Bessel and Volatility-Stabilized Processes

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ABSTRACT

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The work in this thesis expands the study of volatility-stabilized processes introduced in [17]. Using their representation as time-changed Bessel processes and a multidimensional version of the skew-product decomposition theorem, we derive the conclusion that the vector of market weights is a *multidimensional Jacobi diffusion.* The Dirichlet distribution is proved to be the invariant distribution of this diffusion. The fact that the marginals of this vector process are one-dimensional Jacobi diffusions having the Beta distribution as an invariant distribution provides new proofs for limiting behavior results for the individual market weights already established in [17]. Using the spectral representation of the transition density of a one-dimensional diffusion, we establish a series representation involving *Jacobi polynomials* for the transition density of the individual market weights, thus answering one of the open questions in [17]. Using techniques pioneered by M. Yor, we establish the joint distribution of the coordinates of the volatility-stabilized process. We carry out the computation of the moments of these coordinates, as well as that of the moments of the individual market weights. Finally, we discuss

connections with the Cox-Ingersoll-Ross process and present the correct version of a theorem attempted by Gouriéroux on the one-dimensional Jacobi diffusion. We conclude with a discussion of several lines of potential future work.

Contents

To the memory of my high-school Math teacher, Dorin Moraru, and to the memory of my grandmother, Aurelia Sava, both deceased during my years as a graduate student.

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1. **Introduction to Volatility-Stabilized Processes**

1.1. **The cumulative volatility of a financial market and its connection to arbitrage.** The aim of this section is to explain how volatility-stabilized processes appear as natural objects of study within the field of Stochastic Portfolio Theory. This is a branch of Mathematical Finance pioneered by Fernholz in the monograph [15], and more recently studied in-depth by Banner, Fernholz, Karatzas, and Kardaras in the series of papers [3], [17], [18], [19].

In what follows we define the excess growth rate of a portfolio of stocks driven by a multidimensional Brownian motion and further explain how the excess growth rate of the market portfolio provides a measure of the intrinsic volatility available in the market at any given time.

Consider a financial market driven by a multidimensional Brownian motion:

$$dX_i(t) = X_i(t)\left[b_i(t)dt + \sum_{k=1}^{d} \sigma_{ik}(t)dW_k(t)\right], \quad i = 1, \ldots, n. \tag{1.1}$$

The quantity $X_i(t)$ stands for the value of the i^{th} stock at time t and $W_1(\cdot), \ldots, W_d(\cdot)$ are d independent standard Brownian motions. We shall assume $d \geq n$.

The vector valued process $b(\cdot) = (b_1(\cdot), \ldots, b_n(\cdot))$ of rates of return and the $(n \times d)$ matrix valued process $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\leq i\leq n,\, 1\leq j\leq d}$ of volatilities are assumed to satisfy the condition

$$\int_0^T \sum_{i=1}^{n} \left(|b_i(t)| + \sum_{k=1}^{d} \sigma_{ik}^2(t)\right) dt < \infty \;\; a.s. \tag{1.2}$$

for every $T > 0$. This condition is part of the definition of a solution of a stochastic differential equation, as presented in [30, p. 285]. All the processes of this model are defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and are adapted to a given filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t \le \infty}$ with $\mathcal{F}(0) = \{\emptyset, \Omega\}$ modulo P. This filtration satisfies the usual conditions of right continuity and augmentation by $P-$negligible sets.

Under the assumption (1.2), Itô's rule allows us to write formula (1.1) in the equivalent form:

$$d\left(\log X_i(t)\right) = \gamma_i(t)dt + \sum_{k=1}^{d} \sigma_{ik}(t)dW_k(t), \quad i = 1, \ldots, n, \tag{1.3}$$

where

$$\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t), \quad a_{ij}(t) = \sum_{k=1}^{n} \sigma_{ik}(t)\sigma_{jk}(t). \tag{1.4}$$

Here $a(\cdot) = \{a_{ij}(\cdot)\}_{1 \le i,j \le n} = \sigma(\cdot)\sigma^t(\cdot)$ is the variance/covariance matrix-valued process and $\gamma_i(\cdot)$ will be further referred to as the "growth rate" of the i^{th} asset. As pointed out in [29, p. 4], the terminology "growth rate" is justified by the a.s. property

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)dt\right) = 0, \tag{1.5}$$

which is guaranteed to hold when all the eigenvalues of the variance/covariance matrix $a(\cdot)$ are bounded away from infinity; see the right-hand side of condition (1.18). In plain English, the growth rate of an asset is the implicit rate of return that it produces.

In the context of this model, a portfolio rule is an $\mathbb{F}-$progressively measurable process $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_n(\cdot))$ defined on $[0, \infty) \times \Omega$ and with values in the simplex

$$\triangle_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n |\ x_i \geq 0 \text{ for every } i \text{ and} \sum_{i=1}^n x_i = 1\}.$$

The quantity $\pi_i(t)$ is interpreted as the proportion of wealth invested in the i^{th} asset at time t.

To a portfolio rule $\pi(\cdot)$ we associate a value process $Z^\pi(\cdot)$, with the convention that at any time t, a fraction $\pi_i(t)$ of $Z^\pi(t)$ is invested in asset i. Hence

$$\frac{dZ^\pi(t)}{Z^\pi(t)} = \sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)} = b^\pi(t)dt + \sum_{k=1}^d \sigma_k^\pi(t)dW_k(t), \text{ where} \tag{1.6}$$

$$b^\pi(t) := \sum_{i=1}^n \pi_i(t)b_i(t), \quad \sigma_k^\pi(t) := \sum_{i=1}^n \pi_i(t)\sigma_{ik}(t) \text{ for } k = 1, \dots, d, \tag{1.7}$$

$$\text{and} \quad a^{\pi\pi}(t) := \sum_{k=1}^d (\sigma_k^\pi(t))^2 = \sum_{i=1}^n \sum_{j=1}^n \pi_i(t)a_{ij}(t)\pi_j(t) \tag{1.8}$$

are, respectively, the rate-of-return coefficients, the volatility coefficients and the variance of the portfolio.

Using the multidimensional Itô formula again as in (1.3), we are able to write

$$d\left(\log Z^\pi(t)\right) = \gamma^\pi(t)dt + \sum_{k=1}^d \sigma_k^\pi(t)dW_k(t), \tag{1.9}$$

with $\gamma^\pi(t) := \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_*^\pi(t)$ being the growth rate corresponding to the portfolio rule $\pi(\cdot)$. The quantity

$$\gamma_*^\pi(t) := \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)a_{ii}(t) - \sum_{i=1}^n \sum_{j=1}^n \pi_i(t)a_{ij}(t)\pi_j(t)\right) \tag{1.10}$$

is non-negative, and is strictly positive if $\pi_i(t) > 0$ holds a.s. for all $i = 1, \dots, n$ and $t \geq 0$. These details are provided in Proposition 1.3.7 in [15]. It is natural to call $\gamma_*^\pi(\cdot)$ the excess growth rate of the portfolio $\pi(\cdot)$; it is going to play a key role in our further exposition.

Alternatively, the excess growth rate (1.10) can be written as

$$\gamma_*^\pi(t) = \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^\pi(t), \tag{1.11}$$

where we have denoted by $\tau_{ij}^\pi(\cdot)$ the variances/covariances of the portfolio $\pi(\cdot)$, namely

$$\tau_{ij}^\pi(t) := \sum_{k=1}^{n} \big(\sigma_{ik}(t) - \sigma_k^\pi(t)\big)\big(\sigma_{jk}(t) - \sigma_k^\pi(t)\big),\ 1 \le i, j \le n.$$

The derivation of this alternative expression can be found in Lemma 1.3.6 of [15, p. 20].

One of the main problems in Mathematical Finance, and implicitly in Stochastic Portfolio Theory, is the detection and study of riskless opportunities to make a profit, also known as arbitrages. We say that a portfolio rule $\pi(\cdot)$ is a relative arbitrage opportunity relative to a portfolio rule $\rho(\cdot)$ over the time horizon $[0, T]$ if

$$\mathbb{P}\big[Z^\pi(T) \ge Z^\rho(T)\big] = 1 \ \text{ and } \ \mathbb{P}\big[Z^\pi(T) > Z^\rho(T)\big] > 0 \tag{1.12}$$

hold whenever the two portfolio rules start with the same initial fortune $Z^\pi(0) = Z^\rho(0) = z$. If instead of (1.12) we have

$$\mathbb{P}\big[Z^\pi(T) > Z^\rho(T)\big] = 1, \tag{1.13}$$

we say that $\pi(\cdot)$ is a strong arbitrage opportunity relative to $\rho(\cdot)$.

The market portfolio is the natural choice for a reference portfolio with respect to which relative arbitrage in the market (1.1) is going to be studied. The market portfolio is defined by the relative capitalizations

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \quad i = 1, \ldots, n \tag{1.14}$$

and amounts to owning the entire market in proportion to the initial fortune $z > 0$. Indeed, from (1.6), it follows that

$$\frac{dZ^\mu(t)}{Z^\mu(t)} = \frac{d\big(X_1(t) + \cdots + X_n(t)\big)}{X_1(t) + \cdots + X_n(t)}$$

and hence

(1.15)

$$Z^\mu(t) = \frac{z}{x}\big(X_1(t) + \cdots + X_n(t)\big),\ 0 \le t < \infty \ \text{ with } x := X_1(0) + \cdots + X_n(0).$$

The excess growth rate of the market portfolio provides a measure of the amount of available volatility in the market at any given time. If this available volatility is great enough over a period of time, it can be exploited by certain types of portfolios to outperform the market portfolio. The representative result in this direction is Proposition 3.1 from [17], which we restate below:

Proposition 1.1. *Suppose there exists a continuous, strictly increasing function* $\Gamma : [0, \infty) \to [0, \infty)$ *with* $\Gamma(0) = 0$, $\Gamma(\infty) = \infty$ *and such that*

$$\Gamma(t) \le \int_0^t \gamma_*^\mu(s)ds < \infty, \quad (\forall)\ 0 \le t < \infty \tag{1.16}$$

holds almost surely. Then, with the entropy function $S(x) := -\sum_{j=1}^n x_j \log x_j$ *and for any time horizon* $[0, T]$ *that satisfies:*

$$\Gamma^{-1}\left(S\big(\mu(0)\big)\right) =: T_* < T < \infty$$

there exists a sufficiently large real number $c > 0$ *such that the portfolio rule*

$$\pi_i(t) = \frac{c\mu_i(t) - \mu_i(t)\log\mu_i(t)}{c - \sum_{j=1}^n \mu_j(t)\log\mu_j(t)} \qquad i = 1, \ldots, n$$

is a strong arbitrage opportunity relative to the market portfolio; in particular,

$$\mathbb{P}\big[Z^\pi(T) > Z^\mu(T)\big] = 1.$$

A natural candidate for the excess growth rate of the market portfolio that satisfies the hypotheses of the above Proposition is the power function $\gamma_*^\mu(s) = ks^\alpha$, with $\alpha \geq 0$. A key feature of an abstract market based on volatility-stabilized processes is that its cumulative volatility, as measured by the excess growth rate of the corresponding market portfolio, is constant through time:

$$\gamma_*^\mu(s) = k \ \text{ for every } s \text{ and some } k > 0.$$

Market diversity, discussed in [19], is another sufficient condition that guarantees the existence of relative arbitrage opportunities. We are going to state next the main result on market diversity and arbitrage. In the following section we are going to explain how volatility-stabilized markets do not satisfy the diversity condition and yet they do exhibit relative arbitrages, as ensured by Proposition 1.1.

We consider again the market model (1.1), but instead of condition (1.2) we require that

$$\sum_{i=1}^{n} \int_0^T \big(b_i(t)\big)^2 dt < \infty \ \ (\forall)\, T \in (0,\infty) \tag{1.17}$$

and

$$\varepsilon||\xi||^2 \leq \xi^t \sigma(t) \sigma^t(t) \xi \leq M||\xi||^2 \ \ (\forall)\, t \geq 0 \ \text{ and } \ \xi \in \mathbb{R}^n. \tag{1.18}$$

By ξ^t and $\sigma^t(\cdot)$ we have denoted vector and matrix transposes, respectively.

For a portfolio rule $\pi(\cdot)$, we introduce the order-statistics notation

$$\max_{1\leq i\leq n} \pi_i(t) =: \pi_{(1)}(t) \geq \pi_{(2)}(t) \geq \cdots \geq \pi_{(n)}(t) =: \min_{1\leq i\leq n} \pi_i(t) \tag{1.19}$$

for the weights $\pi_i(t)$, ranked at time t from largest $\pi_{(1)}(t)$ to smallest $\pi_{(n)}(t)$.

We say that the market (1.1) is weakly diverse on the time horizon $[0,T]$ if for some $\delta \in (0,1)$ we have almost surely

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)dt < 1-\delta. \tag{1.20}$$

A diversity-weighted portfolio is a portfolio with weights

$$\pi_i^{(p)}(t) := \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p} \qquad (\forall)\ i=1,\dots,n, \tag{1.21}$$

where $\mu_i(t)$ represent the weights of the market portfolio and p is a constant in the interval $(0,1)$.

We now state the main result on market diversity leading to relative arbitrage:

Proposition 1.2. *Suppose that a market of the form* (1.1) *has drift and volatility coefficients that satisfy conditions* (1.17) *and* (1.18)*, and that it is weakly diverse on the time horizon* $[0,T]$*, in the sense that it satisfies condition* (1.20) *a.s.. Then, starting with initial capital* $Z^\mu(0)$*, the value process* $Z^{\pi^{(p)}}(\cdot)$ *of the diversity-weighted portfolio* (1.21) *satisfies*

$$\mathbb{P}\big[Z^{\pi^{(p)}}(T) > Z^\mu(T)\big] = 1 \quad \textit{provided that} \quad T \geq \frac{2}{p\varepsilon\delta}\log n.$$

(Here δ is the constant from the definition of weak-diversity (1.20) *and ε is the constant from condition* (1.18)*.)*

With Propositions 1.1 and 1.2 at hand, it is natural to study the relationship between the excess growth rate criterion and market diversity. An important inequality in this direction, established in [19], is the following:

$$\frac{\varepsilon}{2}\big(1-\pi_{(1)}(t)\big) \leq \gamma_*^\pi(t) \leq M\big(1-\pi_{(1)}(t)\big), \;\; 0 \leq t \leq \infty. \tag{1.22}$$

Here ε and M are the constants from (1.18), $\gamma_*^\pi(t)$ is the excess growth rate corresponding to portfolio rule π and $\pi_{(1)}(t)$ is defined as in (1.19).

By integrating the left hand side of inequality (1.22) it is easy to see that if the market (1.1) satisfies the strong non-degeneracy condition captured by the left hand side of (1.18), then the requirement (1.16) of Proposition 1.1 is satisfied with $\Gamma(t) = \gamma_* t$ and $\gamma_* = \frac{\varepsilon\delta}{2}$. Hence strong arbitrage opportunities relative to the market portfolio do exist.

Conversely, when the right-hand side of (1.18) is satisfied, then (1.16) with $\Gamma(t) = \gamma_* t$ for some γ_* in $(0, M)$ leads to the weak diversity condition (1.20) for $\delta = \frac{\gamma_*}{M}$.

1.2. **The representation of stock prices in terms of Bessel processes in a volatility-stabilized market and related results.** In this section we introduce the system of stochastic differential equations that characterize the volatility-stabilized process and derive, following the exposition in [17], the key fact that the stock prices are time-changed Bessel processes. This result constitutes the motivation and starting point upon which the work in this thesis is based.

We will give a brief overview of the existing results on volatility-stabilized markets from [17], and emphasize that such markets are neither diverse nor do they satisfy the upper bound of condition (1.18) (boundedness away from infinity of the variance/covariance matrix). In particular, the property (1.5) for the quantities (1.4) is not guaranteed for such processes.

Consider the following system of SDEs:

$$d\big(\log X_i(t)\big) = \frac{\alpha}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t), \quad i = 1, \ldots, n \tag{1.23}$$

for some given number $\alpha \geq 0$, or equivalently

(1.24)
$$dX_i(t) = \frac{1+\alpha}{2}\big(X_1(t) + \cdots + X_n(t)\big)dt + \sqrt{X_i(t)\big(X_1(t) + \cdots + X_n(t)\big)}dW_i(t).$$

We know from the work of Bass and Perkins (2002) that the system (1.24) of stochastic differential equations admits a weak solution, and that this solution is unique in distribution. The state-process $X(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))$ of this solution will be our volatility-stabilized process; it takes values in $(0, \infty)^n$. Using Itô's rule, it follows that

$$dX_i(t) = X_i(t)\left(\frac{1+\alpha}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t)\right). \tag{1.25}$$

In [17] the core of the exposition is devoted to the case $\alpha = 0$. The model (1.23) assigns both big variances and big growth rates to the smallest stocks, but in a manner that makes the overall market performance remarkably stable.

Using the notation of the previous section

$$\sigma_{ik}(t) = \frac{\delta_{ik}}{\sqrt{\mu_k(t)}}, \quad \gamma_i(t) = \frac{\alpha}{2\mu_i(t)}. \tag{1.26}$$

Straightforward computations give constant variance and growth rates for the resulting market, namely:

$$a^{\mu\mu}(t) = 1, \quad \gamma_*^{\mu}(t) = \gamma_* := \frac{n-1}{2} \tag{1.27}$$

and

$$\gamma^{\mu}(t) = \sum_{i=1}^{n} \mu_i(t)\frac{\alpha}{2\mu_i(t)} + \gamma_*^{\mu}(t) \equiv \frac{(1+\alpha)n - 1}{2} = \frac{mn}{4} - \frac{1}{2} =: \gamma > 0. \tag{1.28}$$

Here we have defined $m := 2(1+\alpha)$.

From (1.6) and (1.25) it follows that

$$dZ^{\mu}(t) = Z^{\mu}(t)\sum_{i=1}^{n}\left(\frac{1+\alpha}{2}dt + \sqrt{\mu_i(t)}dW_i(t)\right) \quad \text{or} \tag{1.29}$$

$$dZ^{\mu}(t) = Z^{\mu}(t)\left(\frac{n(1+\alpha)}{2}dt + dB(t)\right). \tag{1.30}$$

Here $B(t) := \sum_{i=1}^{n}\int_0^t \sqrt{\mu_i(s)}dW_i(s)$, $0 \le t < \infty$ is a one-dimensional Brownian motion, because it is a continuous local martingale and its quadratic variation is exactly $\langle B\rangle(t) = t$, so the conclusion follows by P. Lévy's theorem.

It follows that $Z^{\mu}(t) = Z^{\mu}(0)\exp(\gamma t + B(t))$, where γ is the constant from (1.28). Recalling (1.15) we get that

$$X(t) = X_1(t) + \cdots + X_n(t) = X(0)\exp(\gamma t + B(t)), \quad 0 \le t < \infty. \tag{1.31}$$

Setting $x := X(0)$, we introduce the continuous, strictly increasing time change

$$\Lambda(t) := \int_0^t \left(\frac{X(s)}{4}\right)ds = \frac{x}{4}\int_0^t \exp(\gamma s + B(s))ds, \quad 0 \le t < \infty \tag{1.32}$$

and the process

$$\widehat{W}_i(t) = \int_0^{\Lambda^{-1}(t)} \sqrt{\Lambda'(u)}dW_i(u), \quad 0 \le t < \infty \tag{1.33}$$

for $i = 1, \ldots, n$. We see that $\langle \widehat{W}_i, \widehat{W}_j\rangle(t) = t\delta_{ij}$, so the processes $\widehat{W}_1(\cdot), \ldots, \widehat{W}_n(\cdot)$ are independent Brownian motions, by P. Lévy's characterization. From Itô's rule we obtain from (1.25)

$$d\sqrt{X_i(t)} = \frac{X(t)(1+2\alpha)}{8\sqrt{X_i(t)}}dt + \frac{1}{2}\sqrt{X(t)}dW_i(t), \tag{1.34}$$

and with the notation of (1.32), (1.33) this equation can be rewritten as

$$\sqrt{X_i(t)} = \sqrt{X_i(0)} + \int_0^t \frac{(m-1)d\Lambda(s)}{2\sqrt{X_i(s)}} + \widehat{W}_i\big(\Lambda(t)\big), \quad i = 1, \dots, n. \tag{1.35}$$

Define $R_i(\cdot) := \sqrt{X_i\big(\Lambda^{-1}(\cdot)\big)}$. After a change of variable formula (1.35) becomes

$$R_i(t) = \sqrt{X_i(0)} + \int_0^t \frac{(m-1)ds}{2R_i(s)} + \widehat{W}_i(t), \quad 0 \le t < \infty. \tag{1.36}$$

This is exactly the stochastic differential equation for a Bessel process of dimension m. For m integer greater or equal than 2, the Bessel process of dimension m can be thought of as the radius of an $m-$dimensional Brownian motion $\big(B_1(\cdot), \dots, B_n(\cdot)\big)$, that is $R^2(\cdot) := \sum_{i=1}^{m} B_i^2(\cdot)$.

It is checked readily that the squared Bessel process $Q_i(\cdot) := \big(R_i(\cdot)\big)^2 = X_i\big(\Lambda^{-1}(\cdot)\big)$ satisfies the equation

$$dQ_i(t) = mdt + 2\sqrt{\big(Q_i(t)\big)^+} d\widehat{W}_i(t). \tag{1.37}$$

General theorems (see Proposition 2.13 in [30, p. 291] and Proposition 3.20 in [30, p. 309]) ensure that this stochastic differential equation has a pathwise unique strong solution for any $Q_i(0) \ge 0$ and $m \ge 0$. For m taking non-integer values, the corresponding solution is still going to be called the squared Bessel process of dimension m.

