Geometry of Rational Curves on Algebraic Varieties

Matthew F. DeLand

Advisor: Johan de Jong

Submitted in partial fulfillment of the
Requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2009

Abstract

Geometry of Rational Curves on Algebraic Varieties

Matthew F. DeLand

In this thesis we study the geometry of the space of rational curves on various projective varieties. These varieties include projective spaces, and smooth hypersurfaces contained within them. The parameter space we will use is the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$. In Chapter 2, we first study the space of conics on hypersurfaces without appealing to the bend and break Lemma. We then undertake a thorough study of rational degree e curves on Fermat hypersurfaces. In the bend and break range, we are able to use a detailed understanding of the space of lines on these hypersurfaces to prove that the moduli spaces are irreducible and have the expected dimension. In Chapter 3, we give an upper bound on the largest dimension of a complete family of linearly non-degenerate rational curves contained in projective space. This bound is an improvement over what the Bend and Break Lemma would imply. In Chapter 4, we study the property of strong rational simple connectedness as it relates to smooth cubic hypersurfaces. Using a naturally defined foliation on the moduli space of pointed lines together with a careful understanding of the variety of lines and planes on such a hypersurface, we are able to conclude that cubic hypersurfaces are strongly rationally simply connected in the best possible range.

Contents

Acknowledgments

First and foremost I would like to thank my advisor Johan de Jong for all of his help and support. I have learned so much from Johan over the years, both mathematically and otherwise, that it would be difficult to describe the amount of gratitude that I feel. For his open-door approach and his willingness to answer questions, patiently explain concepts, and discuss problems, I am truly thankful. I know that as time passes, I will continue to be in awe of his talent both as a mathematician and as an advisor.

I would like to thank my college professors Serge Lang, Igor Frenkel, and Peter Jones for both inspiring and encouraging me to pursue my mathematical studies. Without them I would not have chosen this path. I would like to thank the professors that I have had at Columbia, especially those with whom I have studied algebraic geometry: Michael Thaddeus, Brian Conrad, Bill Fulton, Jason Starr, Frans Oort, Jarod Alper, and Bob Friedman.

I am especially indebted to Jason Starr for his help and guidance, and for taking an interest in my work. His influence is felt throughout this thesis.

I would like to thank Maksym Fedorchuk, Jason Starr, and Aleksey Zinger for their many detailed suggestions on ways to improve the manuscript and for taking the time to read and discuss the thesis with me. It is certainly much improved because of their efforts. I am also thankful to Frans Oort for his time and his insightful suggestions.

To my fellow graduate students at Columbia, especially Daniel, Joe, Danny and the members of IJAGS, I am thankful for all the time spent together learning, struggling, and drinking coffee. I would have never made it alone. It was fun because we were together when class was over.

To Terrance Cope and the department, I could not have made it without you.

Finally, I would also like to thank the algebraic geometry seminars at Stony Brook University, University of Michigan, and Columbia University for inviting me to discuss these results.

To my parents, for opening every door for me, never pushing me through any of them, and supporting me at every stage. I could not have made it without you.

To my brother, you amaze and inspire me.

To Nana, Papa, Grandma D., and the rest of my family, for always believing in me and forgiving my shortcomings. To know that you are there is true strength. To have been so far away from you these years has been painful.

To Sophie, my love. You are there for me even when we've been apart. You make me smile when I'm discouraged, cheer me up when I'm down, and hold my hand when I'm feeling alone. If I have helped you a fraction of the amount you have helped me, then I feel truly lucky. To have you by my side means everything.

To my friends, especially Daniel, Gene, Josh, and Sam - I choose you as my family.

"...comme, dans les lettres de l'algèbre, il ne reste plus la détermination des chiffres de l'arithmétique, lesquels déjà ne contiennent plus les qualités des fruits ou des fleurs additionnés."

-Marcel Proust

Chapter 1

Introduction

In some sense, the simplest example of a projective variety is that of $\mathbb{P}^1$, a rational curve. The study of rational curves on algebraic varieties is a classical subject. Originally, questions about rational curves were posed in the framework of enumerative geometry. Two such question are "How many conics in $\mathbb{P}^2$ pass through two points and are tangent to three lines?" or "How many lines are on a smooth cubic surface in $\mathbb{P}^3$?". The solutions to these questions (4, 27) and ones like them have been know for quite some time. In "modern history", the study of rational curves has seen a resurgence as an important subject within algebraic geometry. As well as playing central roles in modern enumerative geometry and Gromov-Witten theory, an understanding of the geometry of rational curves on varieties has become central to the study of higher dimensional or birational geometry and "arithmetic" over function fields.

When studying rational curves on a given variety, one is asking for information concerning the moduli space of such objects - that is a "space" whose points are in one to one correspondence with the sorts of objects one wishes to study. A beautiful aspect of algebraic geometry is that these spaces are again algebraic objects (whether varieties, schemes, or stacks). There is more than one moduli space attached to the study of rational curves on a variety X. These include the Hilbert scheme, the Chow variety, and the Kontsevich mapping space - which parameterize, in turn, closed subschemes of X, cycles on X, and stable (genus 0) maps to X. Each space is a way of compactifying the collection of objects that we are really interested in studying (smooth rational curves on X of a given degree or homology class). The benefit is that we are able to work with a compact (and often projective) object which provides many tools with which to study the resulting space (for example, the cohomology ring). The downside is that the method we use to compactify the space of smooth rational curves on X may force us to consider many objects "in the boundary" which we never wanted to consider in the first place. This leads us to choose different parameter spaces depending on the sorts of questions that we are interested in answering.

We would like to know everything about these parameter spaces, in some cases to better study X. One could hope to understand their Picard groups or even their cohomology or Chow rings. We could ask questions about the types of singularities they admit. We could also ask refined birational questions like the Kodaira dimension or ask for an understanding of the various cones of curves on the moduli space itself. While these questions are interesting, they presuppose we know the answer to much more "basic" questions.

Question. What is the dimension of this space and what are the irreducible components? In fact, even for relatively simple projective varieties X, much about the geometry of these moduli spaces remains unknown.

In this thesis we choose to work with the Kontsevich moduli space (and sometimes the Hilbert scheme) as our parameter space. This is because of the tools available to study spaces which represent certain functors (as these do). After one proves that a functor (say, on the category of schemes defined over a field) is representable, there is a certain "yoga" available to study the geometry of the resulting moduli space locally at a point (there are also some tools available to study global properties, though this is usually more difficult). We will not give an introduction to deformation theory here, though there are many good references (see [Kol96] or [Ser06] for example). The deformation theory of the two functors allows us to identify the tangent space to the moduli space as a certain cohomology group. Sometimes we can conclude that a point of the moduli space is smooth if there are no obstructions to deforming - this is usually easiest in case the entire obstruction space (which is another cohomology group) vanishes. Such deformation theoretic considerations allow us to conclude that the moduli space has a (global) lower bound on its dimension which we will refer to as the expected dimension (see [Kol96] II.1.13). We immediately restrict our attention to varieties X defined over the complex numbers, because in characteristic zero the deformation theory behaves well.

The first class of varieties where the Kontsevich moduli spaces are well behaved are homogeneous spaces (which include projective spaces, Grassmannians, and flag varieties). This is because these spaces are convex - that is, the tangent bundle restricts to be positive on all rational curves contained inside them. A consequence of this fact, is that the resulting moduli spaces are always smooth and have the expected dimension. The next case one might consider is hypersurfaces inside such homogeneous varieties. Here, even if one only considers smooth hypersurfaces in projective space, the corresponding spaces are more poorly behaved and more poorly understood. These spaces can be singular, have the wrong dimension, and can be "poorly" behaved in families. For example, if we consider smooth degree 100 hypersurfaces in $\mathbb{P}^4$, most of them will not contain a line, but some of them certainly do. By "most", we mean there is an open set in the moduli space of hypersurfaces (which in this case is simply another projective space) where each hypersurface corresponding to a point in this open set has no lines on it. We will often encode this "most" by saying that a general degree 100 hypersurface in $\mathbb{P}^4$ has no lines on it - without referring to the open set - because making the open set explicit is often quite difficult.

This brings us to one of the difficulties of this area of algebraic geometry - it is often much easier to prove properties of the moduli spaces for "general" varieties (where this general will depend on what sort of varieties you are considering - we will mostly be considering smooth hypersurfaces in projective spaces). For example, though we know that a general degree 100 smooth hypersurface in $\mathbb{P}^4$ has no lines on it, given any particular example, we will have a difficult time determining if this is one with a line on it or not. One reason for this is that the proof that the moduli space of rational curves on a general hypersurface satisfies a certain property often makes use of statements like generic smoothness (see [Har77] III.10). This often leaves us with the knowledge that moduli spaces have the correct dimension (or are smooth or irreducible) for a generic hypersurface without being able to identify which hypersurfaces are appropriately generic. As another example, suppose that $X \subset \mathbb{P}^n$ (with $n > 3$) is a smooth hypersurface of degree $d < n$. If X is general, then the space of lines on X is irreducible and has the expected dimension (see [Kol96], Theorem V.4.3).

Conjecture (Debarre-de Jong, see [Deb01], page 51)**.** For arbitrary smooth hypersurfaces X in this degree/dimension range, the space of lines on X is irreducible and has the expected dimension.

In the paper [HRS04], the authors Harris, Roth, and Starr prove that for a general hypersurface of "low degree" ($2d < n + 1$) the space of degree e rational curves on a general hypersurface is irreducible and of the expected dimension. The main idea behind their proof is that for the space of pointed lines in X, the evaluation map to X is flat. Then they use the bend and break lemma (see [Deb01] or [Kol96]), to degenerate to the boundary where the results are known by induction. In Chapter 2, we undertake a study of rational curves on Fermat hypersurfaces - those given by the vanishing of $X_0^d + \dots X_n^d = 0$. Given their symmetry, these hypersurfaces are in some sense "highly" non-generic. The space of lines on a Fermat hypersurface can be computed, and this spaces is irreducible and of the expected dimension as long as

$d < n$. The corresponding evaluation map for pointed lines on a Fermat hypersurface is (very) non-flat. Nevertheless, by applying a careful understanding of how the space of lines through a fixed point behaves, we are able to apply a similar degeneration to the boundary argument in order to prove that the space of degree e rational curves on degree d Fermat's are irreducible and of the expected dimension in a restricted degree range.

Theorem. *(Chapter 2, Corollaries 2.4.23, 2.4.30, and 2.4.36) If X is a degree d Fermat hypersurface in $\mathbb{P}^n$ and if $ed < n$ when $e > 2$ ($2d \leq n+1$ when $e = 2$), then the moduli space $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible and of the expected dimension.*

There is hope that the statement remains true in larger degree ranges - we refer to that section for the relevant discussion.

One main tool in studying the space of rational curves on a variety is the bend and break Lemma which, when it applies, allows the conclusion that components of a Kontesevich moduli space will have to meet the boundary divisor. The Lemma itself states that it is impossible to have a complete family of rational curves passing through two fixed points (say in some projective space). For a projective variety X, when a dimension count implies that there is a one dimensional family of degree e rational curves through two general points, then we can often apply the bend and break Lemma to draw the conclusion above. For $e = 2$ and degree d hypersurfaces in projective space, this will apply in what we refer to as the "bend and break range", $2d < n + 1$. When this numerical condition is not satisfied, it is difficult to say anything about the space of degree $e \geq 2$ curves on a hypersurface because many techniques proceed by degenerating to the boundary and concluding (something) by induction. In the special case that $e = 2$, some arguments become possible because conics (smooth or otherwise) are always complete intersections in projective space. Using this fact, we are able to prove:

Theorem. *(Chapter 2 Theorem 2.3.4 and Theorem 2.3.6) Suppose that $X \subset \mathbb{P}^n$ is* *a smooth degree d hypersurface with $d < n$. If X is general, then the moduli space of conics is irreducible and of the expected dimension. If X is arbitrary (but smooth), we can conclude that there is a unique component which contains a curve passing through a general point of X.*

We mention an application of these Theorems to "modern" enumerative geometry, Gromov-Witten theory. Knowing that the parameter spaces have the expected dimension lets us conclude that Gromov-Witten invariants are "enumerative". In other words we may combine the above Theorems with the Kleiman-Bertini Theorem to make the following statement. If, given general linear spaces $\Lambda_1, \dots \Lambda_r \subset \mathbb{P}^n$ with $\Sigma_i(codim(\Lambda_i \subset \mathbb{P}^n) - 1)$ equal to the virtual dimension of $\overline{\mathcal{M}}_{0,0}(X,e)$, then the Gromov Witten invariant $\int_{[\overline{\mathcal{M}}_{0,1}(X,e)]^{vir}} ev_1^*[\Lambda_i] \cup \dots \cup ev_1^*[\Lambda_r]$ is equal to the actual number of degree e rational curves in X intersecting each of $\Lambda_1, \dots, \Lambda_r$.

