* \in \mathcal{Y}(y)$ such that $\check{Y} \leq {}^{\circ}Y^*$. In particular, $\mathbb{E}\langle C, \check{Y} \rangle \leq \mathbb{E}\langle C, Y^* \rangle$ for all $C \in \mathcal{C}$. Hence, if we denote by $\check{Y}^* \in \check{\mathcal{Y}}$ the process corresponding to Y^* via the representation theorem i.e. such that the identity (39) holds, then $\mathbb{E}V(\check{Y}) \geq \mathbb{E}V(\check{Y}^*)$.

Thus it remains to show that the unique martingale measure is a maximal element of $\mathcal{Y}(1)$. To this end, let Y be an arbitrary element of $\mathcal{Y}(1)$. By right-continuity, it suffices to show that for all t, the set $B_t = \{Y_t > Z_t\}$ has measure zero. This fact was proved by Kramkov and Schachermayer (Lemma 4.3 in [36].)

□

Remark: We can obtain maximality also directly from the process bipolar theorem. In particular, Theorem 2.10 and Corollary 2.11 in [34] show that in general the set of maximal elements is $\mathcal{Y}^{\mathcal{D}} = \{Y^{\mathbb{Q}} \,|\, \mathbb{Q} \in \mathcal{D}\}$ where $\mathcal{D}$ is the $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-closure of $\mathcal{M}$ in $(\mathbb{L})^*$. For a description of $Y^{\mathbb{Q}}$ when $\mathbb{Q}$ is not an element of $\mathcal{M}$, see the discussion preceding Proposition 2.2 in [34] . In particular, in an incomplete market, the set of maximal dual elements includes more than just the densities of equivalent (local)martingale measures.

7 Examples and Extensions

In this section we show how to extend the theory to infinite time-horizon in the form of a verification argument (section 7.4), and use this to establish optimal consumption plans for Wiener driven models (section 7.6). In the first part we show how to extend the model to preferences based on level of satisfaction; this level is a functional of the path of consumption up to date. The examples we discuss are for these more general utilities.

7.1 Extension of the Class of Preferences

In this section we briefly describe how the duality methods can be applied to utilities of the form

$$\mathbb{E}U(C) \triangleq \mathbb{E}\int_0^T F(t, Z(C)_t)dt, \tag{57}$$

where the level of satisfaction

$$Z(C)_t \triangleq \int_0^t \beta e^{-\beta(t-s)}\, dC_s. \tag{58}$$

allows for appropriate discounting of past consumption. This model of preferences, in particular the case $F(t,x) = e^{-\delta t}x^\alpha/\alpha$ of discounted HARA utility, has been studied by various authors. In particular, Hindy and Huang [28] find explicit solutions for a (complete) market in which the risky asset follows a geometric Brownian motion. The optimal solution is a ratio barrier policy: the optimizing agent invests a constant proportion of wealth in the risky asset, and consumes just enough the keep the ratio W/Z of current wealth below a predetermined level k^*, i.e. $W/Z < k^*$ almost surely. A viscocity solution approach to this problem was first considered by Alvarez [2]. Later, Benth, Karlsen, and Reikvam extend these results to exponential Lévy markets [12], [11]. These authors prove that the value function is the unique solution of an associated integro-differential variational equation. They are also able to solve for explicit solutions for spectrally negative Lévy processes. All of these explicit solutions are for the (discounted) HARA utilities in an infinite time-horizon. Solutions for complete markets can be found in Bank and Riedel [9], Bank and Föllmer, [5] and Bank [3].

The duality methods developed in this paper can be extended to these generalized preferences by a change of variables.

Proposition 7.1 *Let the utility function be given as above in (57), and let* $\bar{C}$ *denote the increasing, right-continuous process*

$$\bar{C}_t \triangleq \int_0^t \beta e^{\beta s} dC_s. \tag{59}$$

Also, define the discounted dual variables

$$\mathcal{Y}_d(y) \triangleq \left\{ Y_t^d = \frac{Y_t e^{-\beta t}}{\beta} \;\; s.t. \; Y \in \mathcal{Y}(y) \right\}$$

and define $\check{\mathcal{Y}}_d(y)$ *in analogy with (37), but with respect to these discounted processes.*

Then we have that C^* *is optimal with respect to the preferences expressed in (57) if and only if* $\bar{C}^*$ *is optimal with respect to* $\tilde{U}(C) := \int_0^T \tilde{F}(t, C_t)dt$ *where* $\tilde{F}(t,x) := F(t, e^{-\beta t}x)$.