We also make the important observation that the $m-$dimensional Bessel processes $R_1(\cdot), \dots, R_n(\cdot)$ are independent, being adapted to the filtrations of the independent Brownian motions $\widehat{W}_i(t)$. A comprehensive survey on Bessel processes can be found in Chapter 11 of [49]. A more concise exposition can be found in [10]; several key properties are discussed in this paper, including the additivity property and the Laplace transform. The additivity property establishes that the sum of two squared Bessel processes of dimensions m_1 and m_2 respectively, is a squared Bessel process of dimension $m_1 + m_2$. As in (1.37), m_1 and m_2 may be integers or not. It follows that

$$X\big(\Lambda^{-1}(t)\big) = \sum_{i=1}^{n} X_i\big(\Lambda^{-1}(t)\big) = \sum_{i=1}^{n} R_i^2(t) =: R^2(t), \;\; 0 \le t < \infty \tag{1.38}$$

where $R^2(\cdot)$ is a squared Bessel process of dimension mn.

The representations

$$X_i(T) = R_i^2\big(\Lambda(T)\big), \; i = 1, \dots, n \;\text{ and }\; X(T) = R^2\big(\Lambda(T)\big) \tag{1.39}$$

are the starting key for the results to be presented in the next chapters. Expression (1.32) then becomes $4\Lambda(T) = \int_0^T R^2\big(\Lambda(s)\big)ds$, which translates into

$$\Lambda^{-1}(t) = 4 \int_0^t \frac{ds}{R^2(s)}. \tag{1.40}$$

Combining some of the equalities above, Fernholz and Karatzas obtain in [17, p. 19] a new proof of the Lamperti representation (its classic proof can be found in [10], and extensive generalizations are discussed in [27]). In particular, putting together (1.31) and (1.39) one gets that

$$R^2\big(\Lambda(t)\big) = x \exp(\gamma t + B(t)) \tag{1.41}$$

and further on, using (1.32), we deduce that

$$\boxed{R^2\left(\tfrac{x}{4}\int_0^t \exp(\gamma s + B(s))ds\right) = x \exp(\gamma t + B(t)).} \tag{1.42}$$

Using the new Brownian motion $\widetilde{B}(\cdot) := \frac{1}{2}B(4\cdot)$, this last equality can also be written in the form

(1.43)
$$R\left(x\int_0^\theta \exp\left(2(2\gamma s + \widetilde{B}(s))\right) ds\right) = \sqrt{x}\exp\left(2\gamma\theta + \widetilde{B}(\theta)\right), \quad 0 \leq \theta < \infty.$$

From an intuitive point of view, it is useful to note that the volatility-stabilized process model (1.23) exhibits a striking similarity with a classic Black-Scholes market model consisting of n stocks that can be represented as geometric Brownian motions. According to the Lamperti representation (1.42), the prices of these stocks can be represented as time-changed squared Bessel processes, with the time change being intrinsic to each stock. By contrast, in the volatility-stabilized market the stock prices are time-changed squared Bessel processes, with the time change depending on the entire market and being the same for each stock. In both cases, the clock represented by the time change captures the real life activity, when changes in stock prices occur. The mathematical models capture the fact that these times can be thought of as intrinsic to each company or extrinsic and depending on general economic news.

With the representations (1.39) and (1.40) at hand, the limiting behavior of the volatility-stabilized process $X(\cdot)$ is studied in [17]. We recall two of the main results of this paper:

Proposition 1.3. *For the model* (1.23) *the long-term growth rate for the entire market and for the biggest stock are computed as*

(1.44)
$$\lim_{t\to\infty}\left(\frac{1}{t}\log X(t)\right) = \lim_{t\to\infty}\left(\frac{1}{t}\log X_{(1)}(t)\right) = \gamma \quad a.s.$$

For the model (1.23) *with* $\alpha > 0$*, we have for every* $i = 1, \dots, n$

$$\lim_{t\to\infty} \left(\frac{1}{t} \log X_i(t) \right) = \gamma \quad a.s. \tag{1.45}$$

For the model (1.23) *with* $\alpha = 0$*, we have for every* $i = 1, \dots, n$

$$\limsup_{t\to\infty} \left(\frac{1}{t} \log X_i(t) \right) = \gamma, \quad \liminf_{t\to\infty} \left(\frac{1}{t} \log X_i(t) \right) = -\infty \quad a.s. \tag{1.46}$$

$$\lim_{t\to\infty} \left(\frac{1}{t} \log X_i(t) \right) = \gamma \quad \textit{in probability.}$$

Here γ *is the constant from* (1.28).

A key ingredient in the proof of Proposition 1.3 is the law of large numbers for the Bessel clock $\Lambda^{-1}(t)$, to be found in [61]:

Theorem 1.4. *If* $\big(R^{(\nu)}(t), t \geq 0\big)$ *is a Bessel process with dimension* $d > 2$ *(and index* $\nu := (d/2) - 1 > 0$*) starting at* $R^{(\nu)}(0) \neq 0$*, we have*

$$\frac{1}{\log t} \int_0^t \frac{ds}{\big(R^{(\nu)}(s)\big)^2} \underset{t\to\infty}{\longrightarrow} \frac{1}{d-2} = \frac{1}{2\nu} \quad a.s. \quad \textit{and in } L^p \tag{1.47}$$

$$\sqrt{\log t} \left(\frac{1}{\log t} \int_0^t \frac{ds}{\big(R^{(\nu)}(s)\big)^2} - \frac{1}{d-2} \right) \underset{t\to\infty}{\overset{law}{\longrightarrow}} N, \tag{1.48}$$

where N *is a centered Gaussian variable with variance* $\sigma^2 = 1/(2\nu^3)$*.*

The second result in [17] that characterizes the limiting behavior of the volatility-stabilized market states that:

Proposition 1.5. *For every* $u \in [0, \infty)$*,* $i = 1, \dots, n$ *and* $\delta \in (0, 1)$ *we have*

$$\lim_{u\to\infty} \mathbb{P}\left[\mu_i\left(\Lambda^{-1}(u)\right) \leq 1 - \delta\right] = 1 - \delta^{n-1},$$

where $\Lambda^{-1}(\cdot)$ *is the inverse of the continuous, strictly increasing process* $\Lambda(\cdot)$ *of* (1.32)*.*

This proposition provides in a straightforward manner the important conclusion that the market (1.23) is not diverse. This amounts to the fact that there is no number $\delta \in (0,1)$ such that

$$\mathbb{P}\left[\max_{1\leq i\leq n}\mu_i(t) < 1-\delta,\ (\forall)\ 0 \leq t < \infty\right] = 1. \tag{1.49}$$

The notions of diversity (1.49) and weak diversity (1.20) are studied at large in [19], where an example of a market that is weakly diverse but not diverse is provided.

In light of these examples and recalling the relation between the excess growth rate and the weak diversity arbitrage criteria discussed in Section 1.1, it is natural to ask whether the volatility-stabilized market of (1.23) is weakly diverse. To the best of my knowledge, this is an open question and not one that is easy to settle.

Another key feature of the volatility-stabilized market is that it does not satisfy the boundedness away from infinity represented by the right-hand side of inequality (1.18). In Appendix *A* we provide an argument for this conclusion that relies on the Laplace transform of the squared Bessel process. A reference that includes a derivation of this Laplace transform is the paper [10] by Dufresne.

2. **Bessel and Jacobi Processes**

2.1. **Preliminaries on Bessel processes.** In Section 1.2 we introduced the squared Bessel process $Q(\cdot)$ of dimension $m \geq 2$ started at $q \geq 0$, as the unique strong solution of the stochastic differential equation:

$$dQ(t) = mdt + 2\sqrt{Q(t)}dB(t), \quad Q(0) = q. \tag{2.1}$$

In the literature on Bessel processes, the parameter $\zeta := (m/2) - 1 \in [0, \infty)$ is known as the *index* of the Bessel process. The Bessel process of index ζ, $R^{(\zeta)}(\cdot)$ is defined as $R^{(\zeta)}(t) := \sqrt{Q(t)}$, where $Q(\cdot)$ is governed by (2.1) with $m \equiv 2(1+\zeta)$. A direct application of Itô's rule shows that

$$dR^{(\zeta)}(t) = dB(t) + \left(\zeta + \frac{1}{2}\right)\frac{dt}{R^{(\zeta)}(t)}. \tag{2.2}$$

Let $\mathbb{P}^{(\zeta)}$ be the law of the Bessel process with index $\zeta \geq 0$ and starting point $R(0) > 0$ on the canonical space $C(\mathbb{R}_+, \mathbb{R}_+)$, where $R^{(\zeta)}(t)(\omega) = \omega(t)$. By $\mathbb{E}^{(\zeta)}$ we denote expectations under $\mathbb{P}^{(\zeta)}$. Let $\mathcal{R}^{\zeta}(t) = \sigma\{R^{(\zeta)}(s), s \leq t\}$ be the filtration generated by the Bessel process of index ζ. The process $B(\cdot)$ is a standard one-dimensional Brownian motion under $\big(\mathbb{P}^{(\zeta)}, (\mathcal{R}^{\zeta}(t))_{t\geq 0}\big)$.

From here onwards, we shall write $R(\cdot)$ for $R^{(0)}(\cdot)$, that is, for the process $R(\cdot)$ that satisfies the stochastic differential equation

$$R(t) = R(0) + B(t) + \frac{1}{2}\int_0^t \frac{ds}{R(s)}, \quad 0 \leq t < \infty. \tag{2.3}$$

Let us fix now a real constant $\gamma \geq 0$, and re-write (2.3) as

$$R(t) = R(0) + \left(B(t) - \gamma\int_0^t \frac{ds}{R(s)}\right) + \left(\gamma + \frac{1}{2}\right)\int_0^t \frac{ds}{R(s)}. \tag{2.4}$$

A comparison of the formula (2.4) with (2.2) with suggests that, under a Girsanov change of measure, $R(\cdot)$ can be regarded as a Bessel process of index γ. This is the key idea of the proof that follows.

We start by introducing the exponential process:

$$Z(t) := \exp\left(\gamma \int_0^t \frac{dB(s)}{R(s)} - \frac{\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)}\right), \quad 0 \le t < \infty. \tag{2.5}$$

Standard theorems in stochastic calculus ensure that $Z(\cdot)$ is a positive local martingale and a supermartingale.

From the initial equation (2.3) satisfied by the Bessel process, it follows that

$$\frac{dR(s)}{R(s)} = \frac{dB(s)}{R(s)} + \frac{ds}{2R^2(s)}, \tag{2.6}$$

so (2.5) becomes

$$\log Z(t) = \gamma \int_0^t \frac{dB(s)}{R(s)} - \frac{\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)} = \gamma \int_0^t \frac{dR(s)}{R(s)} - \frac{\gamma+\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)}. \tag{2.7}$$

By Itô's rule

$$d\left(\log R(t)\right) = \frac{dR(t)}{R(t)} - \frac{dt}{2R^2(t)}, \tag{2.8}$$

so from formulas (2.7) and (2.8) together we get

$$\begin{aligned} \log Z(t) &= \gamma\left(\log\left(\frac{R(t)}{R(0)}\right) + \frac{1}{2}\int_0^t \frac{ds}{R^2(s)}\right) - \frac{\gamma+\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)} = \\ &= \gamma \log\left(\frac{R(t)}{R(0)}\right) - \frac{\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)} \end{aligned} \tag{2.9}$$

and hence

$$Z(t) = \left(\frac{R(t)}{R(0)}\right)^{\gamma} \exp\left(-\frac{\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)}\right). \tag{2.10}$$

Next we are going to show that $(Z(t))_{t\geq 0}$ is a true $(\mathbb{P}^{(0)}, (\mathcal{R}^0(t))_{t\geq 0})$ martingale. The argument builds upon ideas in [58]. First we recall that a local martingale $(X(t))_{t\geq 0}$ for which

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X(s)|\right] < \infty \tag{2.11}$$

for every $t > 0$, is a martingale. This criterion can be found in [44, p.16].

The random variable $Z(t)$ is bounded from above by $\left(\frac{R(t)}{R(0)}\right)^{\gamma}$, so it suffices to verify that

$$\mathbb{E}^{(0)}\left[\sup_{0\leq s\leq t} (R(s))^{\gamma}\right] < \infty, \quad \text{for every} \quad t > 0. \tag{2.12}$$

The inequality above calls for an application of Doob's maximal inequality; see Chapter 2 of [49] for a general discussion of martingale inequalities. Doob's maximal inequality for Bessel processes has been studied by J. L. Pedersen in [47]. Inequality (2.12) follows from the main result of [47], which we state below:

Theorem 2.1. *Let $(R(t))_{t\geq 0}$ be a Bessel process of dimension $d > 0$ started at $R(0) \geq 0$. The maximal inequality*

$$\mathbb{E}\left[\max_{0\leq t\leq \tau} \big(R(t)\big)^p\right] \leq \left(\frac{p}{p-(2-d)}\right)^{\frac{p}{2-d}} \mathbb{E}\Big[\big(R(\tau)\big)^p\Big] - \frac{p}{p-(2-d)}\big(R(0)\big)^p \tag{2.13}$$

holds for all $p > (2-d)\vee 0$ and all stopping times τ with respect to the filtration generated by the Bessel process $(R(t))_{t\geq 0}$, satisfying the inequality

$$\mathbb{E}\left[\tau^{\frac{p}{2}}\right] < \infty. \tag{2.14}$$

(Recall the notation $a \vee b := \max(a, b)$).

Since the exponential process $Z(\cdot)$ of (2.5) is a $(\mathbb{P}^{(0)}, \mathcal{R}^0(t))$ martingale, we can apply Girsanov's theorem to obtain that under a change of measure a Bessel process of index 0 becomes a Bessel process of index γ and for any $\mathcal{R}^0(t)$ measurable random variable X

$$\mathbb{E}^{(\gamma)}[X] = \mathbb{E}^{(0)}\left[XZ(t)\right]. \tag{2.15}$$

Throughout the notes, we shall use the formal equality

$$\mathbb{E}[\mathbf{1}_{\{X\in dx\}}Y] = \mathbb{E}[Y|X=x]\cdot\mathbb{P}[X\in dx] \tag{2.16}$$

for arbitrary random variables X and Y, as a shorthand for

$$\mathbb{E}[Y\mathbf{1}_A(X)] = \int_A \mathbb{E}[Y|X=x]\cdot\mathbb{P}[X\in dx], \quad A\in\mathcal{B}(\mathbb{R}). \tag{2.17}$$

From (2.15), (2.16), (2.10) we have then

(2.18)
$$\mathbb{P}^{(\gamma)}[R(t)\in dr] = \mathbb{E}^{(\gamma)}[\mathbf{1}_{\{R(t)\in dr\}}] = \mathbb{E}^{(0)}\left[\mathbf{1}_{\{R(t)\in dr\}}\left(\frac{r}{R(0)}\right)^{\gamma}\exp\left(-\frac{\gamma^2}{2}\int_0^t\frac{ds}{R^2(s)}\right)\right] =$$

$$= \mathbb{E}^{(0)}\left[\left(\frac{r}{R(0)}\right)^{\gamma}\exp\left(-\frac{\gamma^2}{2}\int_0^t\frac{ds}{R^2(s)}\right)\Big|R(t)=r\right]\cdot\mathbb{P}^{(0)}[R(t)\in dr].$$

But the transition probabilities of the Bessel process of index γ are known as

(2.19)
$$\mathbb{P}^{(\gamma)}[R(t)\in dr] = \frac{r}{t}\left(\frac{r}{R(0)}\right)^{\gamma}e^{-\frac{r^2+R^2(0)}{2t}}\,I_\gamma\left(\frac{rR(0)}{t}\right)dr, \quad r>0, \quad \gamma\geq 0;$$

see the paper [10] by Dufresne for a derivation of these transition probabilities. Here

$$I_v(z) = \left(\frac{z}{2}\right)^v\sum_{k=0}^{\infty}\frac{(z^2/4)^k}{k!\Gamma(v+k+1)} \tag{2.20}$$

is the modified Bessel function of the first kind. Hence, the right-hand side of (2.18) becomes

$$\frac{r}{t}\left(\frac{r}{R(0)}\right)^{\gamma}e^{-\frac{r^2+R^2(0)}{2t}}I_\gamma\left(\frac{rR(0)}{t}\right) = \tag{2.21}$$

$$= \frac{r}{t} e^{-\frac{r^2+R^2(0)}{2t}} I_0\left(\frac{rR(0)}{t}\right)\left(\frac{r}{R(0)}\right)^{\gamma} \mathbb{E}^{(0)}\left[\exp\left(-\frac{\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)}\right)\Big|R(t)=r\right]$$

and we are able to deduce the following important formula, first derived by Marc Yor in 1980 in the paper [58]:

$$\mathbb{E}^{(0)}\left[\exp\left(-\frac{\gamma^2}{2}\int_0^t \frac{ds}{R^2(s)}\right)\Big|R(t)=r\right] = \frac{I_\gamma}{I_0}\left(\frac{rR(0)}{t}\right). \tag{2.22}$$

Now suppose $R(\cdot)$ is a Bessel process of index γ. In the same way as above, one can deduce that

$$\mathbb{E}^{(\gamma)}\left[\exp\left(-\lambda\int_0^t \frac{ds}{R^2(s)}\right)\Big|R(t)=r\right] = \frac{I_{\sqrt{2\lambda+\gamma^2}}}{I_\gamma}\left(\frac{rR(0)}{t}\right). \tag{2.23}$$

For a Bessel process $R(\cdot)$ of index γ, we shall try now to find an explicit formula for the conditional density $\mathbb{P}\left[\int_0^t R^{-2}(s)ds \in dT\Big|R(t)=r\right]$ by inverting the above Laplace transform. Following [56, p. 94] we can write

$$I_{|\gamma|}(r) = \int_0^\infty e^{-\frac{\gamma^2 u}{2}}\theta_r(u)du \quad \text{thus} \tag{2.24}$$

$$I_{\sqrt{2\lambda+\gamma^2}}(r) = \int_0^\infty e^{-\lambda u}e^{-\frac{\gamma^2 u}{2}}\theta_r(u)du \quad \text{for} \quad r>0, \tag{2.25}$$

where by [56, p. 42] we have

$$\theta_r(u) := \frac{re^{\frac{\pi^2}{2u}}}{\sqrt{2\pi^3 u}}\int_0^\infty e^{-\frac{y^2}{2u}}e^{-r\cosh y}(\sinh y)\sin\left(\frac{\pi y}{u}\right)dy. \tag{2.26}$$

Therefore

$$\mathbb{P}^{(\gamma)}\left[\int_0^t \frac{ds}{R^2(s)} \in dT\Big|R(t)=r\right] = \frac{e^{-\frac{\gamma^2 T}{2}}\theta_{\frac{rR(0)}{t}}(T)}{I_\gamma(\frac{rR(0)}{t})}dT, \tag{2.27}$$

and we deduce the joint probability density

$$\mathbb{P}^{(\gamma)}\left[\int_0^t \frac{ds}{R^2(s)} \in dT, R(t)\in dr\right] = \frac{r}{t}\left(\frac{r}{R(0)}\right)^{\gamma} e^{-\frac{r^2+R^2(0)}{2t}}e^{-\frac{\gamma^2 T}{2}}\theta_{\frac{rR(0)}{t}}(T)dTdr. \tag{2.28}$$

This leads to the computation of the marginal distribution

(2.29)

$$\mathbb{P}^{(\gamma)}\left[\int_0^t R^{-2}(s)ds \in dT\right] = \int_0^\infty \frac{r}{t}\left(\frac{r}{R(0)}\right)^\gamma e^{-\frac{r^2+R^2(0)}{2t}} e^{-\frac{\gamma^2 T}{2}} \theta_{\frac{rR(0)}{t}}(T)drdT,$$

and, by Bayes's rule, to the conditional distribution

$$\mathbb{P}\left[R(t) \in dr \middle| \int_0^t \frac{ds}{R^2(s)} = T\right] = \frac{r^{\gamma+1}e^{-\frac{r^2}{2t}}\theta_{\frac{rR(0)}{t}}(T)dr}{\int_0^\infty \rho^{\gamma+1}e^{-\frac{\rho^2}{2t}}\theta_{\frac{\rho R(0)}{t}}(T)d\rho}. \tag{2.30}$$

The function $\theta_r(\cdot)$ of (2.26) is often encountered in the literature in connection with the joint distribution of Brownian motion with drift and the integral of its exponential. The classical result in this direction, derived by M. Yor in [57], is the following:

Theorem 2.2. *Let*

$$B^{(\mu)}(t) = B(t) + \mu t, \ \ A^{(\mu)}(t) = \int_0^t \exp\left(2B^{(\mu)}(s)\right)ds,$$

where $B(\cdot)$ is a standard one-dimensional Brownian motion. Then for $t > 0$ fixed, $\mu > 0$ and $x \in \mathbb{R}$, the joint distribution of these two random variables

(2.31)

$$\mathbb{P}\left[A^{(\mu)}(t) \in du, B^{(\mu)}(t) \in dx\right] = \exp\left(\mu x - \frac{\mu^2 t}{2}\right)\theta_{e^x/u}(t)\exp\left(-\frac{1+e^{2x}}{2u}\right)\frac{dudx}{u}.$$

The original proof of this result can be found in [57]; a different proof based on Schrödinger operators, Green's functions and the Feyman-Kac formula can be found in [43].

We conclude this section with two computations that will play a key role in the proofs of the main results. The first of these propositions can be found in [43, p. 320], and in [42, p. 91].