The bend and break Lemma implies an upper bound on the dimension of complete families of smooth rational curves on projective varieties. We may ask though, "What is the largest dimension of a complete family of degree e smooth rational curves on X?". Surprisingly enough, the answer to this question is not known even when X is projective space. There has been related work in the case of higher genus, see [CR94]. When all the rational curves are linearly non-degenerate, we consider an associated, finite map to the appropriate Grassmannian and prove the following:

Theorem. *(Chapter 3 Theorem 3.1.1) If X is the base of a complete family of linearly non-degenerate degree $e \geq 3$ curves in $\mathbb{P}^n$ with maximal moduli, then* $\dim X \leq n - 1$. *For $e = 2$, we can conclude that* $\dim X \leq n$.

Note that for degree 2 and 3 curves, an irreducible rational curve in projective space is smooth if and only if it is linearly non-degenerate. That is, the above theorem gives an upper bound on the dimension on the largest dimension of a complete family of smooth degree 2 and 3 rational curves in projective space. The theorem leaves open a few interesting questions. First, is this bound sharp? Second, can the linearly non-degenerate condition be dropped? And third, and perhaps most relevant to the discussion above, is there a better bound in the case that each rational curve is to lie on a smooth degree d hypersurface in projective space. Notice that the answer to such a question could provide a way of circumventing the numerical restrictions imposed by relying on the bend and break Lemma.

As the geometry of the moduli space of rational curves on X reflects (and is reflected by) the geometry of X, it is interesting to ask questions about curves (especially rational curves) contained in these various moduli spaces. Are these spaces rationally connected, for example? If we treat the projective line as the algebraic analogue of the unit interval in topology, then being rationally connected can be though of as analogous to the topological property of being path connected. Building on this analogy, if the space of rational curves on X passing through two (resp. or more) fixed points is itself rationally connected, then we have a property which we think of as being analogous to the topological property of being simply connected (we refer to Chapter 4 for a more thorough introduction). If X satisfies this property then we say it is rationally simply connected (resp. strongly rationally simply connected).

As motivation for why we would be interested in this property, consider the following problem. Suppose $f : \mathcal{X} \to C$ is a family of rationally connected varieties over a curve. The famous Graber-Harris-Starr theorem [GHS03] guarantees that f has a section, or equivalently that $\mathcal{X}$ has a $k(C)$ point. When $g : \mathcal{X} \to S$ is a family over a surface, the fibers being rationally connected do not guarantee the existence of a section. However, if the fibers satisfy the stronger property of being strongly rationally simply connected, then sections can be proven to exist. What's more, if the fibers of $f : \mathcal{X} \to C$ (over a curve again) are strongly rationally simply connected, then f admits "so many" sections that it satisfies weak approximation. Showing the existence of sections (or of many sections) is what we refer to as "arithmetic over function fields" as an analogy with more number theoretic questions where existence of solutions to Diophantine equations are sought in various number fields.

It turns out though that very few varieties are known to satisfy this stronger geometric property which is quite hard to verify in practice. In [dS06], de Jong and Starr are able to prove that a smooth hypersurface of degree d in $\mathbb{P}^n$ is strongly rationally simply connected if it general and satisfies $d^2 \leq n$, and that an arbitrary smooth hypersurface satisfying $2d^2 - d \leq n + 1$ satisfies this property. It should be the case that this property is satisfied in the range $d^2 \leq n$ for arbitrary smooth hypersurfaces, but as explained above, it is often easier to prove results about general hypersurfaces. When we restrict to $d = 3$, we are able to exploit some geometry which holds on an arbitrary smooth cubic hypersurface. This includes the fact that the moduli space of lines on a smooth cubic hypersurface is always smooth of the expected dimension (which is not true for larger values of d) and the moduli space of planes at least always has the expected dimension. Using these facts, we are able to prove the existence of sufficiently positive surfaces (which are abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) on cubic hypersurfaces from which the method developed in [dS06] allows us to conclude that smooth cubic hypersurfaces are strongly rationally simply connected when they can be:

Theorem. *(Chapter 4 Theorem 4.7.13) If X is a smooth degree 3 hypersurface in $\mathbb{P}^n$ with $n \geq 9$, then X is strongly rationally simply connected.*

Note that if $g : \mathcal{X} \to S$ is a family of cubic hypersurfaces, then the above theorem implies that g has a section. This is already implied by Tsen's theorem (see [Tse36]). That families of such hypersurfaces over curves satisfy weak approximation has already been shown by Hassett and Tschinkel [HT09]. Nevertheless, this stronger geometric property is interesting in its own right, and it is hoped that the techniques to verify it in this case (which are different than those used in [dS06]) will be applicable to other classes of varieties.

Chapter 2

Rational Curves on Hypersurfaces

2.1 Introduction

Throughout this chapter we assume that all schemes, stacks, and morphisms are defined over Spec $\mathbb{C}$.

The study of rational curves on varieties has come to have applications in many areas of algebraic geometry. Indeed, there are applications to Gromov-Witten theory as well as birational geometry and the minimal model program. Nevertheless, on many varieties the geometry of the space of rational curves is not well understood. A careful understanding of the geometry of this space can lead to results in weak approximation, or to proving the existence of sections for families of these varieties (see [dS06] for example). By "the space of rational curves" one (often) means a part of the appropriate Hilbert scheme. Let X be a smooth projective variety over $\mathbb{C}$. Let $R_e(X)$ be the open subscheme of $Hilb^{et+1}(X)$ which parameterizes smooth, degree e rational curves on X. For a given variety, even basic questions about this locus remain unanswered. For example, we often can say very little about its dimension, connectedness, irreducibility, or singularities. In fact, after leaving the class of homogeneous varieties, these questions become quite difficult even for smooth hypersurfaces in projective space, the cases studied this paper.

One recurring theme in the study of these spaces is that it is often easier to answer questions about $R_e(X)$ for a general hypersurface. By this we mean for a fixed n, there is the well known Hilbert scheme of hypersurfaces in projective space, namely $\mathbb{P} = \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))$, and many times something can be said about the $R_e(X)$ for X in an open set $U \subset \mathbb{P}$. This is often a consequence of using results like generic flatness or generic smoothness, and as a consequence, this open set U is rarely made explicit.

For example, in [Kol96] (Chapter V), it shown that for a generic degree d hypersurface X with $d \leq 2n-3$ the space of lines on X is irreducible and smooth of the expected dimension $2n-d-3$. We are able to prove here that for $d < n-1$ the space of conics on a general X is smooth of the expected dimension, see Proposition 2.3.4. As another example, in [HRS04], it shown that if X is a general hypersurface of degree d with $2d < n+1$, then $R_e(X)$ is irreducible and has the expected dimension for all $e \geq 1$. Still, it is difficult to determine which hypersurfaces are general in this sense.

In this Chapter, we are concerned with what can be said about non-general hypersurfaces. See [CS09] for methods which address this issue for cubic hypersurfaces, where the deformation theory works out quite nicely. The known methods for studying the geometry of rational curves on varieties involve using the Kontsevich moduli spaces which sometimes allow the degeneration to the boundary technique. The tool allowing such a degeneration is Mori's bend and break Lemma (see [Deb01] Proposition 3.2). This Lemma can only be applied in a certain numerical range, which explains the degree restrictions used in [HRS04]. We now outline the results of this paper. In Section 2, we review properties of the Kontsevich moduli space and the deformation theory techniques which will be used throughout. In Section 3, we focus back on the Hilbert scheme $Hilb^{2t+1}(X)$ and carefully analyze the deformation theory at points parameterized by this scheme. The first result of this chapter is a study of conics on hypersurfaces in projective space where the bend and break degree range does not hold.

Theorem. *(See Theorem 2.3.6) Suppose X is a smooth degree d hypersurface in $\mathbb{P}^n$ with $d < n-1$ and that the dimension of non-free lines on X is at most $n-3$. Then there is a unique component of $R_2(X)$ whose general point corresponds to a conic through a general point of X.*

This at least implies that if $R_2(X)$ is to be reducible, then any component other than the "good" one (see Section 2) parameterizes conics which are constrained to lie in a subvariety of X. In Section 4, we switch gears to work out what is known about rational curves on Fermat hypersurfaces. These varieties are certainly not general and provide a testing ground to understand what could be true on an arbitrary smooth hypersurface. The fact that enables us to study these varieties is that we are able to understand in depth the lines contained in them (see Section 2.4.2) and even lines passing through a fixed point (see Section 2.4.3). We then restrict ourselves to the bend and break range (in fact, even further depending on e). In Sections 2.4.4-2.4.6, we prove the following theorem:

Theorem. *Suppose that $e > 3$ (resp. $e = 2, 3$) and that $X \subset \mathbb{P}^n$ is the degree d Fermat hypersurface such that $ed < n$ (resp. $2d \leq n+1$, $3d < n+6$) then $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible and has the expected dimension $e(n+1-d)+(n-4)$.*

The proof of the theorem relies on the bend and break Lemma in order to degenerate to the boundary where we can say something inductively. In fact, the theorem only relies on an understanding that while there may be too many lines through some points of X, the locus where this happens occurs in appropriately high codimension, see Definition 2.4.18. It is not known whether any smooth projective hypersurface violates this hypothesis. Outside of the bend and break range, very little is known, even about conics on the Fermat hypersurfaces. We discuss possible problems and approaches in Section 2.4.7 Appendix 2.5 discusses the construction of incidence varieties.

2.2 Kontsevich Moduli Space

To study rational curves on varieties, it is often useful to have a complete parameter space at hand. We will make use of both the Kontsevich moduli space and the Hilbert scheme. Recall that if X is a smooth hypersurface in $\mathbb{P}^n$ $(n \geq 4)$ of degree $2 < d < n$, then $A_1(X) = \mathbb{Z}$. Denote by e, the class of $e[line]$ in $A_1(X)$. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,r}(X, e)$ parameterizes isomorphism classes of data $(C, q_1, \ldots, q_r, f)$ of a proper, connected, at-worst-nodal, arithmetic genus 0 curve C, an ordered collection $q_1, \ldots, q_r$ of distinct smooth points of C and a morphism $f : C \to X$ such that $f_*[C] = e$ and f satisfies the following stability condition:

Stability Condition: Any component C_i of C such that $f(C_i) = pt$ must contain at least three special points. A special point is either one of the q_j or a node of C. This condition is equivalent to the condition that the automorphism group of the map $f : C \to X$ is finite.

In general, $\overline{\mathcal{M}}_{0,r}(X, e)$ is a proper, Deligne-Mumford stack and its points parametrize maps from nodal genus 0 curves to X. The coarse moduli space of $\overline{\mathcal{M}}_{0,r}(X, e)$ is projective, but need not be smooth or irreducible.

There is an evaluation map, which is a projective morphism,

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,r}(X, e) \to X^r$$

sending a datum $(C, q_1, \ldots, q_r, f)$ to the ordered collection $(f(q_1), \ldots, f(q_r))$. For these statements and a more complete discussion, see Sections 0 and 1 of [FP97].

2.2.1 Preliminaries on Deformation Theory

Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $2 < d < n$ ($n > 3$ as above). We recall the following definitions and properties.

Definition 2.2.1. A *rational curve* on X is a map $f : \mathbb{P}^1 \to X$. This is, by definition, a point of the parameter space $Hom(\mathbb{P}^1, X)$; see [Kol96], Section I.1. Recall that if $f(\mathbb{P}^1) = C$ is a local complete intersection in X, then the normal sheaf $N_{C/X}$ is locally free on C of rank $n-2$. If f is an isomorphism onto its image, this bundle splits into the direct sum of $n-2$ line bundles (see [Kol96], II.3.8).

Definition 2.2.2. The *splitting type* of $C \cong \mathbb{P}^1$ is the sequence of integers $a_1 \leq a_2 \leq \ldots \leq a_{n-2}$ such that $N_{C/X} \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \ldots \oplus \mathcal{O}(a_{n-2})$. For $r \geq 0$ a curve C is called *r-free* if $r \leq a_1$. A 0-free curve is often called *free* and a 1-free curve is often called *very free.*

In general, there is a dimension bound on the parameter space $Hom(\mathbb{P}^1, X)$:

Theorem 2.2.3. *([Kol96], Chapter II.1 Theorem 1.2 and 1.3). Let X be smooth and $f : \mathbb{P}^1 \to X$ a point of $M = Hom(\mathbb{P}^1, X)$. Locally around $[f]$, the scheme M can be defined by $h^1(\mathbb{P}^1, f^*T_X)$ equations in a nonsingular variety of dimension $h^0(\mathbb{P}^1, f^*T_X)$. In particular, any irreducible component M of $Hom(\mathbb{P}^1, X)$ has dimension at least*

$$h^0(\mathbb{P}^1, f^*T_X) - h^1(\mathbb{P}^1, f^*T_X) = \chi(\mathbb{P}^1, f^*T_X) = -K_X \cdot f_*[C] + \dim X,$$

where the last equality follows by Riemann-Roch. Further, if $h^1(\mathbb{P}^1, f^*T_X)$ vanishes, then M is smooth at $[f]$.

When X is a smooth degree d hypersurface in $\mathbb{P}^n$, the theorem implies that any irreducible component M of $\overline{\mathcal{M}}_{0,r}(X, e)$ satisfies

$$\dim M \geq e(n + 1 - d) + r + (n - 4).$$

Given a stable, genus 0 map to X, we may associate the corresponding topological type of the curve. This is the graph of C. We will constantly make use of the following fact.