Furthermore, letting $\tilde{V}$ *denote the conjugate of* $\tilde{U}$*, if* $\bar{C}^*(x)$ *is such that*

$$\tilde{u}(x) = \sup_{C \in \mathcal{C}(x)} \mathbb{E}\tilde{U}(C) = \mathbb{E}\tilde{U}(\bar{C}^*(x))$$

and $\check{Y}^*_d(y)$ *satisfies*

$$\tilde{v}(y) = \inf_{\check{Y}_d \in \check{\mathcal{Y}}_d(y)} \mathbb{E}\tilde{V}(\check{Y}_d) = \mathbb{E}\tilde{V}(\check{Y}^*_d(y))$$

for $x,\, y \in \mathbb{R}_+$ *such that* $\tilde{u}'(x) = y$*, then*

$$\check{L}^*(y) = \bar{C}^*(x),$$

where $\check{Y}^*_d(t) = \int_t^T \tilde{F}'(s, \check{L}^*(s))ds$*.*

Proof : The proof follows from two observations. First, we may express the utility associated with C as a functional of the increasing process $\bar{C}$:

$$\mathbb{E}U(C) = \mathbb{E}\int_0^T F(t, e^{-\beta t}\bar{C}_t)dt.$$

Second, from the relation (59) it follows that the budget constraint can be rewritten as

$$\mathbb{E}\langle C, Y\rangle = \mathbb{E}\int_0^T Y_s dC_s \leq x \iff \mathbb{E}\int_0^T \frac{Y_s e^{-\beta s}}{\beta} d\bar{C}_s \leq x.$$

□

In the rest of this section we will use C and $\bar{C}$ interchangeably, taking it for granted that their dynamics are related as in (59). We will also assume that preferences are modeled as in (57), with the felicity function $F(t, x) = e^{-\delta t}x^\alpha/\alpha$ for $\alpha \in (0, 1)$. We assume that the discount factors δ and β are positive.

Remark: The discount factor $e^{-\delta t}$ can be interpreted as accounting for the probability of death at an exponential time τ independent of the market filtration and with parameter δ. More precisely, suppose that $\mathbb{E}U(C) =$

$\mathbb{E}\int_0^\tau F(t, C_t)dt$ so that utility is obtained from the optimal consumption plan until death at the unknown (exponential) time τ. Then we may write

$$\mathbb{E}U(C) = \mathbb{E}\int_0^\tau F(t, C_t)dt = \mathbb{E}\int_0^\infty F(t, C_t)e^{-\delta t}dt.$$

In this way the infinite time model can capture termination at an unknown finite time.

7.2 Budget Constraint in Infinite Time-Horizon

Recall that a consumption plan C is called financiable if there exists a predictable, S-integrable process H such that the value process $V = (V_t)_{t\in[0,T]}$,

$$V_t = x + \int_0^t H_u\, dS_u - C_t, \quad 0 \le t \le T,$$

is nonnegative. In the introduction we explained that when the set $\mathcal{M}$ of equivalent (local) martingale measures is non-empty, a plan C is financiable with initial wealth x if and only if $\mathbb{E}\langle C, Y^{\mathbb{Q}}\rangle \le x$ for all $Y^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}(1)$. When the time-horizon is infinite, however, there generally are no equivalent martingale measures: $\mathcal{M} = \emptyset$.

Often, however, each finite dimensional restriction of the market does have a set of equivalent (local) martingale measures. Furthermore, the densities of these measures can be extended in a consistent manner for an arbitrary (finite) time horizon T. For example, in Wiener driven markets, the Girsanov transforms $Y_t = e^{-\sum_{j=i}^d \int_0^t \lambda_j(s)dW_s^j - \frac{1}{2}\int_0^t \lambda^*\lambda(s)ds}$ give rise to the martingale measures. And while Y is not in general uniformly integrable, and hence does not have a last element, we can define Y_t for all $t \in [0, \infty)$. In an infinite time-horizon these extended processes are what we will need to deal with.

When the constraints for finite time-horizons, the sets $\mathcal{Y}^{\mathcal{M}}(y)$ for a fixed end time $T < \infty$, come from a family of processes restricted to $t \in [0, T]$ as in a Wiener driven market, then it is possible to adapt the finite time-horizon theory. This is because the budget constraint for $T = \infty$ is just a series of finite-time constraints: for each $t < \infty$ we need to be able to find a trading strategy H such that the value process V_s is non-negative for each $s \in [0, t]$. Thus when the set $\mathcal{Y}^{\mathcal{M}}(y)$ is the same for all times, then the condition $\mathbb{E}\langle C, Y^{\mathbb{Q}}\rangle \leq x$ for all $Y^{\mathbb{Q}} \in \mathcal{Y}^{\mathcal{M}}(1)$ is the budget constraint for $T = \infty$ as well.

In the following, when we discuss deflators or refer to $Y \in \mathcal{Y}^{\mathcal{M}}$ in an infinite time-horizon, we mean the extension of the density processes to the infinite time-horizon.