Proposition 2.3. *With $a_c(x) = argcosh(x)$ for $x \geq 1$, we have*

$$\int_0^\infty e^{-xr}\theta_r(t)\frac{dr}{r} = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{a_c(x)^2}{2t}\right) \quad \text{for any } t > 0. \tag{2.32}$$

Proof. Integrating both sides of (2.31) with respect to u yields

$$\frac{\mathbb{P}\left[B^{(\mu)}(t) \in dx\right]}{dx} = \frac{e^{-\frac{(x-\mu t)^2}{2t}}}{\sqrt{2\pi t}} = e^{\mu x - \frac{\mu^2 t}{2}} \int_0^\infty \exp\left(-\frac{1+e^{2x}}{2u}\right)\theta_{\frac{e^x}{u}}(t)\frac{du}{u}, \tag{2.33}$$

or equivalently:

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} = \int_0^\infty \exp\left(-\frac{1+e^{2x}}{2u}\right)\theta_{\frac{e^x}{u}}(t)\frac{du}{u} = \int_0^\infty \theta_r(t)e^{-r\cosh x}\frac{dr}{r} \tag{2.34}$$

which is exactly the desired proposition. □

Proposition 2.4. *For $R(\cdot)$ a Bessel process of index γ starting at $R(0)$ and*

$$C(t) := \int_0^t \frac{ds}{R^2(s)},$$

the following equality holds:

$$\int_0^\infty \mathbb{P}^{(\gamma)}\left[C(t) \in dc, R(t) \in dr\right] dt = \tag{2.35}$$

$$= \frac{r}{\sqrt{2\pi c}}\left(\frac{r}{R(0)}\right)^\gamma \exp\left(-\frac{1}{2c}\left[c^2\gamma^2 + \left(\log\frac{r}{R(0)}\right)^2\right]\right) dcdr =$$

$$= \frac{r}{\sqrt{2\pi c}}\exp\left(-\frac{1}{2c}\left(\log\frac{r}{R(0)} - c\gamma\right)^2\right) dcdr.$$

Proof. Using formula (2.28) for the joint density of $C(t)$ and $R(t)$, we get

$$\int_0^\infty \mathbb{P}^{(\gamma)}\left[C(t) \in dc, R(t) \in dr\right] dt =$$

$$= r\left(\frac{r}{R(0)}\right)^\gamma e^{-\frac{\gamma^2 c}{2}}\left(\int_0^\infty e^{-\frac{r^2+R^2(0)}{2t}}\theta_{\frac{rR(0)}{t}}(c)\frac{dt}{t}\right) dcdr\ .$$

After a change of variable, the above expression becomes

$$r\left(\frac{r}{R(0)}\right)^{\gamma} e^{-\frac{\gamma^2 c}{2}} \int_0^\infty e^{-\frac{(r^2+R^2(0))u}{2rR(0)}} \theta_u(c) \frac{du}{u},$$

and using Proposition 2.3 this is just

$$\frac{r}{\sqrt{2\pi c}}\left(\frac{r}{R(0)}\right)^{\gamma} e^{-\frac{\gamma^2 c}{2}} \exp\left(-\frac{\left[\left(\frac{r^2+R^2(0)}{2rR(0)}\right)\right]^2}{2c}\right).$$

It is straightforward to check that

$$\operatorname{argcosh}\left(\frac{r^2+R^2(0)}{2rR(0)}\right) = \log\left(\frac{r}{R(0)}\right),$$

which concludes the proof. □

2.2. **The skew-product decomposition of Bessel processes, the multidimensional Jacobi process and its connection to the volatility-stabilized market.** In a rather oversimplified way, skew-product decompositions of stochastic processes can be thought of as an analogue of polar coordinates. The classical result in this direction concerns an n-dimensional ($n > 1$) Brownian motion $(B(t),\, t \geq 0)$ starting at $x \neq 0$. Then

$$B(t) = |B(t)| \cdot \Theta_{\left(\int_0^t \frac{ds}{|B(s)|^2}\right)} \tag{2.36}$$

where $(\Theta(u),\, u \geq 0)$ is a standard Brownian motion on S^{n-1} independent of $(|B(t)|,\, t \geq 0)$. This result can be found in [32] or [54].

In this section we consider a family of squared Bessel processes $Q_1(\cdot), \ldots, Q_n(\cdot)$ of dimensions $\delta_1, \ldots, \delta_n$, respectively, as in (1.37). Then $Q(\cdot) := \sum_{i=1}^n Q_i(\cdot)$ is a squared Bessel process of dimension $\delta := \sum_{i=1}^n \delta_i$. Let the clock $C(\cdot)$ be defined as $C(t) := \int_0^t \frac{ds}{Q(s)}$, $t \geq 0$ and let $\alpha(u)$ be its inverse, defined as

$$\alpha(u) := \inf\{t \geq 0 |\, C(t) > u\}, \quad 0 \leq u < \infty. \tag{2.37}$$

In [53], Warren and Yor obtain an analogue of the representation (2.36) by establishing that

$$\frac{Q_i(t)}{Q(t)} = Y_i(C(t)), \quad 0 \le t < \infty, \quad i = 1, \dots, n, \tag{2.38}$$

where $Y_i(\cdot)$ are one-dimensional diffusions taking values in the interval $[0,1]$, independent of the clock $C(\cdot)$ and satisfying the following stochastic differential equations

$$dY_i(t) = \big(\delta_i - \delta Y_i(t)\big)dt + 2\sqrt{Y_i(t)(1 - Y_i(t))}dB_i(t), \quad i = 1, \dots, n \tag{2.39}$$

for suitably correlated Brownian motions $B_1(\cdot), \dots B_n(\cdot)$. Because $Y_1(u) + \cdots + Y_n(u) = 1$, these diffusions are not independent, but rather negatively correlated, with quadratic covariations

$$d\langle Y_i, Y_j\rangle(t) = -4Y_i(t)Y_j(t)dt, \tag{2.40}$$

as it will be transparent from the representations that are going to be obtained in the end of Section 5.2.

Various results on Jacobi processes can be found in [20], [24], and [25]. Due to the fact that they take values in the interval $[0,1]$, such processes are most useful to model dynamic bounded variables, such as probabilities or exchange rates as in [38]. We also note that the Jacobi process exhibits mean-reversion: as the process approaches the boundary points 0 and 1 respectively, the diffusion term becomes small and the drift becomes positive and negative respectively, forcing the process to stay within the interval. Feller's test for explosions (see [30, p. 348]) can be applied to the Jacobi diffusion to make the above statement mathematically rigorous.

Our goal is to generalize the result of Warren and Yor by establishing that the entire vector $(Y_1(\cdot), \dots, Y_n(\cdot))$ is a multidimensional diffusion, which we are going to refer to as the *multidimensional Jacobi process*. We are going to derive another system of stochastic differential equations satisfied by the process $(Y_1(\cdot), \ldots, Y_n(\cdot))$ that will make transparent the correlation structure of the Brownian motions $B_1(\cdot), \ldots, B_n(\cdot)$ from (2.39). The proofs of these results are to be found in Section 5.2, where instead of Bessel processes we are going to use the more general Cox-Ingersoll-Ross process for the derivation of the *multidimensional Jacobi process*.

We are now ready to derive the key fact that the vector of market weights $(\mu_1(\cdot), \ldots, \mu_n(\cdot))$ (defined by formula (1.14)), of the volatility-stabilized market (1.23) is a *multidimensional Jacobi process*. Indeed, according to (1.39) and (1.40) the individual market weights have the following representation

$$\mu_i(T) = \frac{X_i(T)}{X(T)} = \frac{R_i^2\left(\Lambda(T)\right)}{R^2\left(\Lambda(T)\right)}, \quad i = 1, \cdots, n. \tag{2.41}$$

The skew-product decomposition of Bessel processes from Proposition 1 in [53], tells us that

$$R_i^2(t) = R^2(t) Y_i\left(\int_0^t R^{-2}(s)ds\right), \quad 0 \le t < \infty.$$

Letting $t \leftarrow \Lambda(T)$, we get that

$$R_i^2\left(\Lambda(T)\right) = R^2\left(\Lambda(T)\right) Y_i\left(\int_0^{\Lambda(T)} R^{-2}(s)ds\right),$$

which together with formula (1.40) shows that

$$\mu_i(T) = Y_i\left(\frac{T}{4}\right), \quad i = 1, \cdots, n. \tag{2.42}$$

3. The Laws and Moments of Stock Prices and Market Weights

3.1. **The spectral representation of the transition density of a diffusion with applications to the one-dimensional Jacobi process.** One of the main open questions in [17] is to compute the distributions of the market weights, $\mu_i(t)$, $i = 1, \ldots, n$ for the volatility-stabilized market model. In this section we provide an answer to this question using the knowledge from Section 2.2 that each $\mu_i(\cdot)$ is a one-dimensional diffusion. We start by discussing the spectral representation of the transition density of a one-dimensional diffusion ($dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$) following the exposition in [32].

Consider a diffusion over a finite closed interval $I = [l, r]$, with $\sigma^2(x)$ continuous and positive on I and with l and r exit or reflecting boundaries. We recall that a boundary point l is an exit boundary if starting at it, it is impossible to reach any interior state b, no matter how near b is to l. Formally we write $\lim_{b\searrow l}\lim_{x\searrow l} P(T_b < t | X_0 = x) = 0$ for all $t > 0$. As usual, T_b is the first hitting time of the level b by our diffusion. A boundary point l is reflecting if the speed measure assigns the value 0 to $\{l\}$.

We recall that the scale function and speed density of our diffusion are given by the following expressions:

$$s(x) = \exp\left(-\int_{x_0}^{x} \frac{2b(y)}{\sigma^2(y)}dy\right) \quad \text{and} \quad m(x) = \frac{1}{\sigma^2(x)s(x)}. \tag{3.1}$$

Intuitively, $m(x)$ gives the time that $X(t)$ takes to exit a small interval centered at x. See [12, p. 227] for a mathematical explanation of this intuition.

With $f(x)$ bounded and continuous on (l, r) the function

$$u(t,x) := \mathbb{E}_x\left[f(X(t))\right] \tag{3.2}$$

satisfies the Kolmogorov backward differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{1}{m(x)}\frac{\partial}{\partial x}\left(\frac{1}{s(x)}\frac{\partial}{\partial x}\right)u = \mathcal{L}u, \tag{3.3}$$

with the initial condition $u(0,x) = f(x)$, and boundary condition $u(t,l) = 0$ if l is an exit boundary and $u'(t,l) = 0$ if l is reflecting and similarly at the right boundary r. A derivation of this equation can be found in [32, p. 214].

We try to find a solution of equation (3.3) using the method of separation of variables . Writing $u(t,x) = c(t)\phi(x)$ and substituting this into (3.3) gives

$$\frac{c'(t)}{c(t)} = \frac{(\mathcal{L}\phi)(x)}{\phi(x)}. \tag{3.4}$$

This equality can hold if and only if for some constant λ,

$$\begin{cases} c'(t) = -\lambda c(t) \\ (\mathcal{L}\phi)(x) = -\lambda\phi(x). \end{cases} \tag{3.5}$$

From the specified boundary conditions we also get that $\phi(l) = \phi(r) = 0$ if l and r are exit points.

It can be easily checked that the boundary value problem

$$\begin{cases} (\mathcal{L}\phi)(x) = -\lambda\phi(x) \\ \phi(l) = \phi(r) = 0 \end{cases} \tag{3.6}$$

is of elliptic type: $(\mathcal{L}\phi)(x) = \frac{1}{2}\sigma^2(x)\phi''(x) + b(x)\phi'(x)$ and because of the assumption at the start of this section, that σ is continuous, positive, and defined on a finite closed interval, it follows that $\frac{1}{2}\sigma^2$ is bounded away from zero by a positive constant, hence elliptic. An overview of the spectral theory of elliptic operators can be found in Chapter 6 of [40]. It follows that the operator $\mathcal{L}$ has a set of eigenvalues $(\lambda_k)_{k\geq 0}$, $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_k$, $\lambda_k \to \infty$, and corresponding eigenfunctions $\{\phi_k(x)\}_{k\geq 0}$ that are solutions of equation (3.6). The solution of the parabolic equation (3.3) can be represented as an infinite series:

$$u(t,x) = \sum_{k=0}^{\infty} c_k e^{-\lambda_k t} \phi_k(x) \tag{3.7}$$

where the constants c_k are to be determined so that the initial conditions are satisfied.

We further note that the family of eigenfunctions $\{\phi_k(x)\}_{k\geq 0}$ constitutes an orthogonal set with respect to the speed density $m(x)$. Indeed

$$-\lambda_j \int_l^r m(x)\phi_i(x)\phi_j(x)dx = \int_l^r m(x)\phi_i(x)(\mathcal{L}\phi_j)(x)dx = \tag{3.8}$$

$$= \frac{1}{2}\int_l^r \phi_i(x)\left(\frac{\partial}{\partial x}\left(\frac{1}{s(x)}\frac{\partial}{\partial x}\phi_j(x)\right)\right)dx = -\frac{1}{2}\int_l^r \phi_i^{'}(x)\phi_j^{'}(x)\frac{1}{s(x)}dx =$$

$$= -\int_l^r m(x)(\mathcal{L}\phi_i)(x)\phi_j(x)dx = -\lambda_i \int_l^r m(x)\phi_i(x)\phi_j(x)dx.$$

It follows that

$$\int_l^r m(x)\phi_i(x)\phi_j(x)dx = 0 \text{ for } i \neq j, \tag{3.9}$$

hence the conclusion.

From the orthogonality of the family $\{\phi_k(x)\}_{k\geq 0}$ we are able to deduce that

$$c_k = \pi_k \int_l^r m(y)f(y)\phi_k(y)dy, \quad k = 0, 1\ldots \tag{3.10}$$

where

$$f(x) := u(0,x) \text{ and } \pi_k := \frac{1}{\int_l^r m(y)\phi_k^2(y)dy}. \tag{3.11}$$

Denote by $p_t(x,y)$ the transition density of the diffusion $X(t)$, that is $\mathbb{P}^x\left[X(t)\in dy\right] = p_t(x,y)dy$. For $f(\cdot)$ defined as $f(x):=1_{[l,r]}(x)$, equality (3.2) combined with the representation (3.7), in conjunction with (3.11), helps us deduce the equality

$$\int_l^r p_t(x,y)dy = \sum_{k=0}^{\infty} e^{-\lambda_k t}\pi_k\phi_k(x)\left(\int_l^r m(y)\phi_k(y)dy\right). \tag{3.12}$$

Dividing by $(r-l)$ and letting the interval (l,r) shrink to y we obtain formally the spectral representation of the transition density:

$$p_t(x,y) = m(y)\sum_{k=0}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y)\pi_k. \tag{3.13}$$

Now let $Y(\cdot)$ be a Jacobi process satisfying the stochastic differential equation
(3.14)

$$dY(t) = \big(\delta_1-(\delta_1+\delta_2)Y(t)\big)dt+\eta\sqrt{Y(t)(1-Y(t))}dB(t),\ \ Y(0)=y_0\in(0,1).$$

The spectral decomposition of the Jacobi process has been derived by E. Wong in [55]. We recall it here following the presentation of [25]. The eigenvalues are given by:

$$\lambda_k = (\delta_1+\delta_2)k+\frac{\eta^2}{2}k(k-1),\ \ k=0,1\ldots \tag{3.15}$$

and the eigenfunctions are known to be the *Jacobi polynomials* (see Chapter 22 of [1]), which have the following expression:

$$P_k(y) = \left[\frac{\Gamma\left(k+\frac{2\delta_1}{\eta^2}\right)\left(2k+\frac{2\delta_1+2\delta_2}{\eta^2}-1\right)\Gamma\left(\frac{2\delta_1}{\eta^2}\right)\Gamma\left(\frac{2\delta_2}{\eta^2}\right)}{k!\Gamma\left(\frac{2\delta_1+2\delta_2}{\eta^2}+k-1\right)\Gamma\left(\frac{2\delta_1+2\delta_2}{\eta^2}\right)\Gamma\left(\frac{2\delta_2}{\eta^2}+k\right)}\right]^{\frac{1}{2}}. \tag{3.16}$$

$$\cdot\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\Gamma\left(\frac{2\delta_1+2\delta_2}{\eta^2}+k+i-1\right)}{\Gamma\left(\frac{2\delta_1}{\eta^2}+i\right)}y^i.$$

It is shown in [24] that the Jacobi process, the Ornstein-Uhlenbeck process

$$dY(t) = b(Y(t) - \beta)dt + \sqrt{c}\, dW(t) \tag{3.17}$$

and the square-root process

$$dY(t) = b(Y(t) - \beta)dt + \sqrt{c_1 Y(t) + c_0}\, dW(t) \tag{3.18}$$

are the only one-dimensional diffusions whose infinitesimal generator have polynomial eigenfunctions.

• In the particular case of the volatility-stabilized market presented in Section 1.2, the parameters η, δ_1, and δ_2 take the following values

$$\eta = 2, \quad \delta_1 = m, \quad \delta_2 = m(n-1). \tag{3.19}$$

We recall that n represents the number of stocks in the market and m is the dimension of each individual Bessel process that appears in the expression of stocks. The *Jacobi polynomials* of (3.16) and the corresponding eigenvalues of (3.15) take the form below, which we are going to use for the rest of this section.

$$\lambda_k = k(nm + 2k - 2), \quad k = 0, 1 \ldots \tag{3.20}$$

$$P_k(y) = \left[\frac{\Gamma\left(k + \frac{m}{2}\right)\left(2k + \frac{nm}{2} - 1\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m(n-1)}{2}\right)}{k!\Gamma\left(\frac{nm}{2} + k - 1\right)\Gamma\left(\frac{nm}{2}\right)\Gamma\left(\frac{m(n-1)}{2} + k\right)} \right]^{\frac{1}{2}} \cdot \tag{3.21}$$

$$\cdot \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\Gamma\left(\frac{mn}{2} + k + i - 1\right)}{\Gamma\left(\frac{m}{2} + i\right)} y^i.$$

Furthermore the local drift and volatility coefficients are then given as

$$b(y) = m(1 - ny), \quad \sigma(y) = 2\sqrt{y(1-y)}, \tag{3.22}$$

and a straightforward calculation leads to

$$s(y) = \left(2^n y(1-y)^{n-1}\right)^{-\frac{m}{2}} \text{ and } m(y) = 2^{\frac{nm}{2}-2} y^{\frac{m}{2}-1}(1-y)^{(n-1)\frac{m}{2}-1}. \tag{3.23}$$

(We have made the choice $x_0 = \frac{1}{2}$ in the expression of the scale function (3.1).)

• To have an explicit expression for the spectral representation of the Jacobi process we need to compute the quantity π_k of (3.11), where the expression of $m(y)$ and $P_k(y)$ are to be found in (3.23) and (3.21), respectively.

It can be checked through direct computation (and is worth doing as a sanity check) that the Jacobi polynomials are indeed eigenfunctions of the infinitesimal generator corresponding to the Jacobi stochastic differential equation (3.14). This sanity check amounts to verifying that the second equation of the system (3.5) is satisfied when $\mathcal{L} = \frac{1}{2}\sigma^2(y)\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}$ ($\sigma(y)$ and $b(y)$ are given in (3.22)), when ϕ is the polynomial P_k from (3.21), and when λ is the eigenvalue λ_k from (3.20), so we are left to check that the *Jacobi polynomial* of (3.21) is a solution to the Jacobi ordinary differential equation:

$$2y(1-y)P_k^{''}(y) + m(1-ny)P_k^{'}(y) = -k(nm+2k-2)P_k(y). \tag{3.24}$$

We further give an overview of the steps required to compute the quantity π_k from (3.11). The computational details for arbitrary $m \geq 2$ are provided in Appendix B and a different proof for the case $m = 2$ is provided in Appendix C.

The family of polynomials $\{P_k(y)\}_{k\geq 0}$ given by formula (3.21) is orthogonal with respect to the speed measure $m(y)$ (See Appendix B for proof.):

$$\int_0^1 m(y)P_k(y)P_l(y)dy = 0 \text{ for } k \neq l. \tag{3.25}$$

In addition, the *Jacobi polynomials* satisfy the recurrence below:

$$P_{k+1}(y) = (a_k y + b_k)P_k(y) - c_k P_{k-1}(y), \;\; k = 0, 1, \ldots \tag{3.26}$$

The exact values of the coefficients a_k, b_k, c_k, are relevant to the derivation of π_k. Multiplying (3.26) by $a_{k-1}P_{k-1}$ we get

$$a_{k-1}P_{k+1}P_{k-1} = a_{k-1}(a_k y + b_k)P_k P_{k-1} - a_{k-1}c_k P_{k-1}^2. \tag{3.27}$$

Writing (3.26) with $k-1$ instead of k and then multiplying by $a_k P_k$ we get

$$a_k P_k^2 = a_k(a_{k-1}y + b_{k-1})P_k P_{k-1} - c_{k-1}a_k P_{k-2}P_k. \tag{3.28}$$

Taking the difference between equalities (3.27) and (3.28), multiplying it by $m(y)$, integrating from 0 to 1 and using the orthogonality relation (3.25) we get

$$a_k \int_0^1 m(y)P_k^2(y)dy = a_{k-1}c_k \int_0^1 m(y)P_{k-1}^2(y)dy. \tag{3.29}$$

Denoting $I_k := \int_0^1 m(y)P_k^2(y)dy$, we next find out that

$$I_k = \frac{I_0 a_0}{a_k}\prod_{l=1}^{k} c_l\,. \tag{3.30}$$

To simplify notation we write $P_k(y) = p_k Q_k(y)$, where the polynomial $Q_k(y)$ is defined as

$$Q_k(y) := \sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\Gamma\left(\frac{mn}{2}+k+i-1\right)}{\Gamma\left(\frac{m}{2}+i\right)}y^i, \tag{3.31}$$

and the constant p_k is defined so that the above polynomial equality holds, when $P_k(y)$ is given by formula (3.21).