Proposition 2.2.4. *Suppose that $\pi : C \to S$ is a family of semi-stable genus 0 curves with S irreducible. Suppose that the generic fiber has topological type τ. Locally in the étale topology on S, there are normal crossing divisors D_i, $i \in I$, such that the topological type of $\pi : C \to S$ is constant over D_J. Here $J \subset I$ and D_J denotes the locus of points in D_j for $j \in J$ and not in D_i for $i \in I \backslash J$.*

Remark 2.2.5. A more complete discussion can be found in Section 2 of [dJO97] or in Section 1 of [DM69]. We will often say, informally and without referencing this Proposition, that if maps degenerate, then they do so along a divisor.

2.2.2 Remarks on Conics

The study of conics on hypersurfaces is easier than the study of higher degree rational curves on them. One reason is that, as is well known, the Hilbert scheme of plane conics is easy to describe. We have $Hilb^{2t+1}(\mathbb{P}^2) = \mathbb{P}(Sym^2(\mathcal{O}(1))^*) = \mathbb{P}^5$.

Lemma 2.2.6. *Suppose that $[C] \in Hilb^{2t+1}(\mathbb{P}^n)$ is any point. Then the close subscheme $C \subset \mathbb{P}^n$ is contained in a unique linear $P = \mathbb{P}^2 \subset \mathbb{P}^n$. It is a complete intersection in projective space of type $(2, 1, \ldots, 1)$. In particular it is of pure dimension one, and is one of the following types of curves.*

1. *The curve C is defined by an irreducible degree 2 polynomial in P. It is then a smooth degree 2 rational curve in $P \subset \mathbb{P}^n$.*

2. *The curve C is defined by the product of two distinct linear forms in P. It is then the union of two intersecting lines in $P \subset \mathbb{P}^n$. This case will be referred to as a "broken" conic.*

3. *The curve C is defined by the square of a single linear form in P. It is a "double" line in P, and in the ambient projective space.*

Proof. Suppose $C \subset \mathbb{P}^n$ has Hilbert Polynomial $2t+1$. Let $\mathcal{I}$ be the ideal sheaf defining X and consider the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_C \to 0.$$

We have that $\mathcal{O}_{\mathbb{P}^n}(1)$ has $n+1$ global sections, and from the Hilbert polynomial of C, we know that $\mathcal{O}_C(1)$ has at most 3 global sections. This implies that $\mathcal{I}(1)$ has at least $n-2$ independent global sections. If it had more than $n-2$ global sections, then C would be contained (scheme theoretically) in a line (or a point), which is impossible by Hilbert polynomial considerations. This implies that C is contained in a unique $\mathbb{P}^2$, namely the intersection of the hyperplanes in $H^0(\mathbb{P}^n, \mathcal{I}(1))$. The next statement follows from the fact that a subscheme of $\mathbb{P}^n$ having the Hilbert polynomial of a degree d hypersurface must be a degree d hypersurface (see [Ser06] Section 4.3.2). Thus C is defined by a degree 2 polynomial inside P and the only choices for such a polynomial are the three cases listed. □

Remark 2.2.7. For $n > 2$, the above lemma is the key step in showing that $Hilb^{2t+1}(\mathbb{P}^n) = \mathbb{P}(Sym^2(S)^*)$ where S is the universal rank 2 subbundle on the Grassmannian. To see this, we know that $N_{C/\mathbb{P}^n} = \mathcal{O}_C(1)^{n-2} \oplus \mathcal{O}_C(2)^{n-2}$, the Hilbert scheme is smooth at C. The above Lemma implies that we have a map from $Hilb^{2t+1}(\mathbb{P}^n)$ to Grass$(3, n+1,)$ and the fibers are exactly $Hilb^{2t+1}(\mathbb{P}^2) \cong \mathbb{P}^5$. Of course, this $\mathbb{P}^2$ is canonically identified with the fiber of S, and the statement follows. In particular, $Hilb^{2t+1}(\mathbb{P}^n)$ is a $\mathbb{P}^5$ bundle over the Grassmannian Grass$(3, n+1)$, so is smooth and irreducible.

Lemma 2.2.8. *Let $X \subset \mathbb{P}^n$ be a smooth projective variety and let C be a double structure on a line $l \subset X$, see Lemma 2.2.6. The reduced line l is defined by a square zero ideal inside the structure sheaf of C. Said another way, l is a point of*

$Hilb_{t+1}(X)$, C is a point of $Hilb_{2t+1}(X)$ and $C_{red} = l$. There is an exact sequence:

$$0 \to \mathcal{O}_l(-1) \to \mathcal{O}_C \to \mathcal{O}_l \to 0.$$

Proof. As in Lemma 2.2.6, we may choose coordinates of $\mathbb{P}^n$ so that the ideal of C is given by $(x_2^2, x_3, \ldots, x_n)$. Then clearly the ideal defining l is $(x_2, \ldots, x_n)$, a square zero ideal inside $\mathcal{O}_C$. In other words, there is an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_C \to \mathcal{O}_l \to 0,$$

where $\mathcal{I}$ is a sheaf of ideals and $\mathcal{I}^2 = 0$. This implies that $\mathcal{I}$ naturally inherits the structure of a sheaf of $\mathcal{O}_l$ modules, in fact it is locally free. Since C has Hilbert polynomial $2t+1$ and l has Hilbert polynomial $t+1$, we can compute that the Hilbert polynomial of $\mathcal{I}$ is just t (the Hilbert polynomial is additive on exact sequences). However, since $\mathcal{I}$ is a locally free sheaf on l, it can be written in the form $\mathcal{O}_l(\alpha)$ for some integer α. Then the Hilbert polynomial of $\mathcal{O}_l(\alpha)$ is $t+1+\alpha$ and so $\alpha = -1$. □

2.2.3 Canonical Components

The following ideas are established in [dS06], but are included here for completeness and often specialized to the situations that will be encountered in what follows.

Hypothesis 2.2.9. Let X be a smooth, degree d hypersurface in $\mathbb{P}^n$ such that $2 < d < n$. Assume that $M_{1,0} = \overline{\mathcal{M}}_{0,0}(X, 1)$ is irreducible of the expected dimension. There is always a line through a general point of X. This implies that a general point of $M_{1,0}$ parameterizes a free line (see [Kol96] Exercise V.4.6 and Theorem II.3.10.1).

Remark 2.2.10. No hypersurfaces in the given degree range are known to violate this hypothesis. Indeed, it is conjectured [Deb01] (page 51) to always hold for any smooth degree $d < n$ hypersurfaces and it is known in some cases which include $d = 3$ ([Kol96] V.4.4) and $d \leq 6$ ([Beh06]). That $\overline{\mathcal{M}}_{0,0}(X, 1)$ is smooth for a general degree d hypersurface with $d \leq 2n - 3$ follows from [Kol96] Theorem V.4.3.

Lemma 2.2.11. *([dS06] Corollary 4.5) Suppose that X is a degree d hypersurface in $\mathbb{P}^n$ with $d < n-1$ and that X satisfies Hypothesis 2.2.9. Denote by $M_{1,1}$ the irreducible space $\overline{\mathcal{M}}_{0,1}(X,1)$. Then the geometric generic fiber of $ev : M_{1,1} \to X$ is irreducible.*

Lemma 2.2.12. *([dS06] Lemmas 3.3-3.6) Suppose $X \subset \mathbb{P}^n$ is a smooth degree d hypersurface in $d < n-1$ and that X satisfies Hypothesis 2.2.9. For every positive integer e there exists a unique irreducible component $M_{e,0}$ of $\overline{\mathcal{M}}_{0,0}(X,e)$ which contains points that correspond to the following.*

1. *Degree e covers of free lines in $M_{1,0}$.*
2. *Reducible curves $C = C_1 \cup C_2$ where $\deg C_i = e_i$, $e_1 + e_2 = e$, and such that each curve C_i is a free curve parameterized by $M_{e_i,0}$.*
3. *Maps $C \to X$ which are smooth points in $\overline{\mathcal{M}}_{0,0}(X,e)$ such that each non-contracted component C_i of C is in some M_{e_i} and such that at most one of the C_i is not a free curve.*

Further, the general point of M_e parameterizes a smooth, free curve. In addition, for every positive integer e, denote by $M_{e,1}$ the unique irreducible component of $\overline{\mathcal{M}}_{0,1}(X,e)$ dominating $M_{e,0}$. The geometric generic fiber of

$$ev|_{M_{e,1}} : M_{e,1} \to X$$

is also irreducible.

Definition 2.2.13. Suppose X is a smooth degree d hypersurface in $\mathbb{P}^n$ satisfying $d < n-1$ and suppose Hypothesis 2.2.9 holds. The component from Lemma 2.2.12 will be called the *good component*.

For a vector bundle on a broken curve, we can explicitly compute its cohomology via restricting to the separate components in some nice situations.

Lemma 2.2.14. *Let $C = C_1 \cup C_2$ be a reducible genus zero curve on X and let E be a vector bundle on C. If $E|_{C_2}$ is semi-positive, then $H^1(C,E) = H^1(C_1, E|_{C_1})$.*

Proof. Consider the exact sequence

$$0 \to E \otimes \mathcal{O}_{C_2}(-p) \to E \to E \otimes \mathcal{O}_{C_1} \to 0$$

where p is the intersection point of C_1 and C_2. By assumption, E restricts to have only non-negative summands on $C_2 \cong \mathbb{P}^1$ and so twisting down by p does not contribute any h^1. By the long exact sequence in cohomology then, $H^1(C, E) = H^1(C_1, E|_{C_1})$. □

Remark 2.2.15. In the case where E does not restrict to be semi-positive on C_2, the only conclusion that can be drawn is $h^1(C, E) \geq h^1(C_1, E|_{C_1})$.

In the special case of conics, we can recognize when an arbitrary reducible conic is in the good component. This will be a key step in classifying the components of $\overline{\mathcal{M}}_{0,0}(X, 2)$. We will need to know that lines on X are well behaved in the following sense:

Definition 2.2.16. Lines on X are *well-behaved* if the Fano variety (which for lines is the same as the Hilbert Scheme and the same as the Kontsevich moduli space) of lines on X has the expected dimension, and the locus of lines which are not free has dimension at most $n - 3$. Define the open set $L_X \subset X$ to be the set of all points on X through which all lines are free.

Proposition 2.2.17. *On a general hypersurface of degree $d < n$, lines are well-behaved.*

Proof. The proof is very similar to that of Theorem V.4.3 in [Kol96]. Define I to be the incidence correspondence $\{[l], [X] | l \subset X\}$ corresponding to a line contained in a degree d hypersurface. Let H be the projective space which parameterizes degree d hypersurfaces in $\mathbb{P}^n$. There is a natural projection map $p : I \to H$. We also have the projection map $q : I \to \text{Grass}(2, n + 1)$. Let $I^0 \subset I$ be the tuples where X is smooth along l. Let $Z^0 \subset I^0$ denote the locus where l is not a free line on X. We claim that for a point $[l]$ of the Grassmannian, $q^{-1}([l]) \cap Z^0$ has codimension at least $n-d$ in $q^{-1}[l]$.

To prove the claim, we may choose coordinates so that l is given by $x_2 = \ldots = x_n = 0$. If a degree d hypersurface X contains l, its equation may be written as

$$\Sigma_{i=2}^n x_i f_i + \Sigma_{j,k\geq 2} x_j x_k q_{jk}$$

where $f_i = f_i(x_0, x_1)$ and $\deg f_i = d-1$. Lemma 4.3.7 of [Kol96] shows that X is singular at a point contained in l if and only if the f_i have a common zero. It is straightforward to show that if X is smooth along l, then l is free if and only $H^1(l, N_{l/X}(-1)) = 0$. As in the second part of Lemma 4.3.7 [loc. cit], this is equivalent to the condition that $H^0(l, \mathcal{O}(d-1)) = \text{Span}(f_2, \ldots, f_n)$.

For a point $[l] \in \text{Grass}(2, n+1)$, we have

$$I^0 \cap q^{-1}([l]) = \{(f_2, \ldots, f_n, q_{jk}) | f_i \in H^0(l, \mathcal{O}(d-1)) \text{ have no common zeros}\}.$$

We also have

$$Z^0 \cap q^{-1}([l]) = \{(f_2, \ldots, f_n, q_{jk}) | f_i \in H^0(l, \mathcal{O}(d-1)) \text{ and } \text{Span}(f_i) \subsetneq H^0(l, \mathcal{O}(d-1))\}.$$

For a hyperplane $V \subset H^0(l, \mathcal{O}(d-1))$, define

$$Z_V^0 := \{(f_2, \ldots, f_n, q_{jk}) | \text{Span}(f_i) \subset V\} \subset Z^0 \cap q^{-1}([l]).$$

It is clear that $\text{codim}(Z_V^0, I^0 \cap q^{-1}([l])) = n-1$. Then $\text{codim}(Z^0 \cap q^{-1}([l]), q^{-1}([l])) \geq n-d$ because $Z^0 \cap q^{-1}([l]) = \cup_V Z_V^0$. This proves the claim.