7.3 Candidate Optimal Consumption Plans

The first step to finding an optimal consumption plan is to construct solutions to the representation problem (30). The duality theorem, Theorem 6.3, shows that the gradient of the utility functional,

$$\nabla_t \tilde{U}(h) \triangleq \int_t^T \tilde{F}'(s, h(s))\, ds \quad (0 \leq t \leq T),$$

evaluated at the optimal plan C^* must be equal to the auxiliary process $\check{Y}$ arising from the representation of $Y \in \mathcal{Y}(y)$ in terms of the running supremum of some optional process L. In this section we show how to construct this process L in the case that $Y \in \mathcal{Y}^{\mathcal{M}}$ is an exponential Lévy processes.

The construction of the solution for discounted exponential Lévy processes is done in Bank and Föllmer [5]. We include these calculations here for completeness and easy reference.

In our examples we will use discounted HARA utilities $F(t,x) = e^{-\delta t} x^{\alpha}/\alpha$ with $\alpha \in (0,1)$, or, equivalently, $\tilde{F}(t,x) = e^{-t(\delta+\beta\alpha)} x^{\alpha}/\alpha$. Because we will be using preferences based on the level of satisfaction, we are interested in representing the discounted process

$$Y_t = \frac{y}{\beta} e^{-\beta t} e^{X_t}$$

where X_t is a Lévy process. We will also work over an infinite time-horizon and will use the duality theory as a verification argument. This step will be outlined in the next section in Theorem 7.3. In this way it might be sufficient to solve the representation problem for a subset of the deflators only.

The following calculations appear in Bank and Föllmer [5], and we collect these in the proposition below for easy reference.

Proposition 7.2 (Section 3.1.2 in [5])

$$\frac{y}{\beta} e^{-\beta t} e^{X_t} = \mathbb{E}\left[\int_t^{\infty} \tilde{F}'(s, \sup_{v \in [t,s]} L_v) ds \middle| \, \mathbb{F}_t \right]$$

such that

$$L_t = \left(\frac{e^{X_t} e^{t(\delta+\beta(\alpha-1))}}{\kappa(y)} \right)^{\frac{1}{\alpha-1}},$$

where

$$\kappa(y) = \frac{\beta}{y} \mathbb{E} \int_0^{\infty} e^{-s(\delta+\alpha\beta)} \inf_{v \in [0,s]} e^{X_v} e^{v(\delta+\beta(\alpha-1))} ds$$

Proof : See calculations in section 3.1.2 of Bank and Föllmer [5]. □

We can thus associate to each discounted deflator $Y_t = \frac{1}{\beta} e^{-\beta} e^{X_t}$ a family of consumption plans

$$\bar{C}_t^{K,Y} = \sup_{s\in[0,t]} \left[\left(\frac{e^{X_t} e^{t(\delta+\beta(\alpha-1))}}{K} \right)^{\frac{1}{\alpha-1}} \right]. \tag{60}$$

indexed by the constant $K > 0$. When $K = \kappa(y)$ as given above, then we get a plan arising from the representation of yY_t.

Remark: We will show in the next section that if a consumption plan $C^{K,Y}$ is financiable with initial wealth x and is such that $\mathbb{E}\langle C^{K,Y}, Y\rangle = x$ (i.e. there is no other deflator Y' that would result in a higher price: $\mathbb{E}\langle C^{K,Y}, Y'\rangle > x$), then it is optimal. The linearity of the pairing then implies that if we can find such a plan $C^{K(x),Y}$ for a particular choice of x, then optimal consumption plans for any initial wealth are simply constant multiples of this plan. In Section 7.5 we show how to calculate the appropriate constant $K(x)$.

7.4 Duality Result as a Verification Theorem

Sometimes both the dual and primal problems can be very difficult to solve. In these instances we would like to be able to use the duality framework to guess and then verify the optimal solution. The following theorem describes one possibility; it relies on the ability to calculate the price of the consumption plan.

Theorem 7.3 *Let $\bar{C}^{K,Y}$ be one of the consumption plans associated, as described in (60), to the deflator $Y \in \mathcal{Y}(y)$ and suppose that it is possible to choose the constant K such the $\bar{C}^{K,Y}$ is financiable with initial wealth x and such that $\mathbb{E}\langle C^{K,Y}, Y\rangle = xy$, then $C^{K,Y}$ is optimal.*

REMARK: The result is valid for any time horizon $T \leq \infty$. The important property of deflators $Y \in \mathcal{Y}(1)$ we are using here is that if $C \in \mathcal{C}(x)$ then $\mathbb{E}\langle C, Y\rangle \leq x$.