We observe that the family of polynomials $\{Q_k(y)\}_{k\geq 0}$ satisfies a recurrence relation similar to (3.26):

$$Q_{k+1}(y) = \left(\frac{a_k p_k}{p_{k+1}}y + \frac{b_k p_k}{p_{k+1}}\right)Q_k(y) - \frac{c_k p_{k-1}}{p_{k+1}}Q_{k-1}(y). \tag{3.32}$$

Defining $A_k := \frac{a_k p_k}{p_{k+1}}$, $B_k := \frac{b_k p_k}{p_{k+1}}$, $C_k := \frac{c_k p_{k-1}}{p_{k+1}}$, we get that the coefficients of interest a_k and c_k are expressed as

$$a_k = \frac{A_k p_{k+1}}{p_k}, \quad c_k = \frac{C_k p_{k+1}}{p_{k-1}} \tag{3.33}$$

and after a few more algebraic manipulations, the integral I_k turns out to be

$$I_k = \frac{I_0 A_0}{A_k} \cdot \frac{p_k^2}{p_0^2} \prod_{l=1}^{k} C_l \,. \tag{3.34}$$

The values of the coefficients $A_1, \ldots, A_k$ and $C_1, \ldots, C_k$ are derived in Appendix B:

$$A_k = -\frac{\left(\frac{mn}{2} + 2k\right)\left(\frac{mn}{2} + 2k - 1\right)}{\frac{m}{2} + k} \tag{3.35}$$

$$C_k = \frac{k\left(k - 2 + \frac{mn}{2}\right)\left(2k + \frac{mn}{2}\right)\left(\frac{m(n-1)}{2} + k - 1\right)}{\left(k + \frac{m}{2}\right)\left(2k - 2 + \frac{mn}{2}\right)}. \tag{3.36}$$

It follows that the product $\prod_{l=1}^{k} C_l$ can be expressed as follows:

$$\prod_{l=1}^{k} C_l = \frac{k!\left(\frac{mn}{2} + 2k\right)}{\frac{mn}{2}} \frac{\Gamma\left(\frac{m}{2} + 1\right)\Gamma\left(\frac{mn}{2} + k - 1\right)\Gamma\left(\frac{m(n-1)}{2} + k\right)}{\Gamma\left(\frac{mn}{2} - 1\right)\Gamma\left(\frac{m(n-1)}{2}\right)\Gamma\left(k + \frac{m}{2} + 1\right)}. \tag{3.37}$$

A few more computations lead to the rather striking conclusion

$$\frac{A_0}{A_k} \cdot \frac{p_k^2}{p_0^2} \prod_{l=1}^{k} C_l \,. \tag{3.38}$$

Hence $I_k = I_0$ and $\pi_k = \pi_0$. The same conclusion is obtained for the special case $m = 2$ in Appendix C using a method that does not rely on the orthogonality property and recurrence relation satisfied by the *Jacobi polynomials*. Hence

$$\pi_k = \pi_0 = \frac{\Gamma\left(\frac{mn}{2}\right)}{2^{\frac{mn}{2}-2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m(n-1)}{2}\right)}. \tag{3.39}$$

• Putting together formulas (3.13), (3.20), and (3.23) we obtain the final expression for the spectral representation of the transition density of the *Jacobi process*:

$$p_t(x,y) = \frac{y^{\frac{m}{2}-1}(1-y)^{\frac{(n-1)m}{2}-1}\Gamma\left(\frac{mn}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m(n-1)}{2}\right)} \sum_{k=0}^{\infty} e^{-k(nm+2k-2)t} P_k(x) P_k(y) \,, \tag{3.40}$$

where $P_k(y)$ is the *Jacobi polynomial*, as it appears in formula (3.21). In particular, the transition density for the i^{th} market weight $\mu_i(T)$ of the volatility stabilized market, appearing in formulas (2.41) and (2.42), is represented as:

(3.41)

$$\frac{\mathbb{P}\left[\mu_i(T) \in dy\right]}{dy} = \frac{y^{\frac{m}{2}-1}(1-y)^{\frac{(n-1)m}{2}-1}\Gamma\left(\frac{mn}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m(n-1)}{2}\right)} \sum_{k=0}^{\infty} e^{-k(nm+2k-2)\frac{T}{4}} P_k\big(\mu_i(0)\big) P_k(y) \,,$$

$$0 < y < 1, \quad \text{for } i = 1, \ldots, n.$$

The work in this section could be further pursued by studying the spectral theory of the infinitesimal generator $\mathcal{L}$ of (5.36) of the *multivariate Jacobi process*. Multivariate *Jacobi polynomials* do exist in the literature and are a special kind of multivariate orthogonal polynomials, whose properties are discussed for example in [11]. It would be interesting to explore the computational problem as to whether the multivariate *Jacobi polynomials* form a complete family of eigenfunctions for the generator of the multivariate process. Moreover, if the answer would turn out to be affirmative, it would be worth to pursue the computation of mixed moments of the *multivariate Jacobi process* (the case of individual moments will be studied in Section 3.4), in hope of gaining a better understanding of the transition density of this process.

3.2. **The distribution of a single stock price for the volatility-stabilized market model.** In this section we compute the distribution of an individual stock price $X_i(T)$, at a fixed time $T > 0$, for the volatility-stabilized process $\big(X_1(\cdot), \ldots, X_n(\cdot)\big)$ of (1.24). This result complements the conclusions on the asymptotic behaviour of stock prices captured in Proposition 1.3 of Section 1.2.

Using the same notation as in Section 1.2, we write

$$\mathbb{P}\left[\sqrt{X_i(T)} \in dr_i\right] = \mathbb{P}\left[R_i(\Lambda(T)) \in dr_i\right] = \tag{3.42}$$

$$= \int_0^\infty \mathbb{P}\left[R_i(\Lambda(T)) \in dr_i |\, \Lambda(T) = t\right] \mathbb{P}\left[\Lambda(T) \in dt\right] =$$

$$= \int_0^\infty \mathbb{P}\left[R_i(t) \in dr_i |\, \Lambda(T) = t\right] \mathbb{P}\left[\Lambda(T) \in dt\right]$$

by the law of total probability. But

$$\Lambda(T) = \frac{x}{4} \int_0^T e^{\nu s + B(s)} ds \tag{3.43}$$

is the integral of the exponential of Brownian motion with drift; its distribution is well known, from the work of M. Yor. Theorem 2.2 in Section 2 provides the expression of this distribution that we are going to use to get the final result of this section.

From (1.31) and (1.39) it follows that $x = R^2(0)$ and $\nu := \gamma$, where γ is the constant defined in (1.28). From here onwards to the end of this chapter, we redefine the constant γ as the index of the Bessel process $R(\cdot) := \sqrt{\sum_{i=1}^n R_i^2(\cdot)}$. It follows that $\gamma = \frac{mn}{2} - 1$ and $\nu = \frac{\gamma}{2}$.

From (1.40) we know that the process $\Lambda(\cdot)$ is strictly increasing and admits an inverse and hence we can write

$$\mathbb{P}\left[R_i(t) \in dr_i |\, \Lambda(T) = t\right] = \mathbb{P}\left[R_i(t) \in dr_i |\, \Lambda^{-1}(t) = T\right] = \tag{3.44}$$

$$= \mathbb{P}\left[R_i(t) \in dr_i \middle| \, C(t) = \frac{T}{4} \right],$$

where

$$C(t) := \int_0^t \frac{ds}{R^2(s)}.$$

We need to compute the joint density of $R_i(t)$ and $C(t)$, then the conditional density $\mathbb{P}\left[R_i(t) \in dr_i | \, C(t) = c\right]$.

To this end, we shall evaluate in two different ways the expectation

$$\mathbb{E}\left[f\left(R_i^2(t)\right) g\left(C(t)\right) \right]$$

for arbitrary bounded, measurable functions $f(\cdot)$ and $g(\cdot)$, then equate the two results.

• In the first instance, we use the skew product representation

$$R_i^2(t) = R^2(t) Y_i\left(C(t)\right), \quad 0 \le t < \infty$$

discussed in Section 2.2, where $Y_i(\cdot)$ is a one-dimensional Jacobi diffusion with values in $[0,1]$, independent of the process $R(\cdot)$ and satisfying the following stochastic differential equation:

(3.45)

$$dY_i(t) = m\big(1 - nY_i(t)\big)dt + 2\sqrt{Y_i(t)(1 - Y_i(t))}dB_i(t), \quad Y_i(0) = y_i(0) = \frac{R_i^2(0)}{R^2(0)}.$$

We denote by $p_c^{(i)}\left(y_i(0), y\right)$ the transition density of this process:

$$p_c^{(i)}\left(y_i(0), y\right) = \mathbb{P}\left[Y_i(c) \in dy | \, Y_i(0) = y_i(0)\right]. \tag{3.46}$$

All this allows us to write:

$$\mathbb{E}\left[f\left(R_i^2(t)\right) g\left(C(t)\right) \right] = \mathbb{E}\left[f\left(R^2(t) Y_i\left(C(t)\right)\right) g\left(C(t)\right) \right] = \tag{3.47}$$

$$= \int_0^\infty \int_0^\infty \mathbb{E}\left[f\left(r^2 Y_i(c)\right) g(c) \middle| \, R(t) = r, C(t) = c \right] \cdot \mathbb{P}\left[R(t) \in dr, C(t) \in dc\right] =$$

$$= \int_0^\infty g(c) \int_0^\infty \left(\int_0^1 f(r^2 y) p_c^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, y \right) dy \right) \mathbb{P}\left[R(t) \in dr, C(t) \in dc\right].$$

For $r \in (0, \infty)$ fixed, consider the change of variable $y = \frac{r_i^2}{r^2}$, $r^2 dy = 2 r_i dr_i$, for the 'interior integral'. This gives

$$\mathbb{E}\left[f\left(R_i^2(t)\right) g\left(C(t)\right)\right] = \tag{3.48}$$

$$= \int_0^\infty g(c) \left(\int_0^\infty \frac{1}{r^2} \int_0^r f(r_i^2) p_c^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, \frac{r_i^2}{r^2} \right) 2 r_i dr_i \right) \cdot \mathbb{P}\left[R(t) \in dr, C(t) \in dc\right] =$$

$$= \int_0^\infty g(c) \int_0^\infty 2 r_i f(r_i^2) \left(\int_{r_i}^\infty \frac{1}{r^2} p_c^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, \frac{r_i^2}{r^2} \right) \mathbb{P}\left[R(t) \in dr |\, C(t) = c\right] \right) dr_i \cdot$$

$$\cdot \mathbb{P}\left[C(t) \in dc\right],$$

interchanging integrals and conditioning.

• In the second instance, we rely on the law of total probability (conditioning) at the outset, to obtain:

$$\mathbb{E}\left[f\left(R_i^2(t)\right) g\left(C(t)\right)\right] = \int_0^\infty \int_0^\infty f(r_i^2) g(c) \mathbb{P}\left[R_i(t) \in dr_i, C(t) \in dc\right] = \tag{3.49}$$

$$= \int_0^\infty g(c) \left(\int_0^\infty f(r_i^2) \mathbb{P}\left[R_i(t) \in dr_i |\, C(t) = c\right] \right) \cdot \mathbb{P}\left[C(t) \in dc\right].$$

Let us now equate the two expressions (3.48) and (3.49). Because $g(\cdot)$ is arbitrary, we deduce

$$\int_0^\infty f(r_i^2) \mathbb{P}\left[R_i(t) \in dr_i |\, C(t) = c\right] = \tag{3.50}$$

$$\int_0^\infty 2 r_i f(r_i^2) \int_{r_i}^\infty \frac{1}{r^2} p_c^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, \frac{r_i^2}{r^2} \right) \mathbb{P}\left[R(t) \in dr |\, C(t) = c\right] dr_i.$$

Because $f(\cdot)$ is also arbitrary, we further deduce from the above

$$\frac{\mathbb{P}\left[R_i(t) \in dr_i |\, C(t) = c\right]}{dr_i} = \tag{3.51}$$

$$= 2r_i \int_{r_i}^{\infty} \frac{1}{r^2} p_c^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, \frac{r_i^2}{r^2} \right) \mathbb{P}\left[R(t) \in dr | \, C(t) = c \right].$$

The conditional density $\mathbb{P}\left[R(t) \in dr | \, C(t) = c \right]$ has been expressed in formula (2.30) of Section 2. Moreover,

$$\Lambda(T) = \frac{R^2(0)}{4} \int_0^T e^{\frac{\gamma s}{2} + B(s)} ds = R^2(0) \int_0^{\frac{T}{4}} e^{2\left(\gamma s + \frac{B(4s)}{2}\right)} ds = \tag{3.52}$$

$$= R^2(0) \int_0^{\frac{T}{4}} e^{2(\gamma s + \tilde{B}(s))} ds = R^2(0) A^{(\gamma)}(T/4),$$

by using the notation of Theorem 2.2 and the scaling property of Brownian motion. Using the main result of Theorem 2.2, the density $\mathbb{P}\left[\Lambda(T) \in dt\right]$ can be expressed as:

(3.53)

$$\frac{\mathbb{P}\left[\Lambda(T) \in dt\right]}{dt} = \int_{-\infty}^{\infty} \theta_{\frac{e^x R^2(0)}{t}} \left(\frac{T}{4} \right) \exp\left(\gamma x - \frac{(1 + e^{2x}) R^2(0)}{2t} - \frac{\gamma^2 T}{8} \right) \frac{dx}{t} =$$

$$= \frac{1}{t R^{\gamma}(0)} e^{-\frac{\gamma^2 T}{8} - \frac{R^2(0)}{2t}} \int_0^{\infty} \theta_{\frac{z R(0)}{t}} \left(\frac{T}{4} \right) \exp\left(-\frac{z^2}{2t} \right) z^{\gamma - 1} dz,$$

after the change of variable $z = e^x R(0)$.

Putting together formulas (3.42), (3.51), (3.53), and (2.30) we obtain the density of the square root of the i^{th} process $X_i(\cdot)$ at a given time T, as

$$\frac{\mathbb{P}\left[\sqrt{X_i(T)} \in dr_i \right]}{dr_i} = \tag{3.54}$$

$$= 2r_i \int_0^{\infty} \int_{r_i}^{\infty} p_{\frac{T}{4}}^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, \frac{r_i^2}{r^2} \right) \frac{r^{\gamma - 1}}{t R^{\gamma}(0)} e^{-\frac{\gamma^2 T}{8} - \frac{r^2 + R^2(0)}{2t}} \theta_{\frac{r R(0)}{t}} \left(\frac{T}{4} \right) \frac{I_{t,T,R(0)}(\gamma - 1)}{I_{t,T,R(0)}(\gamma + 1)} dr dt,$$

where

$$I_{t,T,R(0)}(\gamma) := \int_0^{\infty} \theta_{\frac{z R(0)}{t}} \left(\frac{T}{4} \right) e^{-\frac{z^2}{2t}} z^{\gamma} dz, \tag{3.55}$$

and the exact expression of $\theta_{\frac{zR(0)}{t}} \left(\frac{T}{4} \right)$ follows from formula (2.26).

With the help of Proposition 2.4, we carry out the sanity check that this density integrates to 1. Finally, we can express the density of the i^{th} stock, $X_i(T)$, as follows:

$$\frac{\mathbb{P}\left[X_i(T) \in dq_i\right]}{dq_i} = \frac{e^{-\frac{\gamma^2 T}{8}}}{R^{\gamma}(0)} \int_0^{\infty} e^{-\frac{R^2(0)}{2t}} \frac{I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} \frac{dt}{t} \cdot \tag{3.56}$$

$$\cdot \int_{q_i}^{\infty} p^{(i)}_{\frac{T}{4}} \left(\frac{R_i^2(0)}{R^2(0)}, \frac{q_i}{q} \right) \theta_{\frac{\sqrt{q}R(0)}{t}} \left(\frac{T}{4} \right) e^{-\frac{q}{2t}} q^{\frac{\gamma-1}{2}} dq.$$

We note that by using the explicit formula of $\theta_{\frac{zR(0)}{t}}\left(\frac{T}{4}\right)$, we can obtain an alternate expression for the function $I_{t,T,R(0)}(\gamma)$, namely

$$I_{t,T,R(0)}(\gamma) = \frac{1}{\pi} \sqrt{\frac{2}{\pi T}} \frac{R(0)}{t} e^{\frac{2\pi^2}{T}} \int_0^{\infty} e^{-\frac{2x^2}{T}} \sinh x \sin\left(\frac{4\pi x}{T}\right) J_{t,R(0),x}(\gamma) dx. \tag{3.57}$$

However, this new expression does not simplify its computational complexity.

The quantity $J_{t,R(0),x}(\gamma)$ can be computed in terms of the confluent hypergeometric function

$$F_1(a,b,z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \quad (a)_k := a(a-1)\dots(a-k+1)$$

using Mathematica:

$$J_{t,R(0),x}(\gamma) = 2^{\frac{\gamma}{2}} t^{\frac{\gamma+2}{2}} \Gamma\left(\frac{\gamma+2}{2}\right) F_1\left(\frac{\gamma+2}{2}, \frac{1}{2}, \frac{R^2(0)\cosh^2 x}{2t}\right) -$$

$$-(2t)^{\frac{\gamma+1}{2}} R(0) \cosh x \, \Gamma\left(\frac{\gamma+3}{2}\right) F_1\left(\frac{\gamma+3}{2}, \frac{3}{2}, \frac{R^2(0)\cosh^2 x}{2t}\right).$$

3.3. **The joint distribution of stock prices for the volatility-stabilized market model.** In this section we compute the joint distribution of stocks, at a fixed time T, by using the same technique that was employed in Section 3.2 to derive the law of an individual stock. We start by writing:

$$\mathbb{P}\left[\sqrt{X_1(T)} \in dr_1, \ldots, \sqrt{X_n(T)} \in dr_n\right] = \tag{3.58}$$

$$= \mathbb{P}\left[R_1(\Lambda(T)) \in dr_1, \ldots, R_n(\Lambda(T)) \in dr_n\right] =$$

$$= \int_0^\infty \mathbb{P}\left[R_1(t) \in dr_1, \ldots, R_n(t) \in dr_n \middle|\, \Lambda(T) = t\right] \cdot \mathbb{P}\left[\Lambda(T) \in dt\right] =$$

$$= \int_0^\infty \mathbb{P}\left[R_1(t) \in dr_1, \ldots, R_n(t) \in dr_n \middle|\, C(t) = \frac{T}{4}\right] \cdot \mathbb{P}\left[\Lambda(T) \in dt\right].$$

Once again, we need to compute the joint distribution

$$\mathbb{P}\left[R_1(t) \in dr_1, \ldots, R_n(t) \in dr_n, C(t) \in dc\right]. \tag{3.59}$$

We shall compute $\mathbb{E}\left[f\left(R_1^2(t), \ldots, R_n^2(t)\right) g\left(C(t)\right)\right]$ in two ways, for bounded measurable $f : [0,\infty)^n \to [0,\infty)$, $g : [0,\infty) \to [0,\infty)$ arbitrary.

- In the first instance, we use the skew product representation

$$R_i^2(t) = R^2(t) Y_i\left(C(t)\right),$$

where $(Y_1(\cdot), \ldots, Y_n(\cdot))$ is a multidimensional Jacobi diffusion with values in the open unit simplex Δ^n, independent of the process $R(\cdot)$ and satisfying the following system of stochastic differential equations:

(3.60)

$$dY_i(t) = \big(m - nmY_i(t)\big)dt + 2\left[\left(1 - Y_i(t)\right)\sqrt{Y_i(t)}dW_i(t) - Y_i(t)\sum_{j\neq i}\sqrt{Y_j(t)}dW_j(t)\right],$$

$$Y_i(0) = \frac{R_i^2(0)}{R^2(0)}.$$

This is exactly the system (5.34) satisfied by a generic *multidimensional Jacobi process*, but considered in the special case of a volatility-stabilized model, for which we have established that the market weights are Jacobi processes as in (2.42).