It is clear then, that $\text{codim}(Z_0, I_0) \geq n-d$. When we consider the fibers of p, this implies that for a general hypersurface X, the locus of lines on X which are not free has codimension at least $n-d$. For a general hypersurface X, the variety of lines contained in X is smooth, irreducible, and has dimension $2n-d-3$ (Theorem V.4.3 of [Kol96]). Thus for a general X, the space of non-free lines has dimension at most $n-3$ as was to be shown. □

Hypothesis 2.2.18. Suppose X is a smooth degree d hypersurface in $\mathbb{P}^n$ satisfying $d < n-1$ and suppose Hypothesis 2.2.9 holds. We assume that lines on X are well behaved.

The main result of this section is the following:

Theorem 2.2.19. *Assume that X satisfies Hypothesis 2.2.18. Suppose M is an irreducible component of $\overline{\mathcal{M}}_{0,0}(X,2)$ containing a reducible conic C through a point of L_X (notation as in Definition 2.2.16), then M is the good component.*

Note that Lemma 2.2.12 does not apply here because it is not a priori true that C is a smooth point in the moduli space.

Proof. Denote by $\Delta = \Delta_{1,1}$ the boundary locus of $\overline{\mathcal{M}}_{0,0}(X,2)$. There are two cases to consider:

Case 1: $B = (M \cap \Delta) \neq M$ and so $B \subset M$ is a divisor (Proposition 2.2.4).

In this case, $\dim(M) = 1 + \dim(B)$. Write $C = C_1 \cup C_2$, where C_1 and C_2 are lines on X which intersect at P and assume that C_1 contains a point of L_X. This implies that C_1 is a free line. If C_2 is also a free line, then M is the good component by Lemma 2.2.12. If C_2 is not a free line, then we count the dimension of such configurations $C = C_1 \cup C_2$ where C_1 is free but C_2 is not.

By assumption, the choice of C_2 is at most an $(n-3)$-dimensional choice. The choice of P on C_2 is of course a one-dimensional choice. The choice of a free line through a fixed point is an $(n-d-1)$-dimensional choice. Thus, the dimension of such configurations is $2n-d-3$. However, by Theorem 2.2.3, $\dim(M) \geq 3n-2d-2$ and so $\dim(B) \geq 3n-2d-3$. For $d < n$, the inequality $(2n-d-3) < (3n-2d-3)$ holds and these configurations of broken conics through a general point cannot account for all the points of B. Choose a general curve T in B which contains C.

First, from Lemma 2.2.14 we know that $H^1(C, T_X|_C) = H^1(C_2, T_X|_{C_2})$ because C_1 is a free line so the tangent bundle of X restricted to C_1 is semi-positive. We claim that the general point of T cannot be the union of two non-free lines specializing to C.

This follows from semi-continuity theorem (see [Har77] III.12.8). Indeed, if the general point of T corresponded to two non-free lines, then we could produce a family of non free lines specializing to C_1. This is impossible though because on the free line C_1 we have $h^1(C_1, T_X|_{C_1}(-1)) = 0$, but on a non free line the corresponding cohomology group has positive dimension.

This implies that a general point of T is a broken conic, each of whose components is a free line. These are smooth points of M by Lemma 2.2.14 and so M is the good component by Lemma 2.2.12.

Case 2: $M \subset \Delta$, that is M is contained entirely in the boundary.

By the same dimension count as above (which works one better in this case), we see that configurations $C = C_1 \cup C_2$ where C_1 is a free line and C_2 is not a free line cannot fill out all of M. By the same argument, M contains points which correspond to free line union free line. However, these are smooth points of the moduli space by Lemma 2.2.14 and so are contained only in the component M. By Lemma 2.2.12, M is the good component. In fact, this shows Case 2 cannot occur. □

2.3 A study of Conics on X Using Hilbert Schemes

2.3.1 Incidence Correspondences

It becomes convenient to use the Hilbert Scheme in order to say something about the space of conics passing through a general point of X.

Before beginning, recall a fact about the irreducibility of fibers:

Lemma 2.3.1. *([dS06], Lemma 3.2) Suppose that $i : N \to M$ and $e : M \to Y$ are* *morphisms of irreducible schemes. If $i(\nu_N)$ is a normal point (here ν_N denotes the generic point of N) of M and if $e \circ i$ is dominant with irreducible geometric generic fiber, then $e : M \to Y$ is also dominant with irreducible geometric generic fiber.*

Proposition 2.3.2. *Fixing $d < n-1$, there is a scheme $\mathcal{I}$ whose points parameterize the data of $\{p \in C \subset X\}$ where X is a smooth degree d hypersurface in $\mathbb{P}^n$, C is a conic on X (by which we will mean a point of $Hilb^{2t+1}(X)$), and p is a point on C. The scheme $\mathcal{I}$ is a smooth quasi-projective variety. There is a forgetful map to a smooth variety, $\mathcal{H}$, whose points parameterize $\{p \in X\}$ (point contained in a smooth hypersurface):*

$$f : \mathcal{I} \to \mathcal{H}, (p \in C \subset X) \longmapsto (p \in X).$$

The general fiber of this map is smooth, projective, and irreducible. In fact every fiber is connected.

Remark 2.3.3. The word parametrize in the above Proposition (and the following proof) should be read as represents the appropriate flag Hilbert functor. Results from Appendix 2.5 will be used.

Proof. The existence of $\mathcal{H}'$ which parameterizes points contained in *any* hypersurface in $\mathbb{P}^n$ follows from Lemma 2.5.6 where $S = \text{Spec}(\mathbb{C})$ and the linear space has dimension 0. The scheme $\mathcal{H}'$ is smooth as it is a projective bundle over a smooth variety. The condition for a hypersurface to be singular is closed in $\mathbb{P}(\text{Sym}^d(V^*))$, call the complement U. The scheme $\mathcal{H}$ then is simply the restriction of $\mathbb{P}(K^*)$ to U, so smooth.

By Lemma 2.5.3 (and the remark following it), the scheme $\mathcal{I}'$ exists (as above the prime is a reminder that all hypersurfaces in $\mathbb{P}^n$ are allowed). By Lemma 2.5.9, $\mathcal{I}'$ is smooth as it is a projective bundle over a flag variety.

Thus, $\mathcal{I}'$ is smooth and so the restriction $\mathcal{I}$ to U is also smooth. By the Yoneda Lemma there is a natural map from $\mathcal{I}$ to $\mathcal{H}$ given by forgetting the conic. By generic smoothness (see [Har77] Corollary III.10.7), a general fiber is smooth.

Consider the scheme $\mathcal{K}'$ which is the locus $\{p \in C \subset X\} \subset \mathbb{P}^n \times Hilb^{2t+1}(\mathbb{P}^n) \times \mathbb{P}(\mathrm{Sym}^d(V^*))$ where each C is a "double line". This is the locus parameterizing a point contained in a non-reduced conic contained in a degree d hypersurface. Call the pullback of $\mathcal{K}'$ to U, simply $\mathcal{K}$. Arguments almost exactly as above show that $\mathcal{K}$ (resp. $\mathcal{K}$') is smooth and irreducible. We will also consider the scheme $\mathcal{J}'$ which is the incidence locus $\{p \in L \subset X\} \subset \mathbb{P}^n \times Hilb^{t+1}(\mathbb{P}^n) \times \mathbb{P}(\mathrm{Sym}^d(V^*))$. This is the locus parameterizing a point contained in a line contained in a degree d hypersurface. It is also smooth and irreducible and we form $\mathcal{J}$ by pulling back to U. There is a natural map from both $\mathcal{K}$ and $\mathcal{J}$ to $\mathcal{H}$, as well as a map $\mathcal{K} \to \mathcal{J}$ which assigns the non-reduced conic to the associated underlying reduced line. The obvious triangle of maps clearly commutes.

We now prove the irreducibility of the general fiber of f. To apply Lemma 2.3.1, let $M = \mathcal{I}$, $Y = \mathcal{H}$, $e = f$, $N = \mathcal{K}$ and i the map described above. To apply the Lemma, we must verify the hypothesis are satisfied. Both N and M are irreducible. The incidence locus M is smooth, so that $i(\nu_N) \in M$ is automatically a normal point. We claim now that $e \circ i$ is dominant with irreducible generic fiber. To check this, first note that the generic fiber of $\mathcal{J} \to \mathcal{H}$ is smooth and irreducible (here we use [Kol96] Theorem V.4.3 as well as Lemma 2.2.11). For a line l on X, the space of doubled lines contained in X having reduced structure l can be identified with the bundle $\mathbb{P}(H^0(N_{l/X}(-p)))$ over l (at least when $d > 1$). For a general line on X, this will be a $\mathbb{P}^{n-d-2}$ bundle over $\mathbb{P}^1$ (see [Kol96] V.4.4.2). This shows that the generic fiber of $\mathcal{K} \to \mathcal{H}$ is dominant with irreducible generic fiber when $d < n - 1$. Composing the two maps then, we have that $e \circ i$ is dominant with irreducible general fiber, and the Lemma implies the same is true for f.

Consider the Stein factorization of $f : I \to H$. That is, f factors $g : I \to Z$ and $h : Z \to H$ where h is finite and g has connected fibers (see [Har77], III.11.5). However, by the argument above, we know that the general fiber of $h \circ g$ is already connected (in fact, irreducible) and so the degree of g is 1. This shows that every fiber of f is connected (see [Har77] Ex.III.11.4). $\square$

Proposition 2.3.4. *For $d < n-1$, the Hilbert scheme $Hilb^{2t+1}(X)$ for a general degree d hypersurface in $\mathbb{P}^n$ is smooth, irreducible and has the expected dimension.*

Proof. The proof is similar to the one given above. Write $\mathbb{P}^n = \mathbb{P}(V)$. Let $\mathcal{H} = Hilb^{2t+1}(\mathbb{P}^n)$. Let $\mathcal{P} = \mathbb{P}(Sym^d(V^*))$ be the Hilbert scheme of degree d hypersurfaces in $\mathbb{P}^n$. Let $\mathcal{I}$ be the incidence correspondence $\{C \subset X\} \subset \mathcal{H} \times \mathcal{P}$. As in Proposition 2.3.2, we have that $\mathcal{I}$ is smooth and irreducible (it is a projective space bundle over $\mathcal{H}$). We can consider the projection $\pi : \mathcal{I} \to \mathcal{P}$. By generic smoothness, a fiber over a general degree d hypersurface X is smooth, and this is isomorphic to $Hilb^{2t+1}(X)$. We now argue that the generic fiber is irreducible.

Consider the scheme $\mathcal{K}$ which is the locus $\{C \subset X\} \subset Hilb^{2t+1}(\mathbb{P}^n) \times \mathbb{P}(\mathrm{Sym}^d(V^*))$ where each C is a "double line". This is the locus parameterizing a non-reduced conic contained in a degree d hypersurface. The variety $\mathcal{K}$ is irreducible and smooth as in Proposition 2.3.2. We have that $\mathcal{K}$ maps to $\mathcal{I}$ and the image of the generic point of $\mathcal{K}$ is a smooth point in $\mathcal{I}$. We must now verify that the map $g : \mathcal{K} \to \mathcal{P}$ is dominant and has irreducible generic fiber. Let X be a general hypersurface. A general X has a smooth $2n - d - 3$ dimensional family of lines on it (see [Kol96] V.4.3.2). As in Proposition 2.3.2, the space of double structures on a fixed general line on a general hypersurface is a projective space bundle of dimension $n - d - 1$. This implies that the map g is dominant with irreducible generic fiber when $d < n - 1$.

As above, we may now apply Lemma 2.3.1 with $N = \mathcal{K}$, $M = \mathcal{I}$ and $Y = \mathcal{P}$. We conclude that the general fiber of $\mathcal{I} \to \mathcal{P}$ is irreducible. The general fiber is smooth by generic smoothness, so the Proposition follows. $\square$

2.3.2 Conics Through General Points

To study conics through a general point, the locus of such curves where the conic on X is a double line must be analyzed. First, it will be shown that they correspond to smooth points in the Hilbert Scheme.

Proposition 2.3.5. *Let C be a subscheme of X with Hilbert polynomial $2t+1$ such that $C_{red} = l$ is a line through a point of L_X (see Definition 2.2.16). Note that l is defined by a square zero sheaf of ideals on C as in Lemma 2.2.8. Let $N_{C/X}$ denote the normal sheaf of C in X. Then we have that $H^1(C, N_{C/X}) = 0$. In particular, C is a smooth point of $Hilb^{2t+1}(X)$.*

Proof. Both l and C are local complete intersections on X (the property of being l.c.i. is intrinsic to subschemes of smooth schemes) because each is a complete intersections in $\mathbb{P}^2$. Denote their ideal sheaves by I_C and I_L respectively. The inclusion of I_C into I_l induces the following exact sequence of sheaves

$$0 \to \mathcal{O}_l(1) \to N_{l/X} \to N_{C/X}|_l \to \mathcal{O}_l(2) \to 0.$$

Using the long exact sequence of cohomology then, it follows that $h^1(l, N_{l/X}(a)) = h^1(l, N_{C/X}|_l(a))$ for each $a \geq -2$. In particular since l is free, $h^1(l, N_{C/X}|_l(a)) = 0$ for $a \geq -1$. The short exact sequence in Lemma 2.2.6 implies that

$$0 \to N_{C/X}|_l(-1) \to N_{C/X} \to N_{C/X}|_l \to 0$$

is short exact. Taking cohomology, $h^1(C, N_{C/X}) = 0$ because $h^1(l, N_{C/X}|_l(a)) = 0$ for $a \geq -1$. This is what was to be shown. □

Theorem 2.3.6. *Suppose that X is a smooth hypersurface in $\mathbb{P}^n$ of degree $d < n-1$. Suppose further that X satisfies Hypothesis 2.2.18. Consider the Hilbert Scheme $\mathcal{H} = Hilb^{2t+1}(X)$. There is a nonempty open set $U_X \subset L_X$ such that there is a unique component Z of $\mathcal{H}$ whose general point corresponds to a smooth conic through a point of U_X. The component Z contains all curves corresponding to*

1. *A nonsingular degree* 2 *rational curve on* X *containing a point of* U_X.