PROOF : The key observation is that the dual function provides an upper bound. In particular, if $C \in \mathcal{C}(x)$ and $\check{Y}(y) \in \check{\mathcal{Y}}(y)$ then we have the following result

$$\begin{aligned} u(x) &\leq \sup_{C \in \mathcal{C}} \mathbb{E}[U(C) - \langle C, \check{Y}\rangle + yx] \\ &= \mathbb{E}V(\check{Y}) + xy. \end{aligned}$$

Subtracting $\mathbb{E}U(C)$ from both sides of this identity gives us an estimate for how close C is to the optimal plan:

$$u(x) - \mathbb{E}U(C) \leq \mathbb{E}V(\check{Y}(y)) + xy - \mathbb{E}U(C). \tag{61}$$

From the definition of the dual functional V and using the notation from the previous section, we have that

$$\mathbb{E}V(y\check{Y}) = \mathbb{E}\tilde{U}(\bar{C}^{\kappa(y),Y}) - y\mathbb{E}\langle C^{\kappa(y),Y}, Y\rangle.$$

Thus if it is possible to choose y such that $\kappa(y) = K$, and such that $\mathbb{E}\langle C^{\kappa(y),Y}, Y\rangle = x$ then the right hand side of (61) is zero and $\mathbb{E}U(C^{K,Y})$ maximizes utility amongst all plans $C \in \mathcal{C}(x)$. Note that K is first chosen so that $C^{K,Y} \in \mathcal{C}(x)$, where membership in $\mathcal{C}(x)$ does not a priori only depend on the "price" with respect to the deflator Y.

□

Remark: If Y is the density of an equivalent martingale measure $\mathbb{Q}$ (for $T < \infty$), then another way to think about the condition of the theorem is that the price of the consumption plan $C^{K,Y}$ is the "price" calculated under the measure $\mathbb{Q}$. One way to verify that a consumption plan is financiable with given capital x is to explicitly construct the financing portfolio. We will use this technique in the examples that follow.

Remark: The identity (61) can also be used to estimate how close to an optimal solution a given plan C is.

Remark: The condition of this theorem can be rephrased as a sensitivity result along the lines of Karatzas, Lehoczky, Shreve, and Xu [32]. In particular, it shows that if $\mathbb{E}\langle C^{K,Y}, Y^{\mathbb{Q}}\rangle \leq x$ for all $\mathbb{Q} \in \mathcal{M}$, and if $\mathbb{E}\langle C^{K,Y}, Y\rangle = x$ then $C^{K,Y}$ is optimal. I.e. the dual optimizer Y maximizes the price. Other equivalent sensitivity results for the utility optimization problem for terminal wealth are discussed in [32].

7.5 How to Calculate the Price, Value, and the Dual Function

In this section we show how to calculate some of the important quantities such as $\mathbb{E}U(C)$ and $\mathbb{E}V(Y)$. The calculations here are based on the results in Bank and Riedel [9]. These authors exploit the special structure of the consumption plans $\bar{C}^{K,Y}$, which are associated to a deflator $Y \in \mathcal{Y}$ that is an exponential Lévy process. A few additional requirements on the jumps of this Lévy process are also made.

The basic budget constraint tells us that the price $w(C)$ of such a consumption plan is given by

$$w(C) = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}} \int_0^T \frac{1}{\beta} e^{-\beta t} d\bar{C}_t.$$

Breaking up this calculation, for each measure $\mathbb{Q}$ we have that

Proposition 7.4 *(Lemma 4.9 in [9]) Let $\mathbb{Q} \in \mathcal{M}$ such that $\mathbb{E}^{\mathbb{Q}} \int_0^T \frac{1}{\beta} e^{-\beta t} d\bar{C}_t < \infty$, then*

$$\mathbb{E}^{\mathbb{Q}} \int_0^T \frac{1}{\beta} e^{-\beta t} d\bar{C}_t = \frac{1}{\beta} \mathbb{E}^{\mathbb{Q}}(\bar{C}(\tau^*))$$

where τ^* is an independent exponential random variable with parameter β.

Remark: When we refer to the equivalent (local) martingale measure $\mathbb{Q} \in \mathcal{M}$ we mean only that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on all finite time-horizons, i.e. on the sigma algebras $\mathbb{F}_T$ for $T < \infty$.

Proof : See proof of Lemma 4.9 in [9]. □

We now turn to look at consumption plans that are our candidates for optimality: those plans that arise from the representation theorem. In the notation of the previous section, for X_t a Lévy process, let

$$\begin{aligned} \bar{C}_t &= \sup_{s\in[0,t]} \left(\frac{e^{X_t} e^{t(\delta+\beta(\alpha-1))}}{\kappa(y)} \right)^{\frac{1}{\alpha-1}} \\ &= \kappa(y)^{\frac{1}{1-\alpha}} \exp\left[\frac{1}{1-\alpha} \sup_{s\in[0,t]} [-X_t - t(\delta+\beta(\alpha-1))] \right] \end{aligned} \tag{62}$$