We denote by $p_c\left(\frac{R_1^2(0)}{R^2(0)}, \dots, \frac{R_n^2(0)}{R^2(0)}; y_1, \dots, y_n\right)$ the transition probability of this process. All this allows us to write:

(3.61)

$$\mathbb{E}\left[f\left(R_1^2(t), \dots, R_n^2(t)\right) g\left(C(t)\right)\right] = \mathbb{E}\left[f\left(R^2(t) Y_1\left(C(t)\right), \dots, R^2(t) Y_n\left(C(t)\right)\right) g\left(C(t)\right)\right]$$

$$= \int_0^\infty \int_0^\infty \mathbb{P}\left[R(t) \in dr, C(t) \in dc\right] \cdot \mathbb{E}\left[f\left(r^2 Y_1(c), \dots, r^2 Y_n(c)\right) g(c) \middle| R(t) = r, C(t) = c\right]$$

$$= \int_0^\infty g(c) \mathbb{P}\left[C(t) \in dc\right] \int_0^\infty \mathbb{E}\left[f\left(r^2 Y_1(c), \dots, r^2 Y_n(c)\right)\right] \cdot \mathbb{P}\left[R(t) \in dr \middle| C(t) = c\right] =$$

$$= \int_0^\infty g(c) \mathbb{P}\left[C(t) \in dc\right] \int_0^\infty \left(\int_{\Delta_n^+} f(r^2 y_1, \dots, r^2 y_n) p_c\left(\frac{R_1^2(0)}{R^2(0)}, \dots, \frac{R_n^2(0)}{R^2(0)}; y_1, \dots, y_n\right) d\tilde{y}\right) \cdot$$

$$\cdot \mathbb{P}\left[R(t) \in dr \middle| C(t) = c\right].$$

• In the second instance we rely on the law of total probability, to write:

$$\mathbb{E}\left[f\left(R_1^2(t), \dots, R_n^2(t)\right) g\left(C(t)\right)\right] = \int_0^\infty g(c) \mathbb{P}\left[C(t) \in dc\right] \cdot \tag{3.62}$$

$$\cdot \int_0^\infty \dots \int_0^\infty f(r_1^2, \dots, r_n^2) \mathbb{P}\left[R_1(t) \in dr_1, \dots, R_n(t) \in dr_n \middle| C(t) \in dc)\right].$$

By equating the expressions (3.61) and (3.62) and since g is arbitrary, we deduce

(3.63)

$$\int_0^\infty \int_{\Delta_n^+} f(r^2 y_1, \dots, r^2 y_n) p_c\left(\frac{R_1^2(0)}{R^2(0)}, \dots, \frac{R_n^2(0)}{R^2(0)}; y_1, \dots, y_n\right) d\tilde{y} \mathbb{P}\left[R(t) \in dr \middle| C(t) = c\right]$$

$$= \int_0^\infty \dots \int_0^\infty f(r_1^2, \dots, r_n^2) \mathbb{P}\left[R_1(t) \in dr_1, \dots, R_n(t) \in dr_n \middle| C(t) = c\right].$$

In the left hand side of equality (3.63), we make the change of variable $(r, y_1, \dots, y_{n-1}) \to (r_1, \dots, r_n)$, with $r_i = r\sqrt{y_i}$ for $i = 1, \dots, n-1$ and $r_n = \sqrt{r^2(1 - y_1 - \dots - y_{n-1})}$. The Jacobian of the determinant of this invertible transformation turns out to be

$$\frac{2^{n-1} r_1 \cdot \ldots \cdot r_n}{(r_1^2 + \dots + r_n^2)^{\frac{2n-1}{2}}}, \tag{3.64}$$

and hence (3.63) becomes

$$\int_0^\infty \dots \int_0^\infty dr_1 \dots dr_n f(r_1^2, \dots, r_n^2) \frac{2^{n-1} r_1 \cdot \ldots \cdot r_n}{(r_1^2 + \dots + r_n^2)^{\frac{2n-1}{2}}} \cdot \tag{3.65}$$
$$\cdot p_c\left(\frac{R_1^2(0)}{R^2(0)}, \dots, \frac{R_n^2(0)}{R^2(0)}; \frac{r_1^2}{\sum r_i^2}, \dots, \frac{r_n^2}{\sum r_i^2}\right) \cdot$$
$$\cdot \left(\frac{\mathbb{P}\left[R(t) \in dr \middle| C(t) = c\right]}{dr}\right)\Big|_{r=\sqrt{\sum r_i^2}}$$
$$= \int_0^\infty \dots \int_0^\infty f(r_1^2, \dots, r_n^2) \mathbb{P}\left[R_1(t) \in dr_1, \dots, R_n(t) \in dr_n \middle| C(t) = c\right].$$

Because $f(\cdot)$ in (3.65) is also arbitrary, we deduce that

$$\frac{\mathbb{P}\left[R_1(t) \in dr_1, \dots, R_n(t) \in dr_n \middle| C(t) = c\right]}{dr_1 \dots dr_n} = \tag{3.66}$$
$$= \frac{2^{n-1} r_1 \dots r_n}{(r_1^2 + \dots + r_n^2)^{\frac{2n-1}{2}}} \cdot p_c\left(\frac{R_1^2(0)}{R^2(0)}, \dots, \frac{R_n^2(0)}{R^2(0)}; \frac{r_1^2}{\sum r_i^2}, \dots, \frac{r_n^2}{\sum r_i^2}\right) \cdot$$
$$\cdot \theta_{\frac{\sqrt{\sum r_i^2} R(0)}{t}}(c) \left(\int_0^\infty \theta_{\frac{zR(0)}{t}}(c) e^{-\frac{z^2}{2t}} z^{\gamma+1} dz\right)^{-1} e^{-\frac{r_1^2 + \dots + r_n^2}{2t}} (r_1^2 + \dots + r_n^2)^{\frac{\gamma+1}{2}}.$$

Putting together formulas (3.53), (3.58), and (3.66), we derive the following expression for the joint law of the square roots of stocks:

$$\mathbb{P}\left[\sqrt{X_1(T)} \in dr_1, \dots, \sqrt{X_n(T)} \in dr_n\right] = \tag{3.67}$$
$$= \frac{e^{-\frac{\gamma^2 T}{8}}}{R^\gamma(0)} 2^{n-1} r_1 \cdot \ldots \cdot r_n (r_1^2 + \dots + r_n^2)^{\frac{\gamma+2-2n}{2}} p_{\frac{T}{4}}\left(\frac{R_1^2(0)}{R^2(0)}, \dots, \frac{R_n^2(0)}{R^2(0)}; \frac{r_1^2}{\sum r_i^2}, \dots, \frac{r_n^2}{\sum r_i^2}\right).$$

$$\cdot \int_0^\infty e^{-\frac{r_1^2+\cdots+r_n^2+R^2(0)}{2t}} \theta_{\frac{\sqrt{\sum r_i^2} R(0)}{t}} \left(\frac{T}{4}\right) \frac{I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} \frac{dt}{t},$$

where $\gamma = \frac{mn}{2} - 1$ and $I_{t,T,R(0)}(\gamma)$ is defined as in formula (3.55).

We note that the expression (3.67) does indeed integrate to 1 and also that the marginals of this joint distribution coincide with the individual distributions of stock prices obtained in formula (3.54). One last straightforward change of variables allows us to write formula (3.67) in the alternative form:

(3.68)

$$\boxed{\begin{gathered} \mathbb{P}\left[X_1(T) \in dq_1, \ldots, X_n(T) \in dq_n\right] = \\ = \frac{e^{-\frac{\gamma^2 T}{8}}}{2R^\gamma(0)} \left(q_1 + \cdots + q_n\right)^{\frac{\gamma+2-2n}{2}} p_{\frac{T}{4}} \left(\frac{R_1^2(0)}{R^2(0)}, \ldots, \frac{R_n^2(0)}{R^2(0)}; \frac{q_1}{\sum q_i}, \ldots, \frac{q_n}{\sum q_i}\right) \cdot \\ \cdot \int_0^\infty e^{-\frac{q_1+\cdots+q_n+R^2(0)}{2t}} \theta_{\frac{\sqrt{\sum q_i} R(0)}{t}} \left(\frac{T}{4}\right) \frac{I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} \frac{dt}{t}. \end{gathered}}$$

3.4. **The moments of stock prices and market weights.** In this section we try to gain a better understanding of the law of the market weights $\mu_i(T)$ for the volatility-stabilized market by studying the moments of the one-dimensional Jacobi diffusion at any time $T > 0$. These moments can be recursively computed using the fact that the eigenfunctions of the Jacobi diffusion are polynomials (discussed in Section 3.1). The knowledge of the transition density of the Jacobi diffusion is not at all needed for the computation of these moments; this comes in handy, since there is no explicit expression for this density in the current literature and any attempt to compute moments using the spectral representation of the density given by formula (3.40) carries a huge computational complexity in it. Finally, the computations in this section are going to help us derive some qualitative statements about the behavior of the volatility-stabilized market.

Let $\mathcal{L}$ be the infinitesimal generator of the one-dimensional diffusion

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \quad X(0) = x \tag{3.69}$$

and let h be an eigenfunction of $\mathcal{L}$ with corresponding eigenvalue λ:

$$\mathcal{L}h = -\lambda h. \tag{3.70}$$

Then, the expectation of $h(X(t))$ has the following expression

$$\mathbb{E}\left[h(X(t))\right] = e^{-\lambda t}h(x), \tag{3.71}$$

under certain 'technical' assumptions that will follow naturally from our proof. This result appears in [20]. As for the proof, a direct application of Itô's rule gives that:

(3.72)

$$dh(X(t)) = \left[\frac{1}{2}\sigma^2(X(t))h''(X(t)) + h(X(t))h'(X(t))\right]dt + h'(X(t))\sigma(X(t))dB(t) =$$

$$= -\lambda h(X(t))dt + h'(X(t))\sigma(X(t))dB(t).$$

$$\text{or } \; h(X(t)) = h(x) - \lambda\int_0^t h(X(s))ds + \int_0^t h'(X(s))\sigma(X(s))dB(s). \tag{3.73}$$

In the case that the stochastic integral $\int_0^t h'(X(s))\sigma(X(s))dB(s)$ is a martingale (this is guaranteed if $\mathbb{E}\left[\int_0^T (h'\sigma)^2(X(t))dt\right] < \infty$, see [30, p.131]), by taking expectations in the above equality we get that

$$\mathbb{E}\left[h(X(t))\right] = h(x) - \lambda\int_0^t \mathbb{E}\left[h(X(s))\right]ds. \tag{3.74}$$

Letting $f(t) := \mathbb{E}\left[h(X(t))\right]$, we see that f is a solution of the very simple ODE $f'(t) + \lambda f(t) = 0$, hence $f(t) = f(0)e^{-\lambda t}$ and $\mathbb{E}\left[h(X(t))\right] = e^{-\lambda t}h(x)$, as claimed.

Next, we are going to apply this result for $\mathcal{L}$ the generator of a Jacobi diffusion and for $h = P_k$, the *Jacobi polynomial*, expressed as in formula (3.21). Observe that in this case $(h'\sigma)$ is a polynomial and because the Jacobi process takes values in $(0,1)$, the quantity $(h'\sigma)^2(Y(t)) = 4Y(t)(1-Y(t))P_k'(Y(t))$ is bounded from above and hence $\mathbb{E}\left[\int_0^T (h'\sigma)^2(Y(t))dt\right] < \infty$ for any $T > 0$. Introducing the polynomial $Q_k(y)$ as in (3.31) we get that

$$\mathbb{E}\left[Q_k(Y(t))\right] = Q_k(Y(0))e^{-k(mn+2k-2)t}. \tag{3.75}$$

To ease our exposition, let us introduce the following notation

$$x_j := \mathbb{E}\left[Y^j(t)\right], \quad Q_k(x) = \sum_{j=0}^{k} a_{kj}x^j, \quad \text{where} \tag{3.76}$$

$$a_{kj} = (-1)^j \binom{k}{j} \frac{\Gamma\left(\frac{mn}{2} + k + j - 1\right)}{\Gamma\left(\frac{m}{2} + j\right)}. \tag{3.77}$$

Then it follows that

$$\sum_{j=1}^{k} a_{kj}\left(x_j - y^j e^{-k(mn+2k-2)t}\right) = a_{k0}\left(e^{-k(mn+2k-2)t-1}\right), \tag{3.78}$$

so we have a recursive formula for the moments x_j. Here

$$dY(t) = \big(m - mnY(t)\big)dt + 2\sqrt{Y(t)(1-Y(t))}dB(t), \quad Y(0) = y,$$

is any of the n one-dimensional marginals of the *multidimensional Jacobi process* from Section 2.2. By recalling that the i^{th} market weight $\mu_i(T)$ can be written $\mu_i(T) = Y\left(\frac{T}{4}\right)$, where the Jacobi process Y satisfies the above equation with parameters m, n, and starting point $Y(0) = \frac{R_i^2(0)}{R^2(0)}$, we derive the following mean and variance formulas:

$$\mathbb{E}\left[\mu_i(T)\right] = \frac{R_i^2(0)}{R^2(0)}e^{-mn\frac{T}{4}} + \frac{1}{n}\left(1 - e^{-mn\frac{T}{4}}\right) \tag{3.79}$$

(3.80)

$$\mathbb{E}\left[\mu_i^2(T)\right] = \frac{m+2}{n(mn+2)} - \frac{2(m+2)}{n(mn+4)}e^{-mn\frac{T}{4}} + \frac{m(m+2)}{(mn+2)(mn+4)}e^{-2(mn+2)\frac{T}{4}} +$$

$$+\frac{R_i^2(0)}{R^2(0)}\frac{2(m+2)}{mn+4}\left(e^{-mn\frac{T}{4}}-e^{-2(mn+2)\frac{T}{4}}\right)+\frac{R_i^4(0)}{R^4(0)}e^{-2(mn+2)\frac{T}{4}}.$$

$$\text{Var}\left[\mu_i(T)\right]=e^{-2mn\frac{T}{4}}\left(e^{-T}-1\right)\frac{R_i^4(0)}{R^4(0)}+G(T)-F(T)+ \tag{3.81}$$

$$+e^{-mn\frac{T}{4}}\left[\frac{4(n-2)}{n(mn+4)}+2e^{-mn\frac{T}{4}}\left(\frac{1}{n}-\frac{m+2}{mn+4}e^{-T}\right)\right]\frac{R_i^2(0)}{R^2(0)},$$

where $G(T)$ and $F(T)$ are given below as

$$G(T)=\frac{m+2}{n(mn+2)}-\frac{2(m+2)}{n(mn+4)}e^{-mn\frac{T}{4}}+\frac{m(m+2)}{(mn+2)(mn+4)}e^{-2(mn+2)\frac{T}{4}} \tag{3.82}$$

$$F(T)=\frac{\left(1-e^{-mn\frac{T}{4}}\right)^2}{n^2}.$$

We note that $\sum_{i=1}^{n}\mathbb{E}\left[\mu_i(T)\right]=1$, as expected since $\sum_{i=1}^{n}\mu_i(T)=1$, and that for fixed i, $\mathbb{E}\left[\mu_i(T)\right]$ is increasing, constant, or decreasing depending on whether $\mu_i(0)=\frac{R_i^2(0)}{R^2(0)}$ is strictly less, equal, or greater than $\frac{1}{n}$, respectively. Also

$$\lim_{T\to\infty}\mathbb{E}\left[\mu_i(T)\right]=\frac{1}{n}\quad\text{and}\quad\lim_{T\to\infty}\mathbb{E}\left[\mu_i^2(T)\right]=\frac{m+2}{n(mn+2)}. \tag{3.83}$$

The same values of these limits are going to be derived via a different method in Chapter 4. We also note that the p^{th} moment of the i^{th} market weight is a polynomial of degree p in $\frac{R_i^2(0)}{R^2(0)}$.

Lastly, we note that the leading coefficient $e^{-2mn\frac{T}{4}}\left(e^{-T}-1\right)$ of $\text{Var}\left[\mu_i(T)\right]$ is negative, so it is worth posing the question for what value of the ratio $\frac{R_i^2(0)}{R^2(0)}$ is the variance of the i^{th} market weight maximized.

We write $\text{Var}\left[\mu_i(T)\right]=\mathcal{P}_T\left(\frac{R_i^2(0)}{R^2(0)}\right)$, where

$$\mathcal{P}_T(x)=A(T)x^2+B(T)x+G(T)-F(T). \tag{3.84}$$

In Appendix 5, by using standard calculus methods, we show that for any $T > 0$, the vertex of the quadratic $\mathcal{P}_T(x)$ has positive x coordinate that is actually greater than $\frac{1}{2}$. Using some more calculus, we are going to conclude in Appendix 5 that for any $T > 0$ the ordering of the variances of the market weights at time T is the same as the ordering of the set of initial data $\{R_1(0), \ldots, R_n(0)\}$.

Finally, we show how knowledge of the moments of a Jacobi diffusion can help towards the computation of moments of stock prices for the volatility stabilized market, by using formula (3.54) obtained in Section 3.2.

For fixed i, let $M_p := \mathbb{E}\left[X_i^p(T)\right]$. Using formula (3.54), M_p has the following expression:

$$M_p = \int_0^\infty 2r_i^{2p+1} \int_0^\infty \int_{r_i}^\infty p_{\frac{T}{4}}^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, \frac{r_i^2}{r^2} \right) \cdot \tag{3.85}$$

$$\cdot \frac{r^{\gamma-1}}{tR^\gamma(0)} e^{-\frac{\gamma^2 T}{8} - \frac{r^2+R^2(0)}{2t}} \theta_{\frac{rR(0)}{t}} \left(\frac{T}{4} \right) \frac{I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} dr dt dr_i.$$

By making the change of variables $(r_i, r) \to (j, r)$, $j = \frac{r_i^2}{r^2}$, we get the following simpler expression for M_p:

$$M_p = \left[\int_0^1 j^p \cdot p_{\frac{T}{4}}^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, j \right) dj \right] \cdot \frac{e^{-\frac{\gamma^2 T}{8}}}{R^\gamma(0)} \cdot \tag{3.86}$$

$$\cdot \int_0^\infty e^{-\frac{R^2(0)}{2t}} \frac{I_{t,T,R(0)}(2p+\gamma+1) I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} \frac{dt}{t}.$$

The integral

$$\int_0^1 j^p \cdot p_{\frac{T}{4}}^{(i)} \left(\frac{R_i^2(0)}{R^2(0)}, j \right) dj$$

represents the p^{th} moment of a Jacobi diffusion, whose computation we have discussed. For $p = 1$ and $p = 2$ we have the explicit formulas (3.79) and (3.80).

In particular, formula (3.86) implies that at any time T, the ordering of the p^{th} moments of the stocks is the same as the ordering of the p^{th} moments of the market weights. For $p = 1$, we have seen that this is just the ordering of the set of initial data $\{R_1(0), \ldots, R_n(0)\}$.

Of possible further interest could be the study, for fixed i, of the monotonicity of M_p with respect to time T.

4. The Invariant Distribution of the Multidimensional Jacobi Process

4.1. **The one-dimensional case: new proofs for existing results.** The aim of this section is to establish the invariant distribution of the multidimensional Jacobi diffusion $(Y_1(\cdot), \ldots, Y_n(\cdot))$ of Section 2.2. We will first discuss the invariant distribution of a one-dimensional Jacobi diffusion and explain how the ergodic theorem for one-dimensional difussions allows us to recover existing results. Next we will prove existence and uniqueness of an invariant density for the multidimensional process, followed by a discussion of the Dirichlet distribution, the multivariate analogue of the Beta distribution, which through one last computation, turns out to be the invariant density of the *multidimensional Jacobi process.*

A general one-dimensional diffusion ($dX(t) = b(X(t))dt + \sigma(X(t))dW(t)$) taking values in an interval with possibly infinite endpoints l and r is known to have good ergodic properties if

$$V(x) := \int_{x_0}^{x} \exp\left(-2\int_{y_0}^{y} \frac{b(v)}{\sigma^2(v)}dv\right) dy \longrightarrow \pm\infty \quad \text{as} \quad x \longrightarrow l, r \tag{4.1}$$

and

$$H := \int_{-\infty}^{\infty} \exp\left(2\int_{x_0}^{x} \frac{b(v)}{\sigma^2(v)}dv\right) \frac{dx}{\sigma^2(x)} < \infty. \tag{4.2}$$

Condition (4.1) guarantees that the time to return to any bounded set is finite with probability one and condition (4.2) assures that this time has finite expectation. A detailed discussion about these results and related ones can be found in Chapter 6 of [12]. Under these assumptions, the ergodic theorem states that for any measurable function $h(\cdot)$, such that $E[h(\xi)] < \infty$, the following limit

$$\frac{1}{T}\int_0^T h(X_t)dt \underset{T\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} h(x)f(x)dx = E[h(\xi)] \tag{4.3}$$

holds with probability one. Here the function

$$f(x) := \frac{1}{H\sigma^2(x)} \exp\left(2\int_{x_0}^{x} \frac{b(v)}{\sigma^2(v)}dv\right) \tag{4.4}$$

is the invariant density of the process and ξ is a random variable with density function $f(\cdot)$.

Now a straightforward computation shows that for a Jacobi diffusion satisfying the stochastic differential equation

$$dY(t) = \big(\delta_1 - \delta Y(t)\big)dt + \eta\sqrt{Y(t)(1-Y(t))}dB(t), \quad Y(0) = y_0 \in (0,1) \tag{4.5}$$

the invariant probability density function $f(\cdot)$ takes the form

$$f(y) = \frac{\Gamma\left(\frac{2\delta}{\eta^2}\right) y^{\frac{2\delta_1}{\eta^2}-1}(1-y)^{\frac{2(\delta-\delta_1)}{\eta^2}-1}}{\Gamma\left(\frac{2\delta_1}{\eta^2}\right)\Gamma\left(\frac{2(\delta-\delta_1)}{\eta^2}\right)}, \quad 0 < y < 1, \tag{4.6}$$

which corresponds to the Beta distribution on the interval $[0,1]$.

Also, it is straightforward to check that condition (4.1) is satisfied. This amounts to verifying that the improper integrals

$$\int_0^{\frac{1}{2}} y^{-\frac{2\delta_1}{\eta^2}}(1-y)^{\frac{2(\delta_1-\delta)}{\eta^2}}dy \quad \text{and} \quad \int_{\frac{1}{2}}^{1} y^{-\frac{2\delta_1}{\eta^2}}(1-y)^{\frac{2(\delta_1-\delta)}{\eta^2}}dy \tag{4.7}$$

are divergent. For the verification it is useful to recall that in the volatility-stabilized model $\eta = 2$, $\delta_1 = m$ and $\delta = mn$, where $m, n \geq 2$.

The first and second moments of a Beta distribution with parameters $\frac{m}{2}$ and $\frac{m(n-1)}{2}$ can be easily computed as $\frac{1}{n}$ and $\frac{m+2}{n(mn+2)}$, respectively. This allows us to recover the conclusion (3.83), which was derived by direct computation of the first and second moments of the market weights at any fixed time.

Also, if X is a random variable having a Beta distribution with parameters $\frac{m}{2}$ and $\frac{m(n-1)}{2}$, it can be easily verified that

$$E[X^{-1}] = \frac{mn-2}{m-2} = n + \frac{n-1}{\frac{m}{2}-1}\,. \tag{4.8}$$

This, together with the ergodic theorem (4.3), recovers the following limiting result for the i^{th} market weight of the volatility stabilized model:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{dt}{\mu_i(t)} = n + \frac{n-1}{\frac{m}{2}-1} = n + \frac{n-1}{\alpha} \quad \text{a.s.} \tag{4.9}$$

Here $\alpha > 0$ and $m > 2$ are the constants introduced in Section 1.2, in (1.23) and (1.28), and n is the number of stocks in the model.

For the volatility-stabilized model described by the system of SDEs (1.23) and (1.24) of Section 1.2, the variance of the i^{th} stock, $a_{ii}(t)$ is exactly $\frac{1}{\mu_i(t)}$, hence the interest in the average long-term behavior of this quantity. Conclusion (4.9) is derived in Proposition 6.1 of [17] via a different method that does not rely on the fact that the market weights are Jacobi processes, as follows:

$$\frac{1}{T}\int_0^T \frac{dt}{\mu_i(t)} = \frac{4}{T}\int_0^T \frac{d\Lambda(t)}{R_i^2\left(\Lambda(t)\right)} = \frac{\log\Lambda(T)}{T}\left(\frac{4}{\log u}\int_0^u \frac{ds}{R_i^2(s)}\right)\Big|_{u=\log\Lambda(T)} \tag{4.10}$$

For a Bessel process $R_i(\cdot)$ of dimension m the following limit is recalled in [17, p. 26] from [48, p. 112] and [60]

$$\lim_{u\to\infty}\left(\frac{1}{\log u}\int_0^u \frac{ds}{R_i^2(s)}\right) = \frac{1}{m-2} \quad \text{a.s.} \tag{4.11}$$

This limit also appears in Theorem 1.4 of Section 1.2 in our exposition and helps one see that

$$\lim_{T\to\infty}\left(\frac{\log\Lambda(T)}{T}\right)=\frac{mn-2}{4}\quad \text{a.s.,} \tag{4.12}$$

after recalling beforehand formula (1.40) for the representation of the inverse clock $\Lambda^{-1}(t)$. Finally, combining the results described by (4.10) – (4.12) one recovers the conclusion (4.9).