2. *A broken conic with at least one line containing a point of* U_X.

3. *A doubled line containing a point of* U_X.

Proof. There is an open set of U such that every irreducible conic passing through U is free, let $U_X = U \cap L_X$. Suppose there are components $Z_1, \ldots, Z_n$ of $\mathcal{H}$ whose general point corresponds to a smooth conic in X through a general point of X, and none of these is the good component, Z.

Consider the variety $\mathcal{I}$ from Proposition 2.3.2. Fixing our X, we have the fiber $\mathcal{I}_X$ for the map $\mathcal{I} \to \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^{n+1})^*)$. There is an evaluation map:

$$\pi : \mathcal{I}_X \longrightarrow \mathcal{H}, (p \in C) \longmapsto C.$$

There is also a map:

$$ev : \mathcal{I}_X \longrightarrow X, (p \in C) \longmapsto p$$

Denote by $\widetilde{Z} = \pi^{-1}(Z)$ and $\widetilde{Z}_i = \pi^{-1}(Z_i)$. Restrict these components to $ev^{-1}(p)$ for a general point $p \in X$, by Proposition 2.3.2 this fiber is connected. By assumption the components $\widetilde{Z}_i$ and $\widetilde{Z}$ are non-empty when restricted to this fiber, and clearly they are the only components that can occur there.

Since the fiber is connected over p, there is some $\widetilde{Z}_i$ that intersects $\widetilde{Z}$ in this fiber, say in the point $(p \in C)$. By applying π we see that C is a conic in $Z \cap Z_i$, and so a singular point of $\mathcal{H}$. However, by the computations above, this is impossible. If C is a smooth conic, it is free and so a smooth point of $\mathcal{H}$. If C is a reducible curve, then by Theorem 2.2.19, C is only contained in Z. And finally, if C is irreducible, but not reduced, then by Proposition 2.3.5 again this is a smooth point of $\mathcal{H}$ and so cannot be contained in the intersection of two irreducible components. This exhausts the options for C, and so such a C cannot exist. This means exactly that $\widetilde{Z}$ intersects none of the $\widetilde{Z}_i$ in the fiber over p. But since this fiber is connected, none of the Z_i can actually occur there. This contradiction implies that none of the Z_i exist to begin with. □

2.4 An Extended Example

2.4.1 Fermat Hypersurfaces

Let $X = X(d, n)$ denote the degree d Fermat hypersurface in $\mathbb{P}^n$. That is, X is defined by the equation $\Sigma_{i=0}^n x_i^d = 0$. These hypersurfaces are not general (highly so, in some sense). The large amount of symmetry they exhibit make them appealing to analyze the rational curves they contain, though this turns out to be a non-trivial task. In this section, rational curves on low degree Fermat Hypersurfaces will be studied. We will illustrate the ranges of degree and dimension where some problems can be solved, but also ranges where even basic questions remain unanswered.

Definition 2.4.1. Let $X = X(d, n)$. Suppose that $x = [a_0, \ldots, a_n] \in X$. We say that a subset $I \subset [0, \ldots, n]$ of the coordinates form a *clump* if $\forall i \in I$ we have $a_i \neq 0$, $\Sigma_{i \in I} a_i^d = 0$ and for no proper subset $J \subset I$ does this condition hold.

We say that $x \in X$ is an *r-clumping point*, if there are r disjoint subsets $I_j \subset \{0, \ldots, n\}$ each of which form a clump, such that if $a_i \in [0, \ldots, n] \setminus \cup I_j$ then $a_i = 0$ and such that there is no $r' > r$ disjoint subsets satisfying these conditions. If $k \geq 0$ and $r \geq 1$, denote the locus $X_{k,r}$ as the subvariety of X consisting of points with k zero coordinates, and such that the $(n + 1 - k)$ non-zero coordinates are r-clumped as above.

We first collect some facts about clumping points whose proofs are straightforward.

Lemma 2.4.2. *Each clump must contain at least two indices. Every point on X has at least 1 clump. Any point $x \in X$ can have at most $(n + 1)/2$ clumps. In general*

$X_{k,r}$, if it is non-empty, has codimension $k+(r-1)$. The locus $X_{0,1}$ is open in X. The locally closed sets $X_{k,r}$ are disjoint and cover X.

Notation 2.4.3. Let $X = X(d,n)$. Given a point $x \in X_{k,r}$, up to a permutation of coordinates, we can write

$$x = [0, \ldots, 0, a_1, \ldots, a_{s(1)}, a_{s(1)+1}, \ldots, a_{s(1)+s(2)}, \ldots, a_{s(r)-1}, \ldots, a_n].$$

This notation means that the coordinates have k zeros and each clump I_j has size $s(j)$. Note that the symmetric group acts on the coordinates of X and stabilizes the set $X_{k,r}$. This notation is a convenient choice of representative for the orbit.

We will invoke the following linear algebra statement a number of times while studying lines on Fermat hypersurfaces.

Proposition 2.4.4. *Let $X = X(d,n)$. Suppose that we are given a point $P = [0, \ldots, 0, a_k, \ldots, a_n] \in X_{k,r}$ (written up to a permutation of coordinates). Let I_m for $1 \leq m \leq r$ be the partition of $(k, \ldots, n)$ into the clumps of the point P. Write $\mathbf{a} = (a_k^d, \ldots, a_n^d)$. Suppose we are given a vector $\mathbf{c} = (c_k, \ldots, c_n) \in \mathbb{C}^{n+1-k}$ such $c_j \neq 0$ for each $k \leq j \leq n$ and such that*

$$a_k^d c_k^i + \ldots + a_n^d c_n^i = 0 \text{ for each } i = 1, \ldots, d-1. \tag{2.1}$$

Then,

1. *if $d-1 \geq n+1-k$ then $c_i = c_j$ for some distinct $k \leq i,j \leq n$.*
2. *if $d-1 \geq n+1-k$, then for each clumping I_m $(1 \leq m \leq r)$ of P we have that $c_i = c_j$ for $i,j \in I_m$.*
3. *if $d-1 < n+1-k$ and there are at most $(d-1)$ distinct values of the c_i, then for each clumping I_m $(1 \leq m \leq r)$ of P we have $c_i = c_j$ for $i,j \in I_m$.*

Proof. The conditions of the Proposition imply that we have the matrix identity

$$\begin{pmatrix} c_k & \dots & c_n \\ c_k^2 & \dots & c_n^2 \\ \vdots & & \vdots \\ c_k^{d-1} & \dots & c_n^{d-1} \end{pmatrix} \begin{pmatrix} a_1^d \\ \vdots \\ a_{n-1}^d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.2}$$

In case $d-1 \geq n+1-k$, then the first $(n+1-k)$ rows of the matrix are linearly dependent. We may then apply the Vandermonde identity to conclude that $\prod_{j=k}^{n} c_j \prod_{k \leq i < j \leq n} (c_i - c_j) = 0$. Because we have each $c_j \neq 0$, this implies $c_i = c_j$ for some pair $i < j$. This implies Part 1.

Let $(S_1, \dots, S_m)$ be a partition of $(k, \dots, n)$ so that $(c(1), \dots, c(m))$ are distinct non-zero numbers satisfying $c_i = c(j)$ for every $i \in S_j$. By assumption $m \leq d-1$. For each $1 \leq j \leq m$, define $a(j) = \Sigma_{i \in S_j} a_i^d$. We have the matrix equation

$$\begin{pmatrix} c(1) & \dots & c(m) \\ c(1)^2 & \dots & c(m)^2 \\ \vdots & & \vdots \\ c(1)^{d-1} & \dots & c(m)^{d-1} \end{pmatrix} \begin{pmatrix} a(1) \\ \vdots \\ a(m) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If the vector $(a(1), \dots, a(m))$ is not the zero vector, then we can apply the Vandermonde identity again to conclude that $c(j) = c(k)$ for distinct j and k. This is a contradiction, so we have that each $a(j) = 0$. We conclude then, that each clump I of P is contained in some S_j. This completes Part 2.

The same argument proves Part 3 because then we have that $m \leq d-1$ and the same Vandermonde identity can be applied. □

2.4.2 Lines on Fermat Hypersurfaces

Definition 2.4.5. Let $x \in X_{k,r}$ be a point. Up to a permutation of coordinates we may write

$$x = [0, \dots, 0, a_1, \dots, a_{s(1)}, a_{s(1)+1}, \dots, a_{s(1)+s(2)}, \dots, a_{s(r)-1}, \dots, a_n]$$

with sets I_j as above. We can construct a linear $\mathbb{P}^{r-1} \subset X(d,n)$ in the following way. For any point $[\lambda_1, \dots, \lambda_r] \in \mathbb{P}^{r-1}$, note that the point (again written up to a permutation of coordinates)

$$[0, \dots, 0, \lambda_1 a_1, \dots, \lambda_1 a_{s(1)}, \lambda_2 a_{s(1)+1}, \dots, \lambda_2 a_{s(1)+s(2)}, \dots, \lambda_r a_{s(r)-1}, \dots, \lambda_r a_n]$$

is also contained in $X(d,n)$. So the λ coordinates give a linear space on X, which we call a *clumped linear space*. Note that there will be many clumped linear spaces on X like this.

Definition 2.4.6. A line contained in one of the above $\mathbb{P}^{r-1} \subset X(d,n)$ (up to a permutation of coordinates) will be called *standard*. (See [Deb01], 2.14).

Lemma 2.4.7. *If $d \geq n$ then any line $l \subset X(d,n)$ passes through a point with two zero coordinates.*

Proof. Suppose l is a line on $X = X(d,n)$. Let $P = [0, a_1, \dots, a_n]$ and $Q = [b_0, \dots, b_{n-1}, 0]$ be points on l. We can always find points on l that have this form (simply the intersection of l with the appropriate coordinate hyperplanes). Note that if l is contained in a coordinate hyperplane, then the Lemma is immediate.

The line l is then given by the linear equation $tP + sQ \in X$ for all values of t, s so the following system of equations must be satisfied (expand the equation for X to l and collect terms in t and s):

$$a_1^{d-i} b_1^i + \dots + a_{n-1}^{d-i} b_{n-1}^i = 0 \text{ for each } i = 1, \dots, d-1$$

If either P or Q has two zero coordinates than the Lemma holds. So we may assume that $a_1 a_2 \cdots a_n \neq 0$ and $b_0 b_1 \cdots b_{n-1} \neq 0$.

Set $c_j = (\frac{b_j}{a_j})$, this is nonzero and the system of equations can be rewritten

$$a_1^d c_1^i + \cdots + a_{n-1}^d c_{n-1}^i = 0 \text{ for each } i = 1, \ldots, d-1.$$

By Part 1 of Proposition 2.4.4, we have $c_k = c_j$ for some $(k < j)$ pair. In this case, we have $\lambda = \frac{b_k}{a_k} = \frac{b_j}{a_j}$. Then the point on l given by $\lambda \cdot P - Q$ satisfies the Lemma. □

Lemma 2.4.8. *For $d \geq n$ any line $l \subset X(d, n)$ is a standard line.*

Proof. By Lemma 2.4.7, l contains a point with two zero coordinates. Let P be a point on l with the maximal number of zero coordinates. Up to a permutation of coordinates, we may write $P = [0, \ldots 0, a_t, \ldots, a_n]$ with $t \geq 2$ and $a_t \ldots a_n \neq 0$. Assume that $Q = [b_0, \ldots, b_n]$ is a general point on l. First, we claim that not each $b_i = 0$ for $i < t$. If this were the case, then the line l would be contained in $X(d, n-t)$. Then by Lemma 2.4.7, this line would contain a point with two zero coordinates, and so we could have found a point on the original l with more zero coordinates than P.