Proposition 7.5 *Suppose that under the measure $\mathbb{Q}$ and the original measure $\mathbb{P}$, the process $-X_t - t(\delta+\beta(\alpha-1))$ is a Lévy process that is not strictly decreasing and has no positive jumps. And let $\bar{C}$ be given as in (62), then* *(i) Then the "price" under the measure $\mathbb{Q}$, denoted by $w^{\mathbb{Q}}(C)$, is given by*

$$w^{\mathbb{Q}}(C) = \begin{cases} \frac{1}{\beta}\kappa(y)^{\frac{1}{1-\alpha}} \frac{\xi^{\mathbb{Q}}(1-\alpha)}{\xi^{\mathbb{Q}}(1-\alpha)-1} & \text{if } \xi^{\mathbb{Q}} > 0 \\ \\ \frac{1}{\beta}\kappa(y)^{\frac{1}{1-\alpha}} & \text{if } \xi^{\mathbb{Q}} = 0 \end{cases}$$

where the parameter $\xi^{\mathbb{Q}}$ is the unique nonnegative solution to

$$\pi^{\mathbb{Q}}(\xi) - \xi(\delta + \beta(\alpha - 1)) = \beta$$

where $\pi^{\mathbb{Q}}$ is the Laplace exponent of $-X_t$ under the measure $\mathbb{Q}$.

(ii) The associated value function $\mathbb{E}U(C)$ is given by

$$\mathbb{E}U(C^{\mathbb{Q}}(x)) = \begin{cases} \frac{1}{\alpha(\delta+\beta\alpha)}\kappa(y)^{\frac{\alpha}{1-\alpha}} \left(\frac{\xi(1-\alpha)}{\xi(1-\alpha)-\alpha}\right) & \text{if } \xi > 0 \\ \\ \frac{1}{\alpha(\delta+\beta\alpha)}\kappa(y)^{\frac{\alpha}{1-\alpha}} & \text{if } \xi = 0 \end{cases}$$

where ξ is the unique nonnegative solution to

$$\pi(\xi) - \xi(\delta + \beta(\alpha - 1)) = \delta + \alpha\beta$$

such that $\pi(\cdot)$ is the Laplace exponent of $-X_t$ under the original measure $\mathbb{P}$.

(iii) The dual function evaluated at $yY^{\mathbb{Q}}$ is given as

$$\mathbb{E}V(yY^{\mathbb{Q}}) = \left(\frac{1}{y}\right)^{\frac{\alpha}{1-\alpha}} \left[\left(\frac{(\kappa^{\mathbb{Q}})^{\frac{\alpha}{1-\alpha}}}{\alpha(\delta+\alpha\beta)}\right)\left(\frac{\xi(1-\alpha)}{\xi(1-\alpha)-\alpha}\right) - \left(\frac{(\kappa^{\mathbb{Q}})^{\frac{1}{1-\alpha}}}{\beta}\right)\left(\frac{\xi^{\mathbb{Q}}(1-\alpha)}{\xi^{\mathbb{Q}}(1-\alpha)-1}\right)\right] \tag{63}$$

Remark: These calculations essentially appear in Bank and Riedel [9]. We include them here for easy reference and to preserve unity of notation.

PROOF : (i) From the previous Proposition 7.4 we have the representation

$$w^{\mathbb{Q}}(C) = \frac{1}{\beta}\mathbb{E}^{\mathbb{Q}}(\bar{C}(\tau^*)).$$

From Lemma 4.11 (ii) in Bank and Riedel [9] we know that if $-X_t - t(\delta+\beta(\alpha-1))$ is a Lévy process with no positive jumps and such that it is not strictly decreasing, then the supremum, $\sup_{t\in[0,\tau^*]}[-X_t - t(\delta+\beta(\alpha-1))]$ is distributed like an exponential random variable with parameter $\xi^{\mathbb{Q}}$, which is the unique nonnegative solution to

$$\pi^{\mathbb{Q}}(\xi) - \xi(\delta+\beta(\alpha-1)) = \beta$$

where $\pi^{\mathbb{Q}}$ is the Laplace exponent of $-X_t$ under the measure $\mathbb{Q}$. Now the calculations are straightforward. If $\xi^{\mathbb{Q}} > 0$ then

$$\begin{aligned} \frac{1}{\beta}\mathbb{E}^{\mathbb{Q}}(\bar{C}(\tau^*)) &= \frac{1}{\beta}\int_0^\infty \kappa(y)^{\frac{1}{\alpha-1}} e^{\frac{1}{1-\alpha}x} e^{-\xi^{\mathbb{Q}}x}\xi\, dx \\ &= \frac{1}{\beta}\kappa(y)^{\frac{1}{1-\alpha}} \frac{\xi^{\mathbb{Q}}(1-\alpha)}{\xi^{\mathbb{Q}}(1-\alpha)-1} \end{aligned}$$

(the constant $\kappa = y\kappa(y)$ depends on the measure $\mathbb{Q}$). If $\xi^{\mathbb{Q}} = 0$ the calculations give $w^{\mathbb{Q}}(C) = \frac{1}{\beta}\kappa(y)^{\frac{1}{1-\alpha}}$.

(ii) This part follows from similar calculations as above, once we realize that the value function can be written as

$$\mathbb{E}U(C) = \frac{1}{\alpha(\delta+\alpha\beta)}\mathbb{E}(\bar{C}(\tau))^{\alpha}$$

where τ is an independent exponential random variable with parameter $\delta+\alpha\beta$.