4.2. **The multidimensional case: existence, uniqueness and explicit computation.** Theorem 2.1 in [35] states that for a recurrent diffusion process there exists an invariant measure and Theorem 3.2 in the same paper ensures that the invariant measure is unique up to a multiplicative constant.

Our next focus is to prove that the *multidimensional Jacobi process* is recurrent. We recall from (5.34) the system of SDEs satisfied by the *multidimensional Jacobi process* $(Y_1(\cdot),\ldots,Y_n(\cdot))$:

(4.13)

$$dY_i(t)=\big(\delta_i-\delta Y_i(t)\big)dt+\eta\left[(1-Y_i(t))\sqrt{Y_i(t)}dW_i(t)-\sum_{j\neq i}Y_i(t)\sqrt{Y_j(t)}dW_j(t)\right],$$

$i=1,\ldots,n$ and the fact that the process takes values in the open unit simplex Δ^n, namely, that we have

$$Y_1(t)+\cdots+Y_n(t)=1,$$

for every $t\geq 0$. A straightforward calculation gives the quadratic variations of the individual components of the process:

$$d\langle Y_i\rangle(t)=\eta^2 Y_i(t)\,(1-Y_i(t))\,dt. \tag{4.14}$$

We start by defining the process $P(t)$ as follows:

$$P(t) := \sum_{i=1}^{n} \left(Y_i(t) - \frac{\delta_i}{\delta} \right)^2 . \tag{4.15}$$

Also, let $\mathcal{S}_r$ be the set of points at Euclidean distance r from the point $\left(\frac{\delta_1}{\delta}, \ldots, \frac{\delta_n}{\delta}\right)$ that also lie on the unit simplex Δ^n, and let S_r be the first time that the process $(Y_1(\cdot), \ldots, Y_n(\cdot))$ hits the set $\mathcal{S}_r$. Our goal is to prove that $P_y[S_r < \infty] = 1$, where $y = (y_1, \ldots, y_n)$ and $y_i = Y_i(0)$.

A direct application of Itô's rule gives the following SDE for P(t):

$$dP(t) = \left(-2\delta P(t) + \sum_{i=1}^{n} \eta^2 Y_i(t) \left(1 - Y_i(t)\right) \right) dt + \tag{4.16}$$

$$+2\eta \sum_{i=1}^{n} \sqrt{Y_i(t)} \left(-\sum_{k=1}^{n} Y_k^2(t) + \sum_{k=1}^{n} \frac{\delta_k}{\delta} Y_k(t) + Y_i(t) - \frac{\delta_i}{\delta} \right) dW_i(t).$$

Next, the quadratic variation of the process $P(\cdot)$ can be calculated:

$$d\langle P \rangle(t) = 4\eta^2 \left(\sum_{k=1}^{n} Y_k(t) \left(Y_k(t) - \frac{\delta_k}{\delta} \right)^2 - \left[\sum_{k=1}^{n} Y_k(t) \left(Y_k(t) - \frac{\delta_k}{\delta} \right) \right]^2 \right) dt. \tag{4.17}$$

For a sufficiently smooth function g, another application of Itô's rule gives:

$$\frac{1}{2} dg(P(t)) = g'(P(t)) \left(-\delta P(t) + \frac{\eta^2}{2} - \frac{\eta^2}{2} \sum_{i=1}^{n} Y_i^2(t) \right) + \tag{4.18}$$

$$+\eta^2 g''(P(t)) \left(\sum_{k=1}^{n} Y_k(t) \left(Y_k(t) - \frac{\delta_k}{\delta} \right)^2 - \left[\sum_{k=1}^{n} Y_k(t) \left(Y_k(t) - \frac{\delta_k}{\delta} \right) \right]^2 \right) dt +$$

$$+(\text{local martingale})$$

Consider the function $f(x) := x^{-\frac{n-1}{2n}} \exp(\frac{\delta x}{\eta^2})$, defined for $x \geq \frac{\eta^2(n-1)}{2n\delta}$ and let g be a primitive of this function: $g(y) := \int_{y_0}^{y} f(x)dx$. It follows that $g'(y)$ and

$g''(y)$ are greater or equal to 0 on the interval $\left[\frac{\eta^2(n-1)}{2n\delta}, \infty\right)$ and also the following differential equation holds for g:

$$g'(x)\left(\frac{\eta^2}{2} - \frac{\eta^2}{2n} - \delta x\right) + \eta^2 x g''(x) = 0. \tag{4.19}$$

For the chosen function g, we are going to show that the drift term in (4.18) is negative when $P(t) \geq \frac{\eta^2(n-1)}{2n\delta}$. This will imply that $g(P(t))$ is going to be a local supermartingale for $P(t) \geq \frac{\eta^2(n-1)}{2n\delta}$.

We note the following inequalities:

$$\left(\sum_{i=1}^{n} Y_i(t)\right)^2 \leq n\sum_{i=1}^{n} Y_i^2(t) \leq n\sum_{i=1}^{n} Y_i(t), \quad \text{hence} \quad \frac{1}{n} \leq \sum_{i=1}^{n} Y_i^2(t) \leq 1, \tag{4.20}$$

and

$$\sum_{k=1}^{n} Y_k(t)\left(Y_k(t) - \frac{\delta_k}{\delta}\right)^2 \leq \sum_{k=1}^{n}\left(Y_k(t) - \frac{\delta_k}{\delta}\right)^2 = P(t). \tag{4.21}$$

With the help of these inequalities, it is easily seen that the drift term of (4.18) is majorized by

$$g'(P(t))\left(-\delta P(t) + \frac{\eta^2}{2} - \frac{\eta^2}{2n}\right) + \eta^2 P(t) g''(P(t)) = 0, \tag{4.22}$$

given our particular choice of g.

Next, we apply the optional sampling theorem to the supermartingale $g(P(t))$ and the stopping time $\tau := S_r \wedge S_s$, where $\frac{\eta^2(n-1)}{2n\delta} \leq r < s < \infty$:

$$g(P(0)) \geq E_y\left[g(P(\tau))\right] = g(r)P_y[S_r < S_s] + g(s)\left(1 - P_y[S_r < S_s]\right). \tag{4.23}$$

Rearranging, we obtain:

$$P_y[S_r < S_s] \geq \frac{g(s) - g(P(0))}{g(s) - g(r)}. \tag{4.24}$$

Finally, because g is strictly increasing and $\lim_{s\to\infty} g(s) = \infty$, we reach the desired conclusion, that $P_y[S_r < \infty] = 1$.

It is a well known fact in elementary probability that the Beta distribution with parameters $\frac{2\delta_1}{\eta^2}$ and $\frac{2(\delta-\delta_1)}{\eta^2}$ can be obtained as a quotient $\frac{X}{X+Y}$, where X and Y are independent random variables with Gamma distributions $\Gamma\left(\frac{2\delta_1}{\eta^2},\theta\right)$ and $\Gamma\left(\frac{2(\delta-\delta_1)}{\eta^2},\theta\right)$, respectively. In addition, the random variables $\frac{X}{X+Y}$ and $X+Y$ are independent. We recall that the probability density function of a $\Gamma(k,\theta)$ variable has the expression

$$\frac{x^{k-1}\exp\left(-\frac{x}{\theta}\right)}{\Gamma(k)\theta^k}. \tag{4.25}$$

In a more general vein, consider a family of independent random variables $X_0, X_1, \ldots, X_q$, where X_j has a $\Gamma(v_j,\theta)$ distribution, $v_j > 0$. We also consider the ratios:

$$Y_j := \frac{X_j}{\sum_{i=0}^q X_i} \quad j = 1, 2, \ldots, q, \tag{4.26}$$

and seek to determine the joint distribution of $Y_1, Y_2, \ldots, Y_q$.

The joint probability density function of $X_0, X_1, \ldots, X_q$ is

$$p_{X_0,\ldots,X_q}(x_0,\ldots,x_q) = \left[\prod_{j=0}^q \Gamma(v_j)\right]^{-1} \theta^{-\sum_{j=0}^q v_j} \left[\prod_{j=0}^q x_j^{v_j-1}\right] \exp\left(-\frac{1}{\theta}\sum_{j=0}^q x_j\right) \tag{4.27}$$

where $0 \le x_j$ for $j = 0, \ldots, q$.

Making the transformation to the new variables $Y_0 = \sum_{i=0}^q X_i, Y_1, Y_2, \ldots, Y_q$, we find:

$$p_{Y_0,\ldots,Y_q}(y_0,\ldots,y_q) = \tag{4.28}$$

$$= \left[\prod_{j=0}^q \Gamma(v_j)\right]^{-1} \theta^{-\sum_{j=0}^q v_j} \left[\left(y_0\left(1-\sum_{j=1}^q y_j\right)\right)^{v_0-1} \prod_{j=1}^q (y_o y_j)^{v_j-1}\right] \exp\left(-\frac{1}{\theta}y_0\right) J,$$

where $0 \leq y_j$ for $j = 0, \ldots, q$ and $\sum_{j=1}^{q} y_j \leq 1$, and J is the Jacobian

$$J = \frac{\partial(x_0, \ldots, x_q)}{\partial(y_0, \ldots, y_q)} = \begin{vmatrix} 1 - \sum_{j=1}^{q} y_j & -y_0 & -y_0 & \ldots & -y_o \\ y_1 & y_0 & 0 & \ldots & 0 \\ y_2 & 0 & y_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ y_q & 0 & 0 & \ldots & y_0 \end{vmatrix} = y_0^q. \tag{4.29}$$

Formula (4.28) can be rearranged in the form

$$p_{Y_0,\ldots,Y_q}(y_0, \ldots, y_q) = \tag{4.30}$$

$$\left[\theta^{\sum_{j=0}^{q} v_j} \prod_{j=0}^{q} \Gamma(v_j)\right]^{-1} \left[\prod_{j=1}^{q} y_j^{v_j - 1}\right] y_0^{\sum_{j=0}^{q} v_j - 1} \left(1 - \sum_{j=1}^{q} y_j\right)^{v_0 - 1} \exp\left(-\frac{y_0}{\theta}\right)$$

defined over $y_j \geq 0$ for $j = 0, \ldots, q$ and $\sum_{j=1}^{q} y_j \leq 1$.

Integrating over the variable y_0, we obtain the joint density of $Y_1, Y_2, \ldots, Y_q$ as

$$\boxed{p_{Y_1,\ldots,Y_q}(y_1, \ldots, y_q) = \frac{\Gamma\left(\sum_{j=0}^{q} v_j\right)}{\prod_{j=0}^{q} \Gamma(v_j)} \left[\prod_{j=1}^{q} y_j^{v_j - 1}\right] \left(1 - \sum_{j=1}^{q} y_j\right)^{v_0 - 1}}. \tag{4.31}$$

This multivariate density is known in the literature as the Dirichlet distribution. See [36, p.486], for a more comprehensive discussion on the topic.

Given that the Dirichlet distribution is the multidimensional generalization of the Beta distribution , which turned out to be the invariant distribution of the one-dimensional Jacobi process, it is reasonable to conjecture that the Dirichlet distribution is the invariant distribution of the *multidimensional Jacobi process.*

We already know that the multivariate Jacobi process of (4.13) has a unique invariant measure. If we denote by $\mu(\overline{x})$, $\overline{x} \in \Delta^n$ the density of this invariant measure, it comes in handy to look for $\mu(\overline{x})$ as a solution to the equation

$$\mathcal{L}^* \mu = 0, \tag{4.32}$$

as pointed in [35, p. 191]. Here $\mathcal{L}^*$ is the adjoint of the infinitesimal generator $\mathcal{L}$ of the multivariate Jacobi process and has the following expression

$$\left(\mathcal{L}^* f\right)(\overline{x}) := \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[a_{ij}(\overline{x}) f(\overline{x})\right] - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[b_i(\overline{x}) f(\overline{x})\right], \tag{4.33}$$

where we recall that the functions $a_{ij}(\overline{x})$ and $b(\overline{x})$ are given by formula (5.37).

Indeed, if we denote by $p(t, \overline{y}, \overline{x})$ the transition probabilities of some process that has an invariant density $\mu(\overline{x})$, it follows that

$$\mu(\overline{x}) = \int \mu(\overline{y}) p(t, \overline{y}, \overline{x}) d\overline{y}. \tag{4.34}$$

We recall (see [30, p. 282]) that under certain assumptions $p(t, \overline{y}, \overline{x})$ considered as a function of $\overline{x}$ and t satisfies the Kolmogorov forward equation

$$\frac{\partial p}{\partial t} = \mathcal{L}^* p. \tag{4.35}$$

Assuming that the right-hand side of (4.34) is twice differentiable with respect to $\overline{x}$ and differentiable with respect to t under the integral sign, the equation (4.32) follows in a straightforward manner.

Appendix E contains the computational details of the proof that the equation (4.32) is satisfied under our circumstances, namely when $\mu(\overline{x})$ is the scaled density corresponding to a Dirichlet distribution:

$$\mu(\overline{x}) = x_1^{\frac{2\delta_1}{\eta^2} - 1} \cdot \dots \cdot x_n^{\frac{2\delta_n}{\eta^2} - 1}, \ \ \overline{x} \in \Delta^n. \tag{4.36}$$

We conclude this section with a non-computational proof of the fact that the Dirichlet distribution is the invariant distribution of the *multidimensional Jacobi process.* This proof belongs to J. Dubedat; see [8].

Consider $Q_1(\cdot), \ldots, Q_n(\cdot)$ to be a family of independent squared Bessel processes of dimensions $\delta_1, \ldots, \delta_n$, or more generally a family of independent CIR processes with parameter sets $(\delta_1, b, \eta), \ldots, (\delta_n, b, \eta)$ respectively, as in (5.1). Also, assume that all these processes start from 0. The process $Q(\cdot) := Q_1(\cdot) + \cdots + Q_n(\cdot)$ is also a squared Bessel process or a CIR process. Define the quotient processes

$$Z_i(\cdot) := \frac{Q_i(\cdot)}{Q(\cdot)}, \quad i = 1, \ldots, n. \tag{4.37}$$

The assumption that the processes $Q_i(\cdot)$ start at 0 guarantees that the random variables $Q_i(t)$ have Gamma distributions $\Gamma\left(\frac{\delta_i}{2}, t\right)$ in the case of a Bessel process and $\Gamma\left(\frac{2\delta_i}{\eta^2}, \frac{\eta^2(1-e^{-bt})}{2b}\right)$ in the case of a CIR process. Our notation is the same as in (4.25).

According to the earlier discussion of this subsection (see page 54), for fixed t, the random vector $Z(t) := (Z_1(t), \ldots, Z_n(t))$ has a Dirichlet distribution $B((2\delta_1)/\eta^2, \ldots, (2\delta_n)/\eta^2)$ and is independent from $Q(t)$. Since $Q(\cdot)$ is Markov, this random vector is also independent from the sigma-algebra generated by the process $Q(\cdot)$ from time t onwards, $\sigma\{Q(t+s)_{s\geq 0}\}$.

By time inversion, the process $\hat{Q}_i(\cdot)$ defined as $\hat{Q}_i(t) = t^2 Q_i(1/t)$ is again a squared Bessel process of dimension δ_i. The previous argument applied to $\hat{Q}(t) := \hat{Q}_1(t) + \cdots + \hat{Q}_n(t)$ shows that the random vector $Z(t)$ is also independent from the sigma-algebra $\sigma\{Q(1/(t^{-1}+s))_{s\geq 0}\}$, hence it is independent of the full process $Q(\cdot)$. This last argument also applies to CIR processes, after recalling from (5.4) that they can be thought of as deterministically time-changed Bessel processes.

Thus the process $Z(\cdot)$ can be constructed in the interval $[t, t+s]$ by sampling $Q(t)$ and an independent Dirichlet distribution to obtain $Z(t)$, and then by evolving the processes $Q_i(\cdot)$ independently up to time $t+s$. The random vector $Z(t+s)$ has again a Dirichlet distribution.

By the skew-product representation from Section 5.2, the process

$$(Z_1(t+s), \ldots, Z_n(t+s))_{s\geq 0}$$

is distributed as a *multidimensional Jacobi process* $(Y_1(u), \ldots, Y_n(u))_{u\geq 0}$ started from a Dirichlet distribution and time-changed by an independent clock $u = C(s)$, $s = \alpha(u)$ (see (2.37)), $du = dt/Q(t)$.

For arbitrary test functions f and g

$$\int \mathbb{E}^{\mu}\left[f\big(Y_1(u), \ldots, Y_n(u)\big)\right] g(u) du = \tag{4.38}$$

$$= \mathbb{E}\left[\int f\big(Z_1(t+\alpha(u)), \ldots, Z_n(t+\alpha(u))\big) g(u) du\right]$$

$$= \mathbb{E}\left[\int_0^{\infty} f\big(Z_1(t+s), \ldots, Z_n(t+s)\big) g\left(C(s)\right) dC(s)\right]$$

$$= \int_0^{\infty} \mathbb{E}\left[f\big(Z_1(t+s), \ldots, Z_n(t+s)\big)\right] \mathbb{E}\left[g\left(C(s)\right)\right] dC(s)$$

$$= \left(\int f d\mu\right)\left(\int g(u) du\right).$$

By μ we have denoted the Dirichlet distribution. For the last two equalities we have used the fact that $(Z_1(t+s), \ldots, Z_n(t+s))_{s\geq 0}$ has a Dirichlet distribution and is independent of the full process $Q(\cdot)$, hence also of the clock $C(s)$.

By disintegration, for fixed $u \geq 0$, we have that

$$\mathbb{E}^{\mu}\left[f\big(Y_1(u), \ldots, Y_n(u)\big)\right] = \int f d\mu, \tag{4.39}$$

which in addition to (4.34) is just another way of saying that the Dirichlet distribution μ is the invariant distribution of the *multidimensional Jacobi process*.

5. **Extensions to Cox-Ingersoll-Ross (CIR) Processes**

5.1. **General facts about Cox-Ingersoll-Ross processes.** The process widely known in the literature as Cox-Ingersoll-Ross has been first introduced and studied by Feller in [14]. Three decades after Feller's inauguration, this process has been studied again in a paper, [7], by Cox, Ingersoll and Ross, as a model for the term structure of interest rates. It satisfies the following stochastic differential equation:

$$dX(t) = \big(a - bX(t)\big)dt + \eta\sqrt{X^+(t)}dW(t), \quad X(0) = x_0. \tag{5.1}$$

The parameters a, b, and η are assumed to be positive. This stochastic differential equation has a unique strong solution for every triplet (a, b, η) and starting point $x_0 > 0$ (see [30, p. 285]). With the help of Feller's test (see [30, p. 348]), it can be seen that the origin is inaccesible if $(2a)/\eta^2 \geq 1$. If $0 < (2a)/\eta^2 < 1$, then the origin is instantaneously reflecting, and for $a = 0$, zero is absorbing and is hit in finite time almost surely. Thus, the process is always nonnegative, and the + sign under the square root in (5.1) can be erased.

The CIR process exhibits mean-reversion, a desirable property of interest rates. For a treatment of interest rate models, including the CIR model, we refer the reader to Chapter 6 of [37].

The squared Bessel process of (1.37) and (2.1), and the CIR process of (5.1) share some striking similarities. These common properties are going to allow us to extend the main result of [53] with only a small amount of additional work.

Like the squared Bessel process, the CIR process exhibits the additivity property in the parameter a: if X_1 and X_2 are CIR processes with parameters (a_1, b, η) and (a_2, b, η), then $X_1 + X_2$ is a CIR process with parameters $(a_1 + a_2, b, \eta)$. As in the case of the squared Bessel process, the additivity property helps towards the computation of the Laplace transform of the CIR process, which has the following expression (see [46]):

(5.2)

$$\mathbb{E}\left[e^{-\lambda X(t)}\right] = \left[1 + \frac{\eta^2 \lambda}{2b}(1 - e^{-bt})\right]^{-\frac{2a}{\eta^2}} \cdot \exp\left(-x_0 \lambda e^{-bt}\left(1 + \frac{\eta^2 \lambda}{2b}(1 - e^{-bt})^{-1}\right)\right),$$

where $X(\cdot)$ is the process of (5.1). Letting $t \to \infty$ in (5.2), we deduce that the CIR process $X(\cdot)$ converges weakly to a Gamma distribution with parameters $\alpha = (2b)/\eta^2$ and $p = (2a)/\eta^2$, that is, a distribution with density $\alpha^p x^{p-1} e^{-\alpha x} (\Gamma(p))^{-1}$.

We remark that the Laplace transforms of the squared Bessel process and of the CIR process, as described by formulas $(A.4)$ of Appendix A and (5.2) respectively, are of the form

$$\mathbb{E}\left[e^{-\lambda X(t)}\right] = \phi(t, \lambda) e^{-x\psi(t,\lambda)}, \quad X(0) = x. \tag{5.3}$$

Markov processes whose Laplace transforms satisfy formula (5.3) can be used to model branching processes with immigration in continuous time; they have been studied extensively in [33].

Another key property is that the CIR process $X(\cdot)$ of (5.1) is a squared Bessel process, time-changed deterministically. (See [21] for this result.) More precisely, if $Q(\cdot)$ is a squared Bessel process of dimension m, then

$$X(t) := e^{bt} Q\left(\frac{\eta^2}{4b}(e^{bt} - 1)\right) \tag{5.4}$$

is a CIR process with parameter set (a, b, η).