In the same manner as in Lemma 2.4.7, we may expand the condition that $l = tP + sQ$ lies on X to arrive at the following system of equations.

$$a_t^{d-i} b_t^i + \ldots + a_n^{d-i} b_n^i = 0 \text{ for each } i = 1, \ldots, d-1,$$

and again we change coordinates $c_j = b_j / a_j$ to arrive at the system:

$$a_t^d c_t^i + \ldots + a_n^d c_n^i = 0 \text{ for each } i = 1, \ldots, d-1.$$

By Part 2 of Proposition 2.4.4, for each clumping I_m of the point P, we have that $c_i = c_j$ for each $i, j \in I_m$. This implies that there is a nonzero λ_m such that $b_i = \lambda_m a_i$ for each $i \in I_m$. Then we have that $b_0^d + \ldots + b_{t-1}^d = 0$ as well (possibly with more than one clump) and so l is a standard line. □

Corollary 2.4.9. *For any value of d the dimension of standard lines on $X(d,n)$ is $n-3$. For $d \geq n$ there is an $n-3$ dimension family of lines on $X = X(d,n)$. This is the expected dimension, $2n-d-3$ when $d = n$ but too large for all $d > n$.*

Proof. The linear $\mathbb{P}^{r-1}$'s from Definition 2.4.5 inside X are parameterized by products of Fermat varieties, each of dimension $|I_j| - 2$. A quick computation shows that $\Sigma_j |I_j| - 2 = n + 1 - 2r$. Each $\mathbb{P}^{r-1}$ has a $2r-4$ dimensional family of lines on it. So when $d \geq n$, this gives an $n-3$ dimensional family of lines on X. Since there are only finitely many ways to choose these clumpings, this is indeed the dimension. The last statement follows immediately. □

Theorem 2.4.10. *The moduli space of lines on $X = X(d,n)$ has the expected dimension $2n-d-3$ whenever $d \leq n$. It is irreducible whenever $d < n$.*

Remark 2.4.11. This is claimed to follow from the exercises in Chapter 2 of [Deb01]; however, it does not appear to follow directly from the results listed there. When $d = n$, it is indeed the case that the space of lines is reducible. For example, $F(X(4,4))$ is the union of 40 curves each with multiplicity 2 and $F(X(5,5))$ is the union of 1960 surfaces (of varying multiplicity) (see [Deb01], Section 2.5).

We delay the proof of Theorem 2.4.10 until we have established a result about the space of lines through points of $X(d,n)$ which are not clumped.

2.4.3 Lines Through Clumped Points

Fixing $d < n$ write $X = X(d,n)$. We now work to understand how many lines can pass through points $x \in X_{k,r}$ for different values of k and r. The variety of lines through a point $x \in X$ will be denoted by $F(X,x)$. This can be given a scheme structure as the fiber of the evaluation morphism $\overline{\mathcal{M}}_{0,1}(X,1) \to X$ over the point $x \in X$.

Suppose that $x = [a_0, \dots, a_n] \in X$. Let $l \subset X$ be a line through x and let $[b_0, \dots, b_n] \in l$ be a point different than x. Then we may choose coordinates $[t, s]$ on $\mathbb{P}^1 \cong l$ such the inclusion map $\mathbb{P}^1 \to X$ is given by $[t, s] \to [a_0 t + b_0 s, \dots, a_n t + b_n s]$. Since $l \subset X$, we have that $\sum (a_j t + b_j s)^d = 0$. We have the normal bundle sequence on l twisted down by the point x,

$$0 \to N_{l/X}(-1) \to \mathcal{O}_l^{n-1} \to \mathcal{O}_l(d-1) \to 0,$$

and the piece of the associated long exact sequence in cohomology:

$$H^0(l, \mathcal{O}_l)^{n-1} \xrightarrow{\alpha'} H^0(l, \mathcal{O}_l(d-1)) \longrightarrow H^1(l, N_{l/X}(-1)) \longrightarrow 0.$$

If the map α' is surjective, then $H^1(l, N_{l/X}(-1)) = 0$ and so $F(X, x)$ is smooth at l (see [Deb01], Section 2.4). This group $H^1(l, N_{l/X}(-1))$ is the cokernel of the map

$$\alpha : \mathbb{C}^{n+1} \to H^0(l, \mathcal{O}_l(d-1))$$

sending

$$(z_0, \dots, z_n) \to \Sigma z_j (a_j t + b_j s)^{d-1}.$$

Fixing the basis $(t^{d-1}, t^{d-2}s, \dots, s^{d-1})$ of $H^0(l, \mathcal{O}_l(d-1))$ this map α is given by the matrix

$$A = \begin{pmatrix} a_0^{d-1} & a_1^{d-1} & \dots & a_n^{d-1} \\ a_0^{d-2}b_0 & a_1^{d-2}b_1 & \dots & a_n^{d-2}b_n \\ \vdots & \vdots & & \vdots \\ b_0^{d-1} & b_1^{d-1} & \dots & b_n^{d-1} \end{pmatrix} \tag{2.3}$$

which has rank at most d. When the rank is d, then the map α' is surjective and l is a smooth point as above. When the rank of this matrix is less than d, then $H^1(l, N_{l/X}(-1)) \neq 0$ and the line is not free on X. This is not a sufficient condition for the point $l \in F(X, x)$ to be a singular point, but it is however necessary.

Definition 2.4.12. A line $l \subset X$ for which $H^1(l, N_{l/X}(-1)) \neq 0$ will be called *obstructed fixing a point.*

Proposition 2.4.13. *Let $X = X(d, n)$ with $d < n$. Suppose that $x \in X_{k,r}$ and let l be a line on X containing x. Suppose that $d - 1 \geq n + 1 - k$, then l is a standard line. If $d - 1 < n + 1 - k$, the obstructed lines through x are standard lines.*

Proof. Up to a permutation of coordinates, write $x = [0, \ldots, 0, a_k, \ldots, a_n]$ with $a_i \neq 0$ for each $k \leq i \leq n$. Let $y = [b_0, \ldots, b_n]$ be a general point on a line l through x. Since y is general, we have $b_i \neq 0$ for each $k \leq i \leq n$.

Because the line l is contained on X, we have that $\sum_{j=0}^{k-1}(sb_j)^d + \sum_{j=k}^{n}(a_j t + sb_j)^d = 0$ identically as a polynomial in s and t. Collecting the coefficients of $s^\alpha t^{n-\alpha}$ for $\alpha < d$ and making the substitution $c_j = b_j/a_j$ for $k \leq j \leq n$, we have the relationship

$$\begin{pmatrix} c_k & \ldots & c_n \\ c_1^2 & \ldots & c_n^2 \\ \vdots & & \vdots \\ c_k^{d-1} & \ldots & c_n^{d-1} \end{pmatrix} \begin{pmatrix} a_k^d \\ \vdots \\ a_n^d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix above is $(d - 1) \times (n + 1 - k)$. There are two cases to consider.

Case 1 ("many zeroes"): $(d - 1) \geq (n + 1 - k)$. By Part 2 of Proposition 2.4.4, we conclude that $c_i = c_j$ inside each clump I_m of the point x. In other words, for each $1 \leq m \leq r$, there is a nonzero λ_m such that $b_i = \lambda_m a_i$ for each $i \in I_m$. Note that there is no condition on the first k coordinates of y other than that they must form their own clump, possibly with zeros, possibly with many clumps. Since the non-zero coordinates of x in each clump are multiples of the corresponding coordinates of the point y, the line l is standard.

Case 2 ("few zeroes"): $d - 1 < n + 1 - k$. In this case, we consider the matrix

$$A = \begin{pmatrix} 0 & \ldots & 0 & a_k^{d-1} & \ldots & a_n^{d-1} \\ 0 & \ldots & 0 & a_k^{d-2}b_1 & \ldots & a_n^{d-2}b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_0^{d-1} & b_1^{d-1} & \ldots & b_k^{d-1} & \ldots & b_n^{d-1} \end{pmatrix}$$

which by assumption has rank less than d. By Part 3 of Proposition 2.4.4, we again conclude that $c_i = c_j$ inside each clump I_m of the point x. The same argument as in Case 1 implies that l is standard. □

Corollary 2.4.14. *Let $X = X(d, n)$ with $d < n$. Suppose that $x \in X_{k,r}$.*

1. *All lines through points with many zeros, that is, when $d - 1 \geq n + 1 - k$ are standard.*
2. *If $l \supset X$ is a line containing x which is not free, then l is a standard line.*
3. *The dimension of the space of non-free lines through x is at most $r + k - 3$ when $k \geq 1$ and $r - 2$ when $k = 0$. When $x \in X_{0,1}$ or $x \in X_{1,1}$, this means there are no non-free lines through x.*
4. *The space of lines through a point $x \in X_{k,r}$ with $r + k - 3 \leq n - d - 1$ has the expected dimension $n - d - 1$. If $r + k - 3 > n - d - 1$, then the space of lines through a point in $X_{k,r}$ has dimension at most $r + k - 3$.*

Proof. The first statement is simply a restatement of Proposition 2.4.13. In the case that $(d - 1) < n + 1 - k$, all obstructed lines through x are standard by the same proposition. This shows 2.

Because all non-free lines containing x are standard lines, we can bound the dimension of standard lines containing x. Suppose first that $k = 0$ and $r > 1$. The point x defines a linear $\mathbb{P}^{r-1}$ on X (see Definition 2.4.5). We claim that any standard line containing X must be contained in this $\mathbb{P}^{r-1}$. This is clear though, because if a l containing x were contained in some other "clumped" $\mathbb{P}^{r'-1}$, then the point x would admit more "clumps". To count the dimension of standard lines through x then, we are counting the dimension of lines in $\mathbb{P}^{r-1}$ through a fixed point, and this has dimension $r - 2$ as claimed.

Suppose then that $0 < k$ and $1 < r$. Up to a permutation of coordinates, write $x = [0, \ldots, 0, a_k, \ldots a_n]$. Suppose l is a standard line containing x, contained in some clumped $\mathbb{P}^s$. As x is a point of this $\mathbb{P}^s$, the clumps defining the linear space must include the clumps occurring in x. In other words, a standard line through x is defined by a line in the $\mathbb{P}^{r-1}$ formed from the clumping of the non-zero coordinates of x, and an arbitrary choice of point $(b_0, \ldots, b_{k-1}) \in \mathbb{A}^k$ such that $\Sigma b_i^d = 0$. The point is taken in affine space to account for the fact that the scaling of the coordinates has already been accounted for in $\mathbb{P}^{r-1}$. This is an $r + k - 3$ dimensional choice as asserted.

In the case $r = 1$ and k arbitrary, the same argument applies, but there is no line in $\mathbb{P}^{r-1}$. So in this case, the choice of point $(b_0, \ldots, b_{k-1})$ must be taken in $\mathbb{P}^{k-1}$. This is a $k - 2 = r + k - 3$ dimensional choice again, and 3 now follows.

Suppose the space of lines through a point $x \in X_{k,r}$ has a component of dimension larger than the expected dimension. Then the generic point of that component is an obstructed point of the moduli space. By 2 though, the obstructed lines are exactly the standard lines, and by 3 we can bound the dimension of these obstructed lines through x. Exactly when $r + k - 3 \leq n - d - 1$, there are not enough standard lines through x that they could form a component of the space of lines through x of too large a dimension. In general, this same argument shows that for any $x \in X_{k,r}$, there cannot be more than $r + k - 3$ dimensional component of the space of lines through x. This finishes 4. □

Remark 2.4.15. The proposition above says that there are points in $X = X(d, n)$ which have "too many" lines through them in the sense that there is more than the expected dimensions worth of $n - d - 1$. For example, suppose ω is a d-th root of -1. Then the point $x = [1, \omega, 0, \ldots, 0]$ has an $n - 3$ dimensional of lines containing it ($r = 1$, $k = n - 1$). In this extreme case, x is a conical point; that is, X contains the cone over the Fermat cut out by $x_0 = x_1 = x_2^d + \ldots x_n^d = 0$ and the vertex of this cone is x. The fact used in the following sections will be that points which contain "too many" lines occur in appropriately high codimension. For example, there are only finitely many conical points on X; see Corollary 2.2 of [CS09].

Remark 2.4.16. Not all standard lines are obstructed. When $d = 3$ the moduli space of lines on $X(d, n)$ is smooth (see [CS09]). Some standard lines will not even be obstructed while fixing a point. For example, a general line l in $X = X(3, n)$ inside a $\mathbb{P}^2$ given by scaling three clumps will have normal bundle $N_{l/X} = \mathcal{O}(1)^{N-4} \oplus \mathcal{O}^2$. In particular, it will not be obstructed while fixing a point. (See [Deb01] 2.5 for this statement). Even for larger values of d, when the number of clumps is large, the rank of the corresponding matrix 2.3 could be full. In any case, we never need a full analysis of when standard lines are obstructed, only a bound on the dimension of lines through points of $X_{k,r}$.

The knowledge of the space of pointed lines on $X = X(d, n)$ can now be used to prove the irreducibility of the space of lines on the Fermat.

Proof of Theorem 2.4.10. We prove the Theorem by induction on n. The case $d = n$ follows from Corollary 2.4.9. Consider the space of lines $F(X)$ on $X = X(d, n + 1)$. This space has dimension at least $2(n + 1) - d - 3$. When X is intersected with a hyperplane of the form $x_i = 0$, the result is $X(d, n)$ and $F(X(d, n))$ has the expected dimension $2n - d - 3$ by the induction hypothesis. Suppose that Y is an irreducible component of $F(X)$. Let H be the subvariety of the Grassmannian corresponding to all lines in the hyperplane $x_i = 0$. By Lemma 4.4 in [Beh06], either $Y \cap H$ has codimension 2 or all the lines parameterized by Y are concurrent. In the first case, Y has codimension 2 in $F(X(d, n))$ and so $\dim(Y) \leq \dim F(X(d, n)) + 2 = 2(n + 1) - d - 3$. But this implies that Y has the expected dimension. In the latter case, all lines parameterized by Y pass through a fixed point $x \in X$. However, since each of these lines are contained in the tangent hyperplane to X at x and X does not contain this hyperplane, $\dim Y \leq n - 2 \leq 2(n + 1) - d - 3$. Note that such a Y cannot form a component under the degree assumption. In any case, Y has the expected dimension.