(iii) The last part is a combination of the results (i) and (ii) and the definition of the dual function V. In particular, we recall from Lemma 4.6 that

$$V(\check{k}) = U(\check{k}) - \langle \check{l}, \check{k}\rangle$$

where the density of $\check{k}$ is given by $F'(t, \check{l}_t)$. The result now follows. □

7.6 Example: Wiener Driven Models

In this section we construct the optimal consumption strategy when the stock dynamics for $i = 1, ...n$ are given by

$$dS_t^i = \mu_i S_t^i dt + \sum_{j=1}^{d} \sigma_{ij} S_t^i dW_t^j.$$

We assume that $d \geq n$ so that the market is possibly incomplete (although see remark at the end). The filtration is the augmentation of the filtration generated by W^j for $j = 1, ...d$. We also assume that the volatility matrix σ is surjective on to its range $\mathbb{R}^n$.

The possible equivalent martingale measures on finite time-horizons correspond to deflators that must be martingales themselves. In a Wiener driven market, all martingales can be written as stochastic integrals with respect to the driving Brownian motions. In particular, Y is a strictly positive (so that $1/Y_t$ is defined) martingale if and only if there exists a vector valued process $\lambda(t)$ such that

$$dY_t = -Y_t \lambda^*(t) dW_t$$

where λ^* denotes the transpose of λ. The solution to this SDE is given by

$$Y_t = e^{-\sum_{j=i}^{d} \int_0^t \lambda_j(s) dW_s^j - \frac{1}{2} \int_0^t \lambda^* \lambda(s) ds}, \quad \text{for } Y_0 = 1.$$

Standard calculations show that $Z^i := (Y\, S^i)_t$ is a martingale under the original measure $\mathbb{P}$ for all $i = 1, .., n$ if and only if

$$\sigma \lambda(t) = \mu.$$

In fact, the Itô formula shows that the dynamics of Z^i are given by

$$dZ_t^i = Z_t^i[\mu_i - \sum_{j=1}^{d} \sigma_{ij}\lambda_j(t)]dt + Z_t^i \sum_{j=1}^{d} (\sigma_{ij} - \lambda_j) dW_t^j.$$

The minimal (in magnitude) solution to this equation is given by

$$\hat{\lambda} = \sigma^*(\sigma\sigma^*)^{-1}\mu,$$

and it is constant in time, while a general solution is of the form

$$\lambda(t) = \hat{\lambda} + \nu(t) \quad \text{for } \nu(t) \in N(\sigma),$$

where $N(\sigma)$ denotes the null space of σ. Another way to state this is that $\nu(t)$ is orthogonal to the rows of σ. Also $\hat{\lambda} \in N(\sigma)^{\perp}$, and hence

$$\parallel \lambda(t) \parallel^2 = \parallel \hat{\lambda} \parallel^2 + \parallel \nu(t) \parallel^2 .$$

For the solutions λ that are constant in time, the corresponding deflator is an exponential Lévy process. We can thus apply the previous analysis to solve the representation problem:

$$\frac{y}{\beta} e^{-\beta t} e^{-\lambda^* W_t - \frac{1}{2}\|\lambda\|^2 t} = \mathbb{E}\left[\int_t^{\infty} e^{-t(\delta+\beta\alpha)} (\sup_{v\in[t,s]} L_v^{\lambda})^{\alpha-1} ds \middle| \mathbb{F}_t \right]$$

where

$$L_t^{\lambda} = \left(\frac{e^{-\lambda^* W_t - \frac{1}{2}\|\lambda\|^2 t} e^{t(\delta+\beta(\alpha-1))}}{\kappa^{\lambda}(y)} \right)^{\frac{1}{\alpha-1}},$$

and

$$\kappa^{\lambda}(y) = \frac{\beta}{y} \mathbb{E}\left[\int_0^{\infty} e^{-s(\delta+\alpha\beta)} \inf_{v\in[0,s]} e^{-\lambda^* W_v - \frac{1}{2}\|\lambda\|^2 v} e^{v(\delta+\beta(\alpha-1))} ds \right].$$

In order to use the verification result in Theorem 7.3 we first look at which deflator Y would be a good candidate for minimizing the dual function. To this end, we parametrize these deflators by $\nu(t)$ where $\nu(t) \in N(\sigma)$ for all $t \in (0, \infty)$, such that

$$Y_t^\nu = e^{-\sum_{j=i}^d \int_0^t \hat{\lambda}_j + \nu(s)_j(s) dW_s^j - \frac{1}{2}\int_0^t \|\hat{\lambda}\|^2 + \|\nu(s)\|^2 ds}.$$

Asociated to the deflators for which ν is constant in time are families of consumption plans

$$\bar{C}_s^{K,\nu} = \sup_{t\in[0,s]} \left(\frac{e^{-(\hat{\lambda}+\nu)^* W_t - \frac{1}{2}(\|\hat{\lambda}\|^2 + \|\nu\|^2)t} e^{t(\delta+\beta(\alpha-1))}}{K} \right)^{\frac{1}{\alpha-1}}.$$