This result combined with the law of large numbers of Theorem 1.4 for the Bessel process allows us to conclude that

$$\int_0^\infty \frac{dt}{X(t)} = \infty, \quad a.s. \tag{5.5}$$

A formula for the price of a zero coupon bond in a CIR environment has been established by H. Geman and M. Yor in [56, p. 84]. The computation of

$$\mathbb{E}\left[e^{-\lambda \int_0^t X(s)ds}\right] \tag{5.6}$$

relies on several techniques, including the reduction of the CIR process with arbitrary parameter η to one with $\eta = 2$. This latter process is also known in the literature as the squared radial Ornstein-Uhlenbeck process, because it can be viewed as the Euclidean norm of a multidimensional Ornstein-Uhlenbeck process, that is a process satisfying the SDE:

$$dZ_i(t) = -bZ_i(t)dt + \sigma dW_i(t), \quad Z_i(0) = z_0^{(i)}, \quad i = 1, \dots, n. \tag{5.7}$$

Further techniques in the computation (5.6) include Girsanov change of measures as in Section 2.1 and the double Laplace transform

$$\mathbb{E}\left[\exp\left(-\frac{a}{2}R^2(t) - \frac{b^2}{2}\int_0^t R^2(s)ds\right)\right] \tag{5.8}$$

for the Bessel process, established by M. Yor in [59].

Our interest in the CIR process has been prompted by the works of Gouriéroux [23] and [25]. In [25] the authors consider the bivariate CIR process with parameter set $(b\beta_i, b, \sqrt{c})$

$$dX_i(t) = -b\big(X_i(t) - \beta_i\big)dt + \sqrt{cX_i(t)}dW_i(t), \quad i = 1, 2, \tag{5.9}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions and b, β_1, β_2, c are strictly positive. The following processes are defined

(5.10)
$$Y_1(t) := \frac{X_1(t)}{X_1(t)+X_2(t)}, \quad Y_2(t) := X_1(t)+X_2(t), \quad Y_1^*(t) := Y_1(\tau(t)), \;\; t \geq 0,$$

where

$$\tau(t) = \int_0^t Y_2(u)du. \tag{5.11}$$

The authors of [25] proceed to claim that $Y_1^*(\cdot)$ is a Jacobi process satisfying the equation:

(5.12)
$$dY_1^*(t) = -b(\beta_1+\beta_2)\left(Y_1^*(t) - \frac{\beta_1}{\beta_1+\beta_2}\right)dt + \sqrt{cY_1^*(t)\left(1-Y_1^*(t)\right)}dW_1^*(t),$$

followed by the statement that the processes $\tau(\cdot)$ and $Y_1(\cdot)$ are independent.

The attempted proof of this result is in the spirit of Proposition 1 from [53]. Upon a closer examination, we have found out that the time changes in the proofs of Gouriéroux and Valéry are not done correctly and the statements made in [25] are not correct. Below we provide the corrected version of the result, but we omit the proof, since it is very similar to that of Proposition 1 in [53] and exploits the similarities between the CIR process and the squared Bessel process.

Theorem 5.1. *Let* $(X_1(t), t \geq 0)$ *and* $(X_2(t), t \geq 0)$ *be two independent CIR processes, with parameter sets* (a_1, b, η) *and* (a_2, b, η)*, respectively, as in* (5.1)*, starting at* x_1 *and* x_2*, with* $x_1 + x_2 > 0$*, and let*

$$T = inf\,\{t \geq 0 |\, X_1(t)+X_2(t) = 0\}.$$

Then there exists a Markov process $(Y(u),\, u \geq 0)$*, also called a Jacobi process, independent of* $(X_1(t) + X_2(t),\, t \geq 0)$*, such that*

$$X_1(t) = \big(X_1(t) + X_2(t)\big) \cdot Y\left(\int_0^t \frac{ds}{X_1(t) + X_2(t)}\right), \quad for\,:\; 0 \leq t < T.$$

The process $Y(\cdot)$ *is a diffusion on* $[0,1]$ *satisfying the following stochastic differential equation*

$$dY(t) = \big(a_1 - (a_1 + a_2)Y(t)\big)dt + \eta\sqrt{Y(t)(1 - Y(t))}dB(t), \tag{5.13}$$

for some Brownian motion $B(\cdot)$*.*

Note that if the process $Y_1^*(\cdot)$ from the work of Gouriéroux were a Jacobi process, representable as $Y_1(\tau(\cdot))$, with $\tau(\cdot)$ and $Y_1(\cdot)$ independent, as claimed, then an explicit formula for the law of the Jacobi process could be obtained through lenghty computations. Indeed, the law of $\tau(\cdot)$ can be obtained by inverting the Laplace transform (5.6) and the transition probabilities for the CIR process are known (see [46]) and can be used to add to the derivation of the law of $Y_1(\cdot)$.

If $p_t(x_0, x)$ is the density of the CIR process started at x_0, then, following [46] we can write

$$p_t(x_0, x) = c\exp\left(-cx - ce^{-bt}x_0\right)\left(\frac{x}{x_0 e^{-bt}}\right)^{\frac{q}{2}} I_q(2c\sqrt{x x_0 e^{-bt}}), \tag{5.14}$$

where $c := (2b)/(\eta^2(1-e^{-bt}))$, $q := (2a)/(\eta^2-1)$ and $I_q(\cdot)$ is the modified Bessel function of the first kind, as in (2.20).

However, an explicit formula for the law of the Jacobi process is not known sofar in the literature.

5.2. **The skew-product decomposition of Cox-Ingersoll-Ross processes and the multidimensional Jacobi process.** The aim of this section is to discuss a multidimensional generalization of the skew-product decomposition result of Warren and Yor from [53]. As in Section 2.2, we are going to consider a family of processes $Q_1(\cdot), \dots, Q_n(\cdot)$, but this time we are going to assume that they are CIR processes with parameter sets $(\delta_1, b, \eta), \dots, (\delta_n, b, \eta)$ (as in (5.1)), instead of Bessel processes of dimensions $\delta_1, \dots, \delta_n$. Note that we can view a squared Bessel process of dimension δ as a CIR process with parameter set $(\delta, 0, 2)$. The clock $C(\cdot)$ and its inverse $\alpha(u)$ are going to be defined as in Section 2.2. (see formula (2.37))

Theorem 5.1 tells us that

$$\frac{Q_i(t)}{Q(t)} = Y_i(C(t)), \quad i = 1, \dots, n, \tag{5.15}$$

where $Y_i(\cdot)$ are one-dimensional diffusions taking values in the interval $[0, 1]$, independent of the clock $C(\cdot)$ and satisfying the following stochastic differential equations

$$dY_i(t) = \big(\delta_i - \delta Y_i(t)\big)dt + \eta\sqrt{Y_i(t)(1 - Y_i(t))}dB_i(t), \quad i = 1, \dots, n, \tag{5.16}$$

for suitably correlated Brownian motions $B_1(\cdot), \dots, B_n(\cdot)$. Because $Y_1(u) + \cdots + Y_n(u) = 1$, these diffusions are not independent, but rather negatively correlated, with quadratic covariations

$$d\langle Y_i, Y_j\rangle(t) = -\eta^2 Y_i(t)Y_j(t)dt \quad i \neq j, \tag{5.17}$$

as it will be transparent from the representations that are going to be obtained in the end of this section.

We are going to establish that the entire vector $(Y_1(\cdot), \dots, Y_n(\cdot))$ is a multi-dimensional diffusion, which we are going to refer to as the *multidimensional*

Jacobi process. We are going to derive the system of stochastic differential equations satisfied by the process $(Y_1(\cdot), \ldots, Y_n(\cdot))$ and write down its infinitesimal generator. We shall start by writing

$$Q_i(t) = x_i + \delta_i t - b\int_0^t Q_i(s)ds + \eta\int_0^t \sqrt{Q_i(s)}d\beta_i(s), \tag{5.18}$$

where $\beta_1(\cdot), \ldots, \beta_n(\cdot)$ are standard independent Brownian motions. Then

$$Q(t) = \sum_{i=1}^n x_i + \delta t - b\int_0^t Q(s)ds + \eta\int_0^t \sqrt{Q(s)}d\beta(s), \tag{5.19}$$

where $\beta(\cdot)$ is a Brownian motion defined by

$$\beta(t) := \sum_{i=1}^n \int_0^t \sqrt{\frac{Q_i(s)}{Q(s)}}\, d\beta_i(s). \tag{5.20}$$

Also define the quotient process

$$\xi_i(t) := \frac{Q_i(t)}{Q(t)}, \quad i = 1, \ldots, n. \tag{5.21}$$

With aid of Itô's rule, we can write

$$d\left(\frac{1}{Q(t)}\right) = \frac{\big((\eta^2 - \delta) + bQ(t)\big)dt - \eta\sqrt{Q(t)}d\beta(t)}{Q^2(t)}. \tag{5.22}$$

Also note the following quadratic covariations:

$$d\langle \beta_i, \beta\rangle(t) = \sqrt{\frac{Q_i(t)}{Q(t)}}dt, \quad d\left\langle Q_i, \frac{1}{Q}\right\rangle(t) = -\frac{\eta^2 Q_i(t)}{Q^2(t)}dt. \tag{5.23}$$

With the help of the stochastic product rule, we find out that

$$d\xi_i(t) = \frac{\delta_i - \delta\xi_i(t)}{Q(t)}dt + \frac{\eta\sqrt{Q_i(t)}}{Q(t)}d\beta_i(t) - \frac{\eta Q_i(t)}{Q(t)\sqrt{Q(t)}}d\beta(t). \tag{5.24}$$

Using the representation (5.20) for $\beta(t)$, we derive:

$$d\xi_i(t) = \frac{\delta_i - \delta\xi_i(t)}{Q(t)}dt + \frac{\eta\sqrt{\xi_i(t)(1-\xi_i(t))}}{\sqrt{Q(t)}}\cdot \tag{5.25}$$

$$\cdot\left(\sqrt{1-\xi_i(t)}d\beta_i(t) - \sqrt{\frac{\xi_i(t)}{1-\xi_i(t)}}\sum_{j\neq i}\sqrt{\xi_j(t)}d\beta_j(t)\right).$$

Introduce the Brownian motions $\widehat{\beta}_i(\cdot)$, $i = 1, \ldots, n$, as follows

$$\widehat{\beta}_i(t) := \int_0^t \sqrt{1-\xi_i(s)}d\beta_i(s) - \sqrt{\frac{\xi_i(s)}{1-\xi_i(s)}}\sum_{j\neq i}\sqrt{\xi_j(s)}d\beta_j(s). \tag{5.26}$$

We now compute the covariation of two such Brownian motions, for $i \neq j$. Writing

$$d\widehat{\beta}_i = \sqrt{1-\xi_i}d\beta_i - \sqrt{\frac{\xi_i}{1-\xi_i}}\sqrt{\xi_j}d\beta_j - \sqrt{\frac{\xi_i}{1-\xi_i}}\sum_{k\neq i,j}\sqrt{\xi_k}d\beta_k, \tag{5.27}$$

$$d\widehat{\beta}_j = \sqrt{1-\xi_j}d\beta_j - \sqrt{\frac{\xi_j}{1-\xi_j}}\sqrt{\xi_i}d\beta_i - \sqrt{\frac{\xi_j}{1-\xi_j}}\sum_{k\neq i,j}\sqrt{\xi_k}d\beta_k,$$

we deduce that

$$d\langle\widehat{\beta}_i, \widehat{\beta}_j\rangle(t) = -\sqrt{\frac{\xi_i(t)\xi_j(t)}{(1-\xi_i(t))\,(1-\xi_j(t))}}\,. \tag{5.28}$$

Introduce the continuous local martingales:

$$M_i(t) := \int_0^t \frac{d\beta_i(s)}{\sqrt{Q(s)}}, \quad \widehat{M}_i(t) := \int_0^t \frac{d\widehat{\beta}_i(s)}{\sqrt{Q(s)}}. \tag{5.29}$$

The quadratic variations of $M_i(t)$ and $\widehat{M}_i(t)$ are $\langle M_i\rangle(t) = \langle\widehat{M}_i\rangle(t) = \int_0^t \frac{ds}{Q(s)}$ and by (5.5), $\langle M_i\rangle(\infty) = \langle\widehat{M}_i\rangle(\infty) = \infty$. Hence $M_i(\cdot)$ and $\widehat{M}_i(\cdot)$ satisfy the hypothesis of the Dambis, Dubins, Schwartz theorem (see [30, p. 174]) and we can conclude that

$$N_i(t) := M_i(\alpha(t)) \quad \text{and} \quad \widehat{N}_i(t) := \widehat{M}_i(\alpha(t)) \tag{5.30}$$

are Brownian motions and again by the time change theorem in stochastic integrals (Proposition 4.8 in [30, p. 176]), it follows that

$$N_i(t) = \int_0^t \frac{d\beta_i(\alpha(s))}{\sqrt{Q(\alpha(s))}}, \quad \widehat{N}_i(t) = \int_0^t \frac{d\widehat{\beta}_i(\alpha(s))}{\sqrt{Q(\alpha(s))}}. \tag{5.31}$$

By introducing the time changed process $Y_i(u) := \xi_i(\alpha(u))$, formula (5.25) becomes

$$Y(u) = Y(0) + \int_0^u (\delta_i - \delta Y(s))ds + \eta \int_0^u \sqrt{Y(s)(1-Y(s))}d\widehat{N}_i(s). \tag{5.32}$$

Observe that

$$d\widehat{N}_i(t) = \sqrt{1-Y_i(t)}dN_i(t) - \sqrt{\frac{Y_i(t)}{1-Y_i(t)}}\sum_{j\neq i}\sqrt{Y_j(t)}dN_j(t). \tag{5.33}$$

Putting together the last two relations, we finally derive the system of stochastic differential equations satisfied by the vector process $(Y_1(\cdot), \ldots, Y_n(\cdot))$:

$$dY_i(t) = \big(\delta_i - \delta Y_i(t)\big)dt + \eta\left[(1-Y_i(t))\sqrt{Y_i(t)}dN_i(t) - Y_i(t)\sum_{j\neq i}\sqrt{Y_j(t)}dN_j(t)\right], \tag{5.34}$$

$$Y_i(0) = \frac{Q_i(0)}{Q(0)} = \frac{x_i}{\sum x_i}.$$

This shows that not only are the individual Jacobi processes $Y_1(\cdot), \ldots, Y_n(\cdot)$ diffusions, with infinitesimal generators

$$\frac{\eta^2}{2}y(1-y)\frac{\partial^2}{\partial y^2} + (\delta_i - \delta y)\frac{\partial}{\partial y}, \quad 0 < y < 1; \tag{5.35}$$

but the entire vector process $(Y_1(\cdot), \ldots, Y_n(\cdot))$ is a diffusion with values in the open unit simplex Δ_n and infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sum_i \sum_j a_{ij}(\bar{y})\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^n b_i(\bar{y})\frac{\partial}{\partial y_i}. \tag{5.36}$$

Here

$$\bar{y} = (y_1, \ldots, y_n), \;\; b_i(\bar{y}) = \delta_i - \delta y_i, \;\; a_{ij}(\bar{y}) = \sum_{k=1}^n \sigma_{ik}(\bar{y})\sigma_{jk}(\bar{y}), \tag{5.37}$$

where

$$\sigma_{ii}(\bar{y}) = \eta(1-y_i)\sqrt{y_i}, \quad \sigma_{ik}(\bar{y}) = -\eta y_i \sqrt{y_k} \;\; (i \neq k), \;\; \text{hence} \tag{5.38}$$

$$a_{ij}(\bar{y}) = \begin{cases} \eta^2 y_i(1-y_i), \text{ for } i = j \\ -\eta^2 y_i y_j, \text{ for } i \neq j. \end{cases}$$

An alternative approach to the multidimensional Jacobi diffusion has been derived by Gouriéroux in [23]. In this paper, the authors start directly with the system of SDEs (5.34) and with the observation that $d\left(\sum_{i=1}^n Y_i\right) = 0$, hence the system is degenerate and $\sum_{i=1}^n Y_i(t) = \sum_{i=1}^n Y_i(0) = 1$.

6. **Open Problems and Potential Future Study**

In this section we recapitulate some of the open questions from the previous chapters and also discuss how our results could be used towards settling other open questions posed in [17]. In the end we list a generalization of the volatility-stabilized model, as suggested by Karatzas in [28].

As discussed at the end of Section 1.2, it would be a desired fact to establish that the volatility-stabilized model is not weakly diverse; the central limit theorem for Bessel processes from (1.48) could be a useful tool in this direction.

At the end of Section 3.4 we have established formula (3.86) for the expectation of the p^{th} moment of the i^{th} component of the volatility-stabilized process, $\mathbb{E}\left[X_i^p(T)\right]$. From a practical point of view it would be useful to study the monotonicity with respect to time T of this expectation.

On page 13 of Section 1.2 we have described a similarity between a market modeled by a volatility-stabilized process and one based on the Black-Scholes market. In light of this similarity, it would be interesting to compare the distribution of the coordinates of the volatility-stabilized process (see formula (3.56)) with the log-normal distribution, in the total variation distance of probability measures. Formula (3.56) could also be used to compute analytically the price of a call option $\mathbb{E}\left[(X_i(T) - K)^+\right]$ in the volatility-stabilized model. Further on, it would be interesting to find a means of comparison between this price and the price of a call option in the Black-Scholes model.

At the end of Section 3.1 we have suggested as an open topic of study, the spectral theory of the infinitesimal generator $\mathcal{L}$ of (5.36) of the *multivariate Jacobi process* and the computation of its mixed moments. This could be an end goal in itself, not requiring knowledge of stochastics, and is also related to the shortest time to beat the market portfolio in a volatility-stabilized model discussed in [17] on pages 11 and 13. More precisely, what is needed is the computation of

$$\mathbb{E}\left[(Y_1(T)\dots Y_n(T))^{-1}\right] \tag{6.1}$$

where $(Y_1(\cdot),\dots,Y_n(\cdot))$ is a *multidimensional Jacobi process.*

To justify this, we denote by $Z(T)$ the exponential local martingale corresponding to the volatility-stabilized model described by the system on SDEs (1.23). See page 5 of [17] for a general discussion of the meaning of this process $Z(\cdot)$. In [28] the following computation is established

$$\log\left(\frac{1}{Z(T)}\right) = \log\left(\frac{X_1(T)\dots X_n(T)}{R_1^2(0)\dots R_n^2(0)}\right)^{\frac{1+\alpha}{2}} + \frac{1-\alpha^2}{8}\sum_{i=1}^{n}\int_0^T \frac{dt}{\mu_i(t)}. \tag{6.2}$$

Here α is the constant from (1.23) and given that we have at hand the joint distribution of $X_1(T),\dots,X_n(T)$, it makes sense to start out with the significantly simpler case $\alpha = 1$. It follows that the Bessel processes used for the volatility-stabilized model (see the representations (1.39)) are $4-$dimensional and

$$Z(T) = R_1^2(0)\dots R_n^2(0)\big(X_1(T)\dots X_n(T)\big)^{-1}. \tag{6.3}$$

The problem of determining the shortest time to guarantee a log-relative-return with respect to the market portfolio is discussed in a general setting in [17, p.11]. The authors establish that the quantity $\mathbb{E}\left[Z(T)X(T)\right]$ is a key ingredient towards the computation of this shortest time. The quantity $X(T)$ for the volatility-stabilized model is given in (1.39). Using the joint distribution of the coordinates of the volatility-stabilized process as it appears in (3.68), after a change of variable we are able to write

$$\mathbb{E}\left[Z(T)X(T)\right] = R_1^2(0)\dots R_n^2(0)\mathbb{E}\left[\frac{X_1(T)+\dots+X_n(T)}{X_1(T)\cdot\dots\cdot X_n(T)}\right] = \tag{6.4}$$

$$= \frac{e^{-\frac{\gamma^2 T}{8}} R_1^2(0)\dots R_n^2(0)}{2R^{\gamma}(0)} \int_0^{\infty}\int_0^{\infty} q^{\frac{\gamma+2-2n}{2}} e^{-\frac{q+R^2(0)}{2t}} \theta_{\frac{\sqrt{q}R(0)}{t}}\left(\frac{T}{4}\right) \frac{I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} \frac{dt}{t} dq\cdot$$
$$\cdot \int_{\Delta^n} \frac{1}{v_1\dots v_n} p_{\frac{T}{4}}\left(\frac{R_1^2(0)}{R^2(0)},\dots,\frac{R_n^2(0)}{R^2(0)}; v_1,\dots,v_n\right).$$

We remark that the above integral over the unit simplex is nothing but the expectation of the inverse of the product of the coordinates of the *multidimensional Jacobi process* $\mathbb{E}\left[\left(Y_1\left(\frac{T}{4}\right)\dots Y_n\left(\frac{T}{4}\right)\right)^{-1}\right]$.

In this context, we also remark that the expectation of the local martingale $Z(T)$ can be easily written down through a formula almost identical to (6.4). A direct proof of the fact that

$$\mathbb{E}\left[Z(T)\right] < 1 \tag{6.5}$$

would provide an alternative solution to the relative arbitrage on any time horizons (formerly) open question for the volatility-stabilized market. This question appeared in [17] and has been settled recently in [3]. See the paper [40] for the connection between arbitrage and the inequality (6.5).

An additional thing to study is whether one could recover any of the limiting results of Proposition 1.3 directly from the distribution of each coordinate of the volatility-stabilized process (formula (3.56)).