To show the irreducibility, note that by Corollary 2.4.14, if $x \in X$ is not a clumped point, then the fiber of the map $\overline{\mathcal{M}}_{0,1}(X,1) \to X$ over X is smooth of the expected dimension, namely $n-d-1$ and all lines through x are free. When $d < n$, the moduli space of lines is connected, (see [Kol96] Theorem V.4.3.3). Now, suppose there is a component of the space of lines $M \subset \overline{\mathcal{M}}_{0,0}(X,1)$ which only parameterizes lines passing through clumped points. For $[l] \in M$ corresponding to $l \subset X$, if l contains a non-clumped point then M meets the smooth locus of the moduli space and so cannot form its own component. So suppose then that l contains only clumped points. Then it is contained in some clumped $\mathbb{P}^{r-1} \subset X$ as discussed above. However, there is only an $n-3$ dimensional family of such lines, and when $d < n$, we have that $2n-d-3 > n-3$ and so again M cannot form its own component. Thus when $d < n$, there is a unique component of the expected dimension. □

2.4.4 Conics in the Bend and Break Range

In this section, fix d such that $2d \leq n+1$. This is what will be called the bend and break range. In this degree range, each component of $\overline{\mathcal{M}}_{0,0}(X,e)$ ($e \geq 2$) must meet the boundary $\Delta \subset \overline{\mathcal{M}}_{0,0}(X,e)$.

Lemma 2.4.17. *Suppose that $X \subset \mathbb{P}^n$ is a degree d smooth hypersurface satisfying $2d \leq n+1$. Let $e \geq 2$ be an integer. Every irreducible component $M \subset \overline{\mathcal{M}}_{0,0}(X,e)$ satisfies either $M \subset \Delta$ or $M \cap \Delta \subset M$ is nonempty and has codimension one.*

Proof. Suppose M as above is not contained in the boundary Δ. There is a unique component $M_2 \subset \overline{\mathcal{M}}_{0,2}(X,e)$ dominating M. The dimension of M_2 is at least $e(n+1-d)+n-2$. The dimension of every non-empty fiber of the evaluation map $ev : M_2 \to X \times X$ is at least $e(n+1-d)-n$ which is greater than 0 for all $e \geq 2$ in this range of d and n. However, there can be no complete curve contained in any fiber by the bend and break Lemma (see [Deb01], Proposition 3.2). This implies that $M_2 \subset \overline{\mathcal{M}}_{0,2}(X,e)$ meets the boundary in a divisor by Proposition 2.2.4. The result then follows for M as well. □

We can abstract the properties satisfied by lines on the Fermat Hypersurfaces into a general definition.

Definition 2.4.18. Suppose that $X \subset \mathbb{P}^n$ is a degree d smooth hypersurface with $d < n$. Recall that $F(X, x)$ denotes the scheme of lines on X through x. Consider the stratification of X given by $X = \coprod_e X_e$ where $X_e = \{x \in X | \dim F(X, x) = n - d - 1 + e\}$ for $e > 0$. Note then that X_e is the locus on X where the dimension of the space of lines is e more than expected. Suppose that $\overline{\mathcal{M}}_{0,0}(X, 1)$ is irreducible of the expected dimension and that lines on X are well behaved. We say that X is *well-stratified* if $\mathrm{codim}(X_e) \geq e + 1 + n - d$ whenever $e > 0$ and X_e is non-empty. When X is well-stratified, denote by Z_e the space of lines through points of X_e. That is, $Z_e = ev^{-1}(X_e)$ for $ev : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$.

Remark 2.4.19. Suppose X as above is well-stratified. When $e > d-2$, this implies the dimension of X_e is negative, so that X_e is empty. In other words, we only need to ever consider $0 \leq e \leq d - 2$. If we do not assume that the space of lines on X is irreducible and has the expected dimension, then the remaining condition implies the dimension statement. It does not imply the irreducibility. We may then ask if every smooth hypersurface is well-stratified (in this weaker sense). We know of no counterexamples.

We record the following straightforward fact.

Lemma 2.4.20. *If $X \subset \mathbb{P}^n$ is well stratified, then* $\dim Z_e = \dim(X_e) + (n-d-1+e)$.

Lemma 2.4.21. *Suppose that $X = X(d, n)$ is a Fermat Hypersurface and $d < n$. Then X is well-stratified.*

Proof. By Theorem 2.4.10, the variety $F(X)$ is irreducible and has the expected dimension $2n - d - 3$. By Corollary 2.4.14, the dimension of the space of lines through $x \in X_{k,r}$ is at most $max(r + k + 3, n - d - 1)$. If $r + k + 3 = n - d - 1 + e$ then $\mathrm{codim}(X_{k,r}) = e + 1 + n - d$ by Lemma 2.4.2. □

Theorem 2.4.22. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $2d \leq n+1$. Every irreducible component $M \subset \overline{\mathcal{M}}_{0,0}(X,2)$ has the expected dimension $3n-2d-2$.*

Proof. Let $Y = \overline{\mathcal{M}}_{0,1}(X,1) \times_{ev,X,ev} \overline{\mathcal{M}}_{0,1}(X,1)$. As a stack, Y admits a 2 to 1 cover of the boundary Δ. Let $ev : \overline{\mathcal{M}}_{0,1}(X,1) \to X$ be the evaluation map. By Lemma 2.4.20, the space of pointed lines $\overline{\mathcal{M}}_{0,1}(X,1)$ has the expected dimension $2n-d-3+1 = 2n-d-2$.

Consider a point x in the locus X_e. Through a point $x \in X_e$, there is a $n-d-1+e$ dimensional family of lines. We have that $\dim(Z_e \times_{X_e} Z_e) = 2\dim(Z_e) - \dim(X_e)$. Using Lemma 2.4.20 and the definition, this is at most $2n-d-4+e$ when $e > 0$. When $e = 0$, this dimension reads $3n-2d-2$. When $e > 0$, this dimension is at most $2n-6$ because $e \leq d-2$ (see Remark 2.4.19). This dimension is always strictly less than $3n-2d-3$ because of our assumption that $2d \leq n+1$.

The stratification of X gives a stratification $Y = \coprod Y_e$ where $Y_e = Z_e \times_{X_e} Z_e$. Each irreducible component of Y has dimension bigger than or equal to $2(2n-d-2) - (n-1) = 3n-2d-3$. But from the stratification of Y and the computations above, each Y_e has dimension less than or equal to this expected dimension. The locus Y_e cannot contribute a component to Y when $e > 0$. In fact, the expected dimension is only achieved over the stratum $Z_0 \times_{X_0} Z_0$. In any case we see that Y has the expected dimension.

Because we are in the bend and break range, $M \cap \Delta \neq \emptyset$ and we know that $M \cap \Delta \to M$ has image that is codimension 1, or is all of M (see Lemma 2.4.17). From the dimension count above though, we know that Δ has dimension $3n-2d-3$, but $\dim(M) \geq 3n-2d-2$, so the case $M \subset \Delta$ is ruled out. That is, Δ is a divisor in M and so M has dimension one bigger, namely, the expected dimension. □

Corollary 2.4.23. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $2d \leq n+1$. Then $M = \overline{\mathcal{M}}_{0,0}(X,2)$ is irreducible and has the expected* *dimension.*

Proof. Every irreducible component of M has the expected dimension by Theorem 2.4.22. Because $X \subset \mathbb{P}^n$ is well-stratified, by Lemma 2.2.12 it suffices to prove that any component $V \subset M$ contains a point parameterizing a reducible curve $C = C_1 \cup C_2$ where each C_i is a free line. To show this, we use the fact that any component of M meets the boundary component Δ in a divisor by Lemma 2.4.17. Suppose that the general point in $V \cap \Delta$ is not given by free curves. This general point cannot correspond to a point in Y_e for $e > 0$ because $\dim(Y_e) < 3n - 2d - 3$ (see the proof of Theorem 2.4.22). Then we see that the general point of $V \cap \Delta$ must correspond to a point in Y_0. However, for a general point of X_0, every line through $x \in X_0$ is free. Thus the general point of $V \cap \Delta$ must correspond to a point of a proper subvariety Y' of Y_0. But this implies that $\dim(Y') < 3n - 2d - 3$ so that the general point of $V \cap \Delta$ cannot be contained in Y' either. This gives a contradiction, and the Corollary follows. □

Corollary 2.4.24. *If $X = X(d, n)$ with $2d \leq n + 1$, then $\overline{\mathcal{M}}_{0,0}(X, 2)$ is irreducible and has the expected dimension.*

2.4.5 Cubics in the Bend and Break Range

In this section, assume that $3d < n + 6$ and again suppose that X is well-stratified. As in the case of conics, in this degree range, each component of $\overline{\mathcal{M}}_{0,0}(X, 3)$ must meet the boundary $\Delta \subset \overline{\mathcal{M}}_{0,0}(X, 3)$ (Lemma 2.4.17). We are not able to prove irreducibility of $\overline{\mathcal{M}}_{0,0}(X, 3)$ in the range $2d \leq n + 1$ because of the possibility of components completely contained in the boundary. One could still hope there is a unique component parameterizing smooth cubics, but the methods used here will not show this. In any case, in this restricted range we prove that the moduli space of cubics is irreducible and of the expected dimension.

Remark 2.4.25. In this section and the next, we will freely use the terminology of a stable genus 0 graph. This represents the dual graph to a stable map. At times, the two ideas will be (slightly) conflated but we trust no confusion will result. For a thorough treatment of these objects, see [BM96] Section 1.

Proposition 2.4.26. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $3d < n+6$. There can be no irreducible component M of $\overline{\mathcal{M}}_{0,0}(X, 3)$ contained completely in the boundary.*

Proof. Consider the following dual graphs (which represent stable maps): $\tau_{(1,1,1)} = \textit{line} \cup \textit{line} \cup \textit{line}$, $\tau_{1,2} = \textit{line} \cup \textit{conic}$, and τ_3 which consists of a contracted $\mathbb{P}^1$ with 3 lines attached to it. In each of these cases, we may consider $\overline{\mathcal{M}}(X, \tau)$ which is a proper Deligne-Mumford Stack [BM96]. Note that these are the only dual graphs which can occur in the boundary.

The stack $\overline{\mathcal{M}}(X, \tau_{1,1,1})$ admits an evaluation map ev to X by sending a graph to the first node (implicitly the nodes are ordered). Writing $M(\tau_{1,1})$ (resp. $M(\tau_1)$ to denote the stable degree two maps of graph type $\textit{line} \cup \textit{line}$ (resp. $\textit{line}$) along with a point on one of the lines, we have the following fiber product diagram.

$$\begin{array}{ccc} M(\tau_{1,1,1}) & \longrightarrow & M(\tau_{1,1}) \\ \downarrow & & \downarrow{\scriptstyle ev'} \\ M(\tau_1) & \xrightarrow{ev} & X \end{array}$$

This evaluation map stratifies $M(\tau_{1,1,1})$ as $M(\tau_{1,1,1}) = \coprod M(\tau_{1,1,1})_e$. We first check that the space of chains of three free lines has the expected dimension. The choice of a free line is $2n - d - 3$ parameters, indeed this is open in the space of lines on X. The choice of two points on such a line is two parameters and the choice of a free line through each of these points is $n - d - 1$ parameters. This implies this locus has dimension at most $4n - 3d - 3$, the expected one.

We must also consider the cases where we have glued a non-free line to a pair of free lines or glued any line to a pair of lines where at least one is not free. In fact, we may directly bound the dimension of chains of three lines where at least 1 is not free. The space of non-free lines on X has dimension $n-3$. Suppose we are considering the locus of such chains where the "middle" line is not free. This is an $(n-3)$ dimensional choice, and the choice of two points on this line is of course, another 2 dimensional choice. The choice of lines through these points is at most an $n-3$ dimensional. It is only equal to $n-3$ in case we have chosen a conical point on our line, but because there are only finitely many of these, we see that this locus has dimension bounded by $3(n-3)$. The case where the non-free line occurs on the "end" is bounded similarly by $3(n-3)$.

Now consider the stack $\overline{\mathcal{M}}(X, \tau_3)$. It admits a map ev to X sending a map to its evaluation at the contracted point. Let $B_e := ev^{-1}(X_e)$. The dimension of B_e is bounded from above by $\dim(X_e)$ plus three times the dimension of the space of lines through a point of X_e. If $e > 0$, then $\dim(B_e) \leq d - e - 2 + 3(n - d - 1 + e) = 3n-2d-5+2e \leq 3n-9$. If $e = 0$, then $\dim(B_e) = (n-1)+3(n-d-1) = 4n-3d-4$, the expected dimension.

We have proven the following Lemma which we note for future reference.