For each such consumption plan, we can calculate the expected value using Proposition 7.5:

$$\mathbb{E}U(C^{K,\nu}) = \frac{1}{\beta} K^{\frac{1}{1-\alpha}} \frac{\xi^\nu(1-\alpha)}{\xi^\nu(1-\alpha)-\alpha}$$

where ξ^ν is the unique nonnegative solution of

$$\frac{1}{2}(\| \hat{\lambda} \|^2 + \| \nu \|^2)\xi(\xi+1) - \xi(\delta + \beta(\alpha-1)) = \delta + \alpha\beta.$$

We can also calculate the "price" under each measure $\mathbb{Q} \in \mathcal{M}$ that preserves the Lévy property. These measures correspond to deflators Y^μ where μ is constant in time. The price $w^\mu(C^{K,\nu})$ is then given by

$$w^\mu(C^{K,\nu}) = \frac{1}{\beta} K^{\frac{1}{1-\alpha}} \frac{\zeta^{\nu.\mu}(1-\alpha)}{\zeta^{\nu,\mu}(1-\alpha)-1},$$

where $\zeta^{\mu,\nu}$ solves

$$(\zeta-1)^2 \| \hat{\lambda} \|^2 + (\zeta-1) \| \hat{\lambda} \|^2 + \zeta \| \nu \|^2 - \| \mu \|^2 + \| \zeta\nu - \mu \|^2 - 2\zeta(\delta + \beta(\alpha-1)) = 2\beta.$$

Observe that if $\nu = 0$ then the dependence on μ drops out, in other words, the price of $C^{K,\nu}$ is the same under all of the Lévy property preserving measures. This suggests that Y^0 is a good candidate for the dual optimizer. In fact, we can verify directly, by exhibiting the financing portfolio for each finite time-horizon T, that if $\mathbb{E}\langle C, Y^0\rangle = x$ then $C \in \mathcal{C}(x)$.

To this end we first recall that a consumption plan C is financiable up to time $T < \infty$ and with initial capital x if and only if there exists an admissible portfolio strategy H such that

$$V_t = x + \sum_{i=1}^{n} \int_0^t H_s^i dS_s^i - C_t$$

is nonnegative for all $t \in [0, T]$. Let $\mathbb{E}^0$ denote expectation under the measure $\mathbb{Q}^0$ associated to Y^0. In order to find such a strategy H, we first observe that our assumptions ensure that the process

$$V_t \triangleq \mathbb{E}^0 \left[\int_t^T dC_s \,\middle|\, \mathbb{F}_t \right] + x - \mathbb{E}^0 C_T$$

is nonnegative. Furthermore, it can be written as the sum of a martingale $M_t := \mathbb{E}^0[C_T|\mathbb{F}_t] - \mathbb{E}^0 C_T$ and a decreasing process:

$$V_t = x + M_t - C_t.$$

Re-writing the portfolio dynamics, so that we know what we are looking for, observe first that $\tilde{W}_t := W_t + \hat{\lambda} t$ is a Brownian motion under $\mathbb{Q}^0$. Let π_t denote the amount of money invested in each stock, i.e. $\pi_t^i = S_t^i H_t^i$, then in terms of this new Brownian motion, our goal is to find a portfolio π such that

$$V_t = x + \int_0^t \pi^*(s) \sigma d\tilde{W}_s - C_t^{0,K}. \tag{64}$$

From the form of the consumption plan $\bar{C}^{K,\hat{\lambda}}$ we know that M_t is adapted to the filtration generated by $\hat{\lambda}^* W_t$. Following the calculations found in Karatzas and Shreve (Proposition 5.8.6 in [33]), we can then show that M_t can be written as the stochastic integral

$$M_t = \int_0^t \psi(s)\hat{\lambda} d\tilde{W}_t.$$

By construction, however, $\hat{\lambda}$ is in the range of σ^*, the transpose of the volatility matrix. In particular, the least squares solution to the problem $\sigma^*\pi = \psi(s)\hat{\lambda}$ is an exact solution. In fact, we have that $\pi(s) = \psi(s)(\sigma\sigma^*)^{-1}\sigma\hat{\lambda}$ is the desired financing portfolio.

These calculations show that with initial capital $x = \mathbb{E}^0\langle \bar{C}^{0,K}, Y^0\rangle$ we can find a portfolio that finances this consumption plan up to time T. The choice of T, however, is arbitrary, so we have that C is financiable for the full infinite-time horizon. We can now use the verification theorem, Theorem 7.3, to conclude that the consumption plan associated with $\hat{\lambda}$ is optimal. Proposition 7.5 can then be used to calculate the appropriate constant K for the initial wealth x, and to calculate the value of the utility optimization problem.