Last, but not least, a study of a generalization of the volatility-stabilized process has been started by I. Karatzas. The more general version of the system (1.23) is taken to be

$$d\left(\log X_i(t)\right) = \frac{\alpha}{2\left(\mu_i(t)\right)^{2\beta}} + \frac{\sigma}{\left(\mu_i(t)\right)^{\beta}} dW_i(t), \quad i = 1,\dots,n. \tag{6.6}$$

Appendix A. Proof that the volatility-stabilized model does not satisfy the upper bound of condition (1.18)

If the volatility-stabilized market model satisfied the nondegeneracy condition, then for some $M > 0$ and every choice of $x_1, \ldots, x_n$, one would have

$$\sum_{i=1}^{n} \frac{x_i^2}{\mu_i(t)} \leq M \sum_{i=1}^{n} x_i^2, \quad (\forall) t > 0. \tag{A.1}$$

With the choice $x_i = 1$ and $x_j = 0$ for $j \neq i$, it would follow that $\frac{1}{M} \leq \mu_i(t)$ for every t and some constant $M > 1$. Letting t be the inverse clock $\Lambda^{-1}(t)$ from (1.40) it follows that

$$\frac{1}{M} \leq \frac{R_i^2(t)}{R^2(t)}. \tag{A.2}$$

Recall that $R_i(\cdot)$ and $R(\cdot)$ are Bessel processes of dimension m and mn, respectively. Taking expectations in the above inequality we get that

$$\mathbb{E}\left[\exp(-R^2(t))\right] \geq \mathbb{E}\left[\exp(-MR_i^2(t))\right]. \tag{A.3}$$

If $Q(t)$ is a squared Bessel process of dimension δ, with $Q(0) = x$, then it is shown in [10] that

$$\mathbb{E}\left[\exp(-\lambda Q(t))\right] = \frac{\exp(-\frac{\lambda x}{1+2\lambda t})}{(1+2\lambda t)^{\frac{\delta}{2}}}. \tag{A.4}$$

Hence $(A.3)$ becomes

$$\frac{\exp(-\frac{R^2(0)}{1+2t})}{(1+2t)^{\frac{mn}{2}}} \geq \frac{\exp(-\frac{MR_i^2(0)}{1+2Mt})}{(1+2Mt)^{\frac{m}{2}}} \tag{A.5}$$

or, written differently:

$$\left(\frac{(1+2t)^n}{1+2Mt}\right)^m \leq \exp\left(\frac{2MR_i^2(0)}{1+2Mt} - \frac{2R^2(0)}{1+2t}\right). \tag{A.6}$$

Since $n \geq 2$, taking $t \to \infty$ on both sides gives $\infty \leq 1$, a contradiction.

Appendix B. Some properties of the Jacobi polynomials

In this Appendix we provide several computational details on Jacobi polynomials, details that have been omitted in Section 3.1.

We start by proving the orthogonality relation for Jacobi polynomials, (3.25). Let $a := \frac{m}{2}$ and $b := \frac{m(n-1)}{2}$. Then

(B.1)
$$\left(y^a(1-y)^b P_k'(y)\right)' = y^{a-1}(1-y)^{b-1}\left(y(1-y)P_k''(y) + (a-(a+b)y)\,P_k'(y)\right) =$$
$$= y^{a-1}(1-y)^{b-1}(-k)\left(\frac{mn}{2}+k-1\right)P_k(y),$$

where the last equality follows from the differential equation (3.24) satisfied by the Jacobi polynomial. Multiply $(B.1)$ by $P_k(y)$ to obtain:

(B.2) $$\left(y^a(1-y)^b P_k'(y)\right)' P_k(y) + k y^{a-1}(1-y)^{b-1}\left(\frac{mn}{2}+k-1\right)P_k(y)P_l(y) = 0.$$

Rewriting this last equality with k and l reversed and then substracting the two equations we get:

(B.3) $$P_l(y)\left(y^a(1-y)^b P_k'(y)\right)' - P_k(y)\left(y^a(1-y)^b P_l'(y)\right)' +$$
$$+P_k(y)P_l(y)y^{a-1}(1-y)^{b-1}\left((k-l)\left(\frac{mn}{2}+k+l-1\right)\right) = 0.$$

Integration by parts shows that

(B.4) $$\int_0^1 P_l(y)\left(y^a(1-y)^b P_k'(y)\right)' dy = -\int_0^1 P_l'(y)y^a(1-y)^b P_k'(y)dy =$$
$$= \int_0^1 P_k(y)\left(y^a(1-y)^b P_l'(y)\right)' dy$$

and next the orthogonality property follows as a direct consequence of $(B.3)$ and $(B.4)$.

Next, we go through the main steps of the derivation of the coefficients A_k and C_k from the recurrence relation

$$Q_{k+1}(y) = (A_k y + B_k)Q_k(y) - C_k Q_{k-1}(y), \tag{B.5}$$

where the polynomial $Q_k(y)$ is defined through formula (3.31). We start by equating the coefficients y^i, on the left and right hand side of $(B.5)$, where i is any arbitrary integer between 0 and $k+1$:

$$(k+1)k\left(\frac{mn}{2}+k+i-1\right)\left(\frac{mn}{2}+k+i-2\right) = \tag{B.6}$$

$$= -A_k ki\left(\frac{m}{2}+i-1\right) + B_k k(k-i+1)\left(\frac{mn}{2}+k+i-2\right) - C_k(k-i+1)(k-i).$$

Regarding formula $(B.6)$ as a polynomial equality in the variable i and identifying the coefficients of i^2, i, and 1 on the left and right hand side respectively, we get the following system of three equations, with A_k, B_k and C_k being the three unknowns to be determined:

(B.7)
$$\begin{cases} kA_k + kB_k + C_k = -k(k+1) \\ -k\left(k-2+\frac{mn}{2}\right)B_k + kC_k = -k\left(\frac{mn}{2}+k-1\right)\left(\frac{mn}{2}+k-2\right) \\ kA_k\left(\frac{m}{2}-1\right) + kB_k\left(\frac{mn}{2}-3\right) - C_k(2k+1) = -k(k+1)(mn+2k-3). \end{cases}$$

Only the values of A_k and C_k are relevant for the computation of the integral I_k from Section 3.1 and these values captured in the formulas (3.35) and (3.36).

Appendix C. More computations with the Jacobi polynomials

In this Appendix we give a different proof, for the case $m = 2$, that the quantity π_k from Section 3.1 does not depend on n.

The Jacobi polynomial $P_k(y)$ of (3.21) takes the following expression:

$$P_k(y) = \left(\frac{2k+n-1}{n-1}\right)^{\frac{1}{2}} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n+k+i-2}{n+k-2} y^i. \tag{C.1}$$

We recall a combinatorial identity (see [52]) that is going to help us express the Jacobi polynomial in a different form. This identity states that

$$\sum_{j=0}^{p} \binom{p}{j} \binom{q}{j} a^{p-j} b^j = \sum_{j=0}^{p} \binom{p}{j} \binom{q+j}{j} (a-b)^{p-j} b^j. \tag{C.2}$$

Letting $a = y - 1$ and $b = y$ we get that

$$P_k(y) = (-1)^k \sqrt{\frac{2k+n-1}{n-1}} \sum_{i=0}^{k} \binom{k}{i} \binom{n+k-2}{k-i} (y-1)^i y^{k-i}. \tag{C.3}$$

The Leibniz rule for the higher derivative of a product of two functions implies that

$$\frac{\partial}{\partial y^k} \left[y^k (y-1)^{n+k-2} \right] = k!(y-1)^{n-2} \sum_{i=0}^{k} \binom{k}{i} \binom{n+k-2}{k-i} (y-1)^i y^{k-i}. \tag{C.4}$$

Hence we derive the following identity:

$$P_k(y) = (1-y)^{2-n} \frac{1}{k!} \sqrt{\frac{2k+n-1}{n-1}} \frac{\partial}{\partial y^k} \left[y^k (1-y)^{n+k-2} \right] \tag{C.5}$$

Similar equalities for other families of orthogonal polynomials go in the literature under the name of Rodrigues formula. Letting $R := \frac{1}{k!}\sqrt{\frac{2k+n-1}{n-1}}$, formula $(C.5)$

combined with integration by parts gives:

$$\int_0^1 (1-y)^{n-2} P_k^2(y)dy = R\int_0^1 P_k(y)\frac{\partial}{\partial y^k}\left[y^k(1-y)^{n+k-2}\right]dy = \tag{C.6}$$

$$= (-1)^k R\int_0^1 y^k(1-y)^{n+k-2}\frac{\partial P_k(y)}{\partial y^k}dy =$$

$$= (-1)^k R\sqrt{\frac{2k+n-1}{n-1}}k!(-1)^k\binom{n+2k-2}{k}\int_0^1 y^k(1-y)^{n+k-2}dy.$$

It is known that (see for instance [39])

$$\int_0^1 y^k(1-y)^{n+k-2}dy = \frac{\Gamma(k+1)\Gamma(n+k-1)}{\Gamma(n+2k)}, \tag{C.7}$$

and hence equality $(C.6)$ turns out to be $\frac{1}{n-1}$. So

$$\pi_k = \frac{1}{\int_0^1 m(y)P_k^2(y)dy} = \frac{1}{2^{n-2}\int_0^1 (1-y)^{n-2}P_k^2(y)dy} = \frac{n-1}{2^{n-2}}, \tag{C.8}$$

which is the same quantity that one obtains by letting $m = 2$ in formula (3.39) from Section 3.1.

Appendix D. The moments of the market weights for the volatility-stabilized market model: some calculus details

In this Appendix we study the behavior of the quadratic $\mathcal{P}_T(x) = A(T)x^2 + B(T)x + G(T) - F(T)$ from Section 3.4 by using standard calculus methods. The coefficients of this quadratic, as functions of T, are given below:

$$A(T) = e^{-2mn\frac{T}{4}}\left(e^{-T} - 1\right) \tag{D.1}$$

$$B(T) = e^{-mn\frac{T}{4}}\left[\frac{4(n-2)}{n(mn+4)} + 2e^{-mn\frac{T}{4}}\left(\frac{1}{n} - \frac{m+2}{mn+4}e^{-T}\right)\right]$$

$$G(T) = \frac{m+2}{n(mn+2)} - \frac{2(m+2)}{n(mn+4)}e^{-mn\frac{T}{4}} + \frac{m(m+2)}{(mn+2)(mn+4)}e^{-2(mn+2)\frac{T}{4}}$$

$$F(T) = \frac{\left(1 - e^{-mn\frac{T}{4}}\right)^2}{n^2}.$$

In what follows we are going to show that $-\frac{B(T)}{2A(T)} > \frac{1}{2}$ for every $T > 0$, by studying the function

$$-\frac{B(T)}{2A(T)} = \frac{e^{mn\frac{T}{4}}\left[\frac{2(n-2)}{n(mn+4)} + e^{-mn\frac{T}{4}}\left(\frac{1}{n} - \frac{m+2}{mn+4}e^{-T}\right)\right]}{1 - e^{-T}}. \tag{D.2}$$

To ease the notation, let:

$$k := \frac{mn}{4}, \quad a := \frac{1}{n}, \quad c := \frac{2(n-2)}{n(mn+4)}. \tag{D.3}$$

We want to show that:

$$ce^{kT} + a - (a+c)e^{-T} - \frac{1}{2} + \frac{1}{2}e^{-T} > 0 \quad \text{for every} \quad T > 0, \quad \text{or that} \tag{D.4}$$

$$f(T) := ce^{kT} + e^{-T}\left(\frac{1}{2} - a - c\right) > \frac{1}{2} - a. \tag{D.5}$$

We compute the derivative $f'(T) = cke^{kT} - e^{-T}\left(\frac{1}{2} - a - c\right)$ and note that $\frac{1}{2} - a - c = ck$, hence $f'(T) = ck\left(e^{kT} - e^{-T}\right) > 0$ for $T > 0$.

Hence $f(T)$ is strictly increasing on $[0, \infty)$ and $f(T) > f(0) = \frac{1}{2} - a$. Since the argument x of $\mathcal{P}_T(x)$ takes values in $(0,1)$, it is natural to ask next for what times T, $-\frac{B(T)}{2A(T)} > 1$. We want to find T such that

$$ce^{kT} + a - (a+c)e^{-T} - 1 + e^{-T} > 0. \tag{D.6}$$

Define $g(T) := ce^{kT} + e^{-T}(1-a-c)$. Then $g'(T) = kce^{kT} - e^{-T}(1-a-c)$. The equation $g'(T) = 0$ has a unique solution T_0 in $(0, \infty)$, with

$$T_0 = \frac{1}{k+1}\ln\left(\frac{1-a-c}{ck}\right). \tag{D.7}$$

For $T > T_0$, $g'(T) > 0$ and for $T < T_0$, $g'(T) < 0$. Hence $g(T)$ starts at $g(0) = 1 - a$, decreases until T_0 and then increases, with $\lim_{T\to\infty} g(T) = \infty$. It follows that for some unique $T_1 > T_0$, $g(T_1) = 1 - a$. For $T > T_1$, $g(T) > 1 - a$ and for $T < T_1$, $g(T) < 1 - a$. Hence for $T > T_1$, the vertex of the quadratic $\mathcal{P}_T(x)$ has $x-$coordinate greater than 1 and for $T < T_1$ the same vertex has $x-$coordinate in the interval $\left(\frac{1}{2}, 1\right)$.

In the first instance $\mathcal{P}_T(x)$ is increasing on $(0,1)$, so we conclude that for any $T > T_1$, the ordering of the variances of market weights at time T is the same as the ordering of the set of initial data $\{R_1(0), \ldots, R_n(0)\}$.

The same conclusion follows for all times $T \leq T_1$ if all the market weights at time 0, $\mu_1(0), \ldots, \mu_n(0)$, are less than $V(T) := -\frac{B(T)}{2A(T)}$. In that case they all belong to the subdomain on which the quadratic is increasing.

If $T \leq T_1$ and if some market weight at time 0, say $\mu_1(0)$, is greater than $V(T)$, and hence also than $\frac{1}{2}$, then $\mu_2(0), \ldots, \mu_n(0)$ are all less or equal to $\frac{1}{2}$ and the market weight with the highest variance at time T is the one for which the absolute value $|V(T) - \mu_i(0)|$ is minimized, where $V(T) := -\frac{B(T)}{2A(T)}$. We see that

$$|V(T) - \mu_i(0)| = \begin{cases} \mu_1(0) - V(T), & \text{for } i = 1 \\ V(T) - \mu_i(0), & \text{for } i \geq 2. \end{cases}$$

But $\mu_1(0) - V(T) < V(T) - \mu_i(0)$, since $\mu_1(0) + \mu_i(0) \leq 1 < 2V(T)$, and hence the same conclusion follows, that the ordering of the variances of market weights at time T is the same as the ordering of the set of initial data $\{R_1(0), \ldots, R_n(0)\}$.

Appendix E. The invariant distribution of the multidimensional Jacobi process: some computational details

In this appendix we perform the computations that show that equation (4.32) is satisfied when the operator $\mathcal{L}^*$ is given by formula (4.34) and the multivariate density μ is given by formula (4.36). The coeffiecients of the operator $\mathcal{L}^*$ appear in (5.37) and (5.38). To ease our notation, we define the variables

$$S := x_1 + \dots x_{n-1} \tag{E.1}$$

$$P := x_1^{\frac{2\delta_1}{\eta^2}-1} \cdot \dots \cdot x_{n-1}^{\frac{2\delta_{n-1}}{\eta^2}-1}.$$

Then $\mu(\overline{x}) = P(1-S)^{\frac{2\delta_n}{\eta^2}-1}$. It follows that

$$\frac{1}{2}\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}\left[a_{ii}(\overline{x})\mu(\overline{x})\right] = \frac{\eta^2}{2}\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\left[x_i(1-x_i)P(1-S)^{\frac{2\delta_n}{\eta^2}-1}\right] = \tag{E.2}$$

$$= \frac{\eta^2}{2}\sum_{i=1}^{n-1}\left[x_i\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2} - \frac{4\delta_i}{\eta^2}P(1-S)^{\frac{2\delta_n}{\eta^2}-1}\right.$$

$$+(1-x_i)\frac{2\delta_i}{\eta^2}\left(\frac{2\delta_i}{\eta^2}-1\right)\frac{P}{x_i}(1-S)^{\frac{2\delta_n}{\eta^2}-1} - (1-x_i)\frac{2\delta_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2}$$

$$+x_i\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2} - (1-x_i)\frac{2\delta_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2}$$

$$\left.+(1-x_i)x_i\left(\frac{2\delta_n}{\eta^2}-1\right)\left(\frac{2\delta_n}{\eta^2}-2\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-3}\right]$$

$$= \frac{\eta^2}{2}\sum_{i=1}^{n-1}\left[2x_i\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2} - \frac{4\delta_i}{\eta^2}P(1-S)^{\frac{2\delta_n}{\eta^2}-1}\right.$$

$$+\underline{\frac{2\delta_i}{\eta^2}\left(\frac{2\delta_i}{\eta^2}-1\right)\frac{P}{x_i}(1-S)^{\frac{2\delta_n}{\eta^2}-1}} - \underline{\left(\frac{2\delta_i}{\eta^2}\right)^2 P(1-S)^{\frac{2\delta_n}{\eta^2}-1}} + \frac{2\delta_i}{\eta^2}P(1-S)^{\frac{2\delta_n}{\eta^2}-1}$$

$$-\frac{2\delta_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2} + \underline{\frac{2\delta_i x_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2}}$$

$$-\frac{2\delta_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2} + \underline{\frac{2\delta_i x_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2}}$$

$$\left.+x_i\left(\frac{2\delta_n}{\eta^2}-1\right)\left(\frac{2\delta_n}{\eta^2}-2\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-3} - \underline{x_i^2\left(\frac{2\delta_n}{\eta^2}-1\right)\left(\frac{2\delta_n}{\eta^2}-2\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-3}}\right].$$

Also

(E.3)

$$\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j\neq i}^{n-1}\frac{\partial^2}{\partial x_i\partial x_j}\left[a_{ij}(\overline{x})\mu(\overline{x})\right] = -\frac{\eta^2}{2}\sum_{i=1}^{n-1}\sum_{j\neq i}^{n-1}\frac{\partial^2}{\partial x_i\partial x_j}\left[x_i x_j P(1-S)^{\frac{2\delta_n}{\eta^2}-1}\right] =$$

$$= -\frac{\eta^2}{2}\sum_{i=1}^{n-1}\left[\frac{2\delta_i}{\eta^2}\frac{2(\delta-\delta_n-\delta_i)}{\eta^2}P(1-S)^{\frac{2\delta_n}{\eta^2}-1} - \frac{2\delta_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)(S-x_i)P(1-S)^{\frac{2\delta_n}{\eta^2}-2}\right.$$

$$-\left(\frac{2\delta_n}{\eta^2}-1\right)\frac{2(\delta-\delta_n-\delta_i)}{\eta^2}x_i P(1-S)^{\frac{2\delta_n}{\eta^2}-2}$$

$$\left.+x_i(S-x_i)\left(\frac{2\delta_n}{\eta^2}-1\right)\left(\frac{2\delta_n}{\eta^2}-2\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-3}\right]$$

$$= -\frac{\eta^2}{2}\sum_{i=1}^{n-1}\left[\left(\frac{2}{\eta^2}\right)^2\delta_i(\delta-\delta_n)P(1-S)^{\frac{2\delta_n}{\eta^2}-1} - \underline{\left(\frac{2\delta_i}{\eta^2}\right)^2 P(1-S)^{\frac{2\delta_n}{\eta^2}-1}}\right.$$

$$-\frac{2\delta_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)PS(1-S)^{\frac{2\delta_n}{\eta^2}-2}+\underline{\frac{2\delta_i x_i}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2}}$$
$$-x_i\left(\frac{2\delta_n}{\eta^2}-1\right)\frac{2(\delta-\delta_n)}{\eta^2}P(1-S)^{\frac{2\delta_n}{\eta^2}-2}+\underline{\left(\frac{2\delta_n}{\eta^2}-1\right)\frac{2\delta_i x_i}{\eta^2}P(1-S)^{\frac{2\delta_n}{\eta^2}-2}}$$
$$\left.+x_i\left(\frac{2\delta_n}{\eta^2}-1\right)\left(\frac{2\delta_n}{\eta^2}-2\right)PS(1-S)^{\frac{2\delta_n}{\eta^2}-3}-\underline{x_i^2\left(\frac{2\delta_n}{\eta^2}-1\right)\left(\frac{2\delta_n}{\eta^2}-2\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-3}}\right].$$

Finally

$$-\sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}\left[b_i(\overline{x})\mu(\overline{x})\right]= \tag{E.4}$$

$$=\frac{2\delta_i\delta}{\eta^2}P(1-S)^{\frac{2\delta_n}{\eta^2}-1}-\delta\left(\frac{2\delta_n}{\eta^2}-1\right)x_iP(1-S)^{\frac{2\delta_n}{\eta^2}-2}$$
$$+\delta_i\left(\frac{2\delta_n}{\eta^2}-1\right)P(1-S)^{\frac{2\delta_n}{\eta^2}-2}-\underline{\delta_i\left(\frac{2\delta_i}{\eta^2}-1\right)\frac{P}{x_i}(1-S)^{\frac{2\delta_n}{\eta^2}-1}}.$$

Summing up the expressions $(E.2)$, $(E.3)$ and $(E.4)$, we see that the underlined terms cancel and we get that

$$\mathcal{L}^*\mu=\frac{\eta^2}{2}P(1-S)^{\frac{2\delta_n}{\eta^2}-2}\left[2\left(\frac{2\delta_n}{\eta^2}-1\right)S-\frac{2(\delta-\delta_n)}{\eta^2}(1-S)\right.$$
$$-\frac{4(\delta-\delta_n)}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)+\left(\frac{2\delta_n}{\eta^2}-1\right)\left(\frac{2\delta_n}{\eta^2}-2\right)S-\left(\frac{2}{\eta^2}\right)^2(\delta-\delta_n)^2(1-S)$$
$$+\frac{2(\delta-\delta_n)}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)S+\left(\frac{2\delta_n}{\eta^2}-1\right)\frac{2(\delta-\delta_n)}{\eta^2}S+\frac{4\delta}{\eta^2}\frac{(\delta-\delta_n)}{\eta^2}(1-S)$$
$$\left.-\frac{2\delta}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)S+\frac{2(\delta-\delta_n)}{\eta^2}\left(\frac{2\delta_n}{\eta^2}-1\right)(1-S)\right].$$

The expression in-between the square parenthesis above can be regarded as a polynomial of degree one in the variable S, and by checking that this polynomial takes the value 0 at the points 0 and 1, we conclude that it is identically zero, hence $\mathcal{L}^*\mu \equiv 0$, and this finalizes the proof that the Dirichlet distribution is the invariant distribution of the *multidimensional Jacobi process*.

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