Observe that we could have also approached our guess of the optimal dual variable by performing a partial minimization of the dual problem (partial because we only know how to solve the representation problem for the time-homogeneous deflators). Using Proposition 7.5 we can evaluate the dual function as follows

$$\mathbb{E}V(yY^\nu) = \left(\frac{\beta}{y}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{1}{\delta+\alpha\beta}\right)^{\frac{1}{1-\alpha}} \left(\frac{\xi^\nu}{\xi^\nu+1}\right) \times \left[\left(\frac{(\xi^\nu+1)(1-\alpha)}{\xi^\nu(1-\alpha)-\alpha}\right) - \left(\frac{\zeta^{\nu,\nu}(1-\alpha)}{\zeta^{\nu,\nu}(1-\alpha)-1}\right)\right].$$

Looking at this expression we observe that the dual function depends on ν only through its magnitude. Thus if a minimum occurs for a deflator such that ν is constant (i.e. such that the above calculations are valid), then it must be that

$\nu = 0$. If this were not the case, then the minimum would not be unique (This is true at least in the general case that the dimension of the null space $N(\sigma)$ is greater than one.).

REMARK: There is also a third way to look at this problem, and the question of why $\hat{\lambda}$ should be the overall minimizer: for the purposes of portfolio optimization, this market is complete. In fact, if we choose an orthonormal basis $v_1, ..., v_n$ for the range $R(\sigma^*)$ of σ^*, then we can see from the covariance structures (and the Lévy characterization of Brownian motion) that $W' = (v_1^* W, ..., v_n^* W)$ is a standard n-dimensional Brownian motion. Furthermore, the stock dynamics can be written solely in terms of W', and, with respect to the (augmented) filtration $\mathcal{F}^{W'}$, the market is complete.

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A Appendix

In this section we provide a proof of the minimax theorem used. The proof follows the basic outline of theorem 3.1 in [49] with appropriate modifications to account for convex compactness.

Theorem A.1 (Minimax, Theorem 6.2) *Let A be a nonempty convex subset of a topological vector space, and B a nonempty, closed, convex, and convexly compact subset of a topological vector space. Let $h : A \times B \to \mathbb{R}$ be convex on A, and concave and upper-semicontinuous on B. Then*

$$\inf_A \sup_B h = \sup_B \inf_A h$$

Proof. As a first step we note that

$$\inf_A \sup_B h \geq \sup_B \inf_A h.$$

We show that the reverse inequality also holds. Define $\alpha := \inf_A \sup_B h$. Next, consider any finite collection of elements $a_1, ..., a_m \in A$ and define $g_i := h(a_i, \cdot)$, $i = 1, ..., m$. A consequence of the Mazur-Orlicz theorem (lemma 2.1 (b) in [49]) is that there exists $\lambda_1, ..., \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that

$$\sup_{b\in B}[h(a_1, b) \wedge ... \wedge h(a_m, b)] = \sup_{b\in B}[\lambda_1 h(a_1, b) + \cdots + \lambda_m h(a_m, b)].$$

By assumption, the function h is convex on the set A, and hence

$$\sup_{b\in B}[h(a_1, b) \wedge ... \wedge h(a_m, b)] \geq \sup_{b\in B} h\left(\sum_{i=1}^m \lambda_i a_i, b\right) \geq \alpha.$$

Furthermore, the supremum is achieved. In fact, for a fixed $a \in A$, suppose that $\{b_n\}_{n\in\mathbb{N}}$ is a sequence such that

$$\lim_{n\to\infty} h(a, b_n) = \sup_{b\in B} h(a, b).$$

Then because B is convexly compact, there exists a subnet of convex combinations $\{y_\beta\}_{\beta \in D}$ and $y \in B$ such that $y_\beta \to y$. In particular, each y_β is a finite convex combination $y_\beta = \sum \gamma_i b_i$. Because the function h is concave on B,

$$h(a, y_\beta) \geq \sum \gamma_i h(a, b_i)$$

In addition, because h is upper-semicontinuous in B we know that

$$\limsup_\beta h(a, y_\beta) \leq h(a, y).$$

In particular, combining these two results,

$$h(a, y) \geq \sup_{b \in B} h(a, b) \geq h(a, y).$$

This shows that the supremum is achieved, in particular combining this with the previous result,

$$\{b \in B \,|\, , h(a_1, b) \geq \alpha\} \cap \cdots \cap \{b \in B \mid h(a_m, b) \geq \alpha\} \neq \emptyset.$$

Each of these sets is closed and convex, because h is concave and upper-semicontinuous on B. Thus the collection $[\{b \in B \mid h(a, b) \geq \alpha\}]_{a \in A}$ of closed and convex sets satisfies the finite intersection property. Convex compactness implies that

$$\bigcap_{a \in A} \{b \in B \mid h(a, b) \geq \alpha\} \neq \emptyset.$$

In particular,

$$\sup_B \inf_A h \geq \alpha := \inf_A \sup_B h.$$

□