Lemma 2.4.27. *Suppose that $M(\tau) \subset \overline{\mathcal{M}}_{0,0}(X, 3)$ is the Deligne Mumford Stack associated to a graph τ that has exactly three degree one components. Consider the locus in $M(\tau)$ where each component maps to a free line. Then the dimension of this locus is the expected one, $4n - 3d - 1 - \#n(\tau)$, where $n(\tau)$ denotes the number of nodes of the graph type τ. If instead at least one of the components maps to a non-free line on X, then this locus has dimension at most $3(n-3)$. In particular, no component of $\overline{\mathcal{M}}_{0,0}(X, 3)$ is contained in $M(\tau)$ (in fact, no codimension two locus of any component can be either) because $3d < n + 6$.*

Finally consider the substack $M(\tau_{1,2})$. This stack has dimension at least $4n - 3d - 2$. Suppose though that there is $V \subset M(\tau_{1,2})$ which forms an open substack of a component of $\overline{\mathcal{M}}_{0,0}(X, 3)$. Note that the curves in V cannot be mapped to free curves on X. We mark two points on $\tau_{1,2}$, one on the line, one on the conic to create the new stable graph type $\tau'_{1,2}$. We have an evaluation map from $M(\tau'_{1,2}) \to X^3$, where we evaluate at the two points just marked and at the node. Note that under the evaluation map, the image of the node cannot be a general point on X and the image of the two marked points cannot be a general point of X^2. Because $4n - 3d + 1 - (3(n-1) - 2) > 0$ in the assumed degree range, the fibers of this map are positive dimensional. But the fibers correspond exactly to a line through two fixed points and a conic through two fixed points. As the line clearly can not deform, it must be the conic that moves on X. By the bend and break Lemma then, the component $\overline{V}$ meets the boundary (so in codimension one). When it does meet the boundary, it does so at a graph type which is the union of lines (possible with a contracted component), but because these lines cannot all be free, such a locus has dimension at most $3(n-3)$ as shown above. In the assumed degree range though, this cannot be a codimension one locus in any component. □

Theorem 2.4.28. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $3d < n + 6$. Every irreducible component $M \subset \overline{\mathcal{M}}_{0,0}(X, 3)$ has the expected dimension $4n - 3d - 1$.*

Proof. By the above proposition, every component M intersects the boundary in codimension one. Suppose there is a component M of too large a dimension, $\dim(M) > 4n - 3d - 1$. In particular, the general point of M parameterizes a non free curve. However, by the explicit computation in Proposition 2.4.26, the boundary locus consisting of curves where at least one of the components is not free is bounded by $3(n-3)+1$. In the restricted degree range, this cannot be a divisor instead of M, so that such an M cannot exist. □

Corollary 2.4.29. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $3d < n + 6$. Then the moduli space $M = \overline{\mathcal{M}}_{0,0}(X, 3)$ is irreducible and has the expected dimension.*

Proof. Every irreducible component of M has the expected dimension by the first proposition. It suffices to prove by Lemma 2.2.12 again, that any irreducible component $V \subset M$ contains a point parameterizing a reducible curve $C = C_1 \cup C_2 \cup C_3$ where each C_i is smooth and free as in the statement. For this, we use the fact that any component of M meets the boundary component Δ in a divisor. Applying the bend and break lemma again as in Proposition 2.4.26, we may assume that in codimension at most two, M contains graphs which are the union of "three lines" (possibly concurrent). If the general point in $V \cap \Delta$ is not given by free curves, then it must be entirely contained in a locus, which, by Lemma 2.4.27, cannot constitute an entire codimension two locus inside V. □

Corollary 2.4.30. *If $X = X(d, n)$ is a Fermat hypersurface with $3d < n + 6$, then $\overline{\mathcal{M}}_{0,0}(X, 3)$ is irreducible and has the expected dimension.*

2.4.6 High Degree Curves on Low Degree Fermat Hypersurfaces

In this section, we focus on curves of degree $\alpha > 3$. Given such an α, we work in the range $\alpha \cdot d < n$. Using similar methods to the preceding two sections, we prove the following theorem.

Theorem 2.4.31. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $\alpha \cdot d < n$. Every irreducible component $M \subset \overline{\mathcal{M}}_{0,0}(X, \alpha)$ has the expected dimension $E(\alpha) = \alpha(n + 1 - d) + n - 4$.*

Remark 2.4.32. This theorem says heuristically that whenever the most degenerate configurations of curves on X cannot contribute a component of too large a dimension, then every component has the expected dimension.

First we show:

Proposition 2.4.33. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $\alpha \cdot d < n$. There can be no irreducible component of M contained completely in the boundary.*

Proof. Suppose that τ is a degree α stable graph type with at least two vertices (that is, of boundary type). Suppose further that $V \subset M(\tau)$ is open inside an irreducible component of $\overline{\mathcal{M}}_{0,0}(X, \alpha)$. This component is clearly not the good component and so the maps at each vertex of the generic graph type of V cannot all be free. Because we are in the bend and break range ("well within" in fact), we may apply the bend and break lemma repeatedly to arrive eventually at a stable graph type where each vertex has degree at most 1. The goal is to construct a map $V \to X^r$ such that the fibers have positive dimension and such that in the general fiber, there is a degree greater than one component which must "move" while fixing two points. This will allow us to conclude that $M \backslash V$ consists of more degenerate boundary strata.

Let $f : C \to X$ be a generic map in V (so of type τ). We claim that we can mark α special points on C such that the following properties hold.

1. Each component C_i of C which has degree greater than one has two special points on it; whenever possible, the special points are always nodes of C rather than smooth points of C_i.
2. Every leaf (that is, component of C meeting the rest of C in a single point) has a special point in its smooth locus.
3. Every positive degree component of C has a special point on it.
4. There are at most $\alpha - 2$ degree 1 components containing a single special point.

To arrange this, suppose that there are m components of C with positive degree and that m' of them have degree greater than one. We simply choose 2 special points on each of these m' components, choosing nodes whenever possible (there may be many ways to do this). We then have $\alpha - 2m'$ special points left to distribute, but this number is at least $m' - m$, that is, the number of degree 1 components. Choose a special point in the smooth locus of each degree 1 leaf, then choose special points on the remaining degree 1 components as nodes. With any remaining special points to choose, place them at nodes of degree 1 components which only have one special point on them already.

Let $V' \subset M'$ be the locus where we have marked the non-nodal special points of τ as described above. We have an evaluation map $V' \to X^\alpha$ given by evaluating at all the special points. The dimension of V' is at least $\alpha(n+1-d)+(n-4)+k$, where k is the number of special points which are not nodes of τ. Then the fibers of this evaluation have dimension at least $2\alpha - \alpha \cdot d + n + (k-4)$. By assumption, this number is at least α (since k is at least 2). We wish to show that in the fibers of this map, the graph type becomes more reducible. Suppose that, in the fibers of this map, none of the components with degree greater than one move. We have then that the only possible deformations in fibers are given by the lines which can move through a single fixed point. However, each line with a single special point on it must meet the rest of the curve in the specified way. By arguing inductively beginning with components which do not move, such a line can only contribute one moduli to the fiber. As there are at most $(\alpha - 2)$ lines with only one marked point on them, we have that these arrangements do not account for all the moduli in the fiber. Thus at least one of the components of degree greater than one must move in the fiber. But when it does, we have a rational curve deforming while fixing two points, so it must break. This implies that the component M contains a more degenerate locus (in codimension one as usual).

We successively apply the same argument to the next locus until a graph type of α lines is reached (possibly with contracted components). Note that we are able to do this because of the assumption that $\alpha \cdot d < n$ and we may have to apply the argument $(\alpha - 1)$ times. By the following Lemma though, in the assumed degree range, this locus cannot have enough moduli so that V could have contributed an entire component, which finishes the proof of the proposition. Note that when we apply the Lemma, we know that the arrangement of lines reached by successively breaking inside M cannot be an arrangement of free lines - otherwise we would already be in the good component. □

Lemma 2.4.34. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $\alpha \cdot d < n$. Let τ be a stable degree α graph such that each vertex of τ has degree at most 1. Then the dimension of $M(\tau)$ is at most $E(\alpha) - \alpha$. Moreover, the locus of maps in $M(\tau)$ containing a non-free line has dimension bounded by $\alpha(n-3) + \alpha - 3$ which is strictly less than $E(\alpha) - \alpha$ by assumption.*

Proof. The proof will proceed by induction on α. The base case, $\alpha = 2$ follows from a brief analysis of the proof of Theorem 2.4.22. The case $\alpha = 3$ was treated explicitly above. Thus we suppose that the Lemma is true for all $3 \leq k < \alpha$. Choose a leaf v of the graph τ (necessarily of degree 1) and note that there is a fiber product diagram:

$$\begin{array}{ccc} M(\tau) & \longrightarrow & M(\tau') \\ \downarrow & & \downarrow{\scriptstyle ev'} \\ M(\tau_1) & \xrightarrow{ev} & X \end{array}$$

Here τ_1 is the stable graph which corresponds to a single degree one vertex and a single marked point on it. The evaluation map to X is denoted ev. The graph τ' is the one obtained by replacing v with a single marked point, and the map to X (ev') is the evaluation map at that marked point. By considering the stratification $X = \coprod X_e$, we may also stratify $M(\tau)$ according to which locus the node corresponding to v maps to. In other words, write $Ev : M(\tau) \to X$, the evaluation at the node v, and then we have $M(\tau) = \coprod M(\tau)_e = Ev^{-1}(X_e)$.

Let $U \subset M(\tau')$ be the open substack corresponding to maps where each line of τ' maps to a free chain of lines on X. By induction, U has at most the expected dimension $E(\alpha-1) - (\alpha-1) + 1$, and is equal to this if and only if there are no contracted components. Then (with the obvious notation) $\dim(U_e) \leq E(\alpha-1)-\alpha+\dim(X_e)-\dim(X)$.

Let $W_e \subset M(\tau_1)_e$ be the locus of free lines. This locus the expected dimension $(n+1-d)+\dim(X_e)$. Then we see that the locus $V_e \subset M(\tau)_e$ corresponding to chains all of which are free lines, has dimension $\dim(U_e)+\dim(W_e)-\dim(X_e) \leq E(\alpha-1)-\alpha+(n+1-d)+(\dim(X_e)-\dim(X))$. Again, there is equality if and only if there are no contracted components. In particular, we see that the locus of free chains of lines in $M(\tau)$ has dimension $E(\alpha)-\alpha$.

There are two further cases to consider. We must consider the locus inside $M(\tau)_e$ where we have glued a line to a chain where at least one line is not free, or a non-free line to a chain of free lines.

In the first case, we may at least say that, by induction, the dimension of the locus in $M(\tau')_e$ of chains which contain at least one non-free line have dimension $(\alpha-1)(n-3)+(\alpha-4)+1$. Then the space of lines in $M(\tau_1)_e$ has dimension $\dim(X_e)+n-d-1+e$, and so the dimension in $M(\tau)_e$ of glued chains of this type has dimension at most $\alpha(n-3)+(\alpha-3)$ as desired (because $e \leq d-2$).

In the second case, we know that $\dim(U_e) = E(\alpha-1)-\alpha+\dim(X_e)-\dim(X)$ and the space of non free lines containing a point of X_e has dimension $\dim(X_e)+n-d-1+e$. Then the dimension in $M(\tau)_e$ of glued chains of this form has dimension at most $E(\alpha-1)-\alpha+n-3$ which we can immediately check is less than $\alpha(n-3)+(\alpha-3)$. Since $M(\tau) = \coprod M(\tau)_e$, the dimension statements are immediate. □

Proof of Theorem 2.4.31. Suppose M is an irreducible component of $\overline{\mathcal{M}}_{0,0}(X,\alpha)$. By the bend and break Lemma, $M \cap \Delta \neq \emptyset$. By Proposition 2.4.33, we know that $M \cap \Delta$ is not contained in the boundary, and so is a divisor on M. By the same Proposition, we know that $\dim \Delta \leq e(n+1-d)+(n-4)-1$, because no component is contained inside the boundary. Since $\dim M \geq \alpha(n+1-d)+(n-4)$ and $M \cap \Delta$ is a divisor on M, we get that $\dim(M) = \alpha(n+1-d) + (n-4)$ as desired. □

Corollary 2.4.35. *Suppose that $X \subset \mathbb{P}^n$ is a well-stratified, smooth, degree d hypersurface with $\alpha \cdot d < n$. Then the moduli space $\overline{\mathcal{M}}_{0,0}(X, \alpha)$ is irreducible and has the expected dimension.*

Proof. By Lemma 2.4.17, we know $\overline{\mathcal{M}}_{0,0}(X, \alpha)$ has a unique component M parameterizing reducible chains of maps $C_1 \cup \ldots \cup C_e \to X$ where each C_i is free line.

Suppose that V is any component of $\overline{\mathcal{M}}_{0,0}(X, e)$ which is not the good component. By Theorem 2.4.31, we know that V has the expected dimension. Because V is not the good component, a general point of V cannot parameterize a free curve. By the complicated bend and break method used in Proposition 2.4.33, in codimension at most $\alpha - 1$, the component V contains graph types which correspond to chains of lines on X. But by Lemma 2.4.34, this locus does not have large enough dimension to contribute a codimension e locus to an entire component. This contradiction finishes the proof. □

Corollary 2.4.36. *If $X = X(d, n)$ is a Fermat hypersurface with $\alpha \cdot d < n$, then $\overline{\mathcal{M}}_{0,0}(X, \alpha)$ is irreducible and has the expected dimension.*

2.4.7 Remarks Concerning Conics on Fermat Hypersurfaces not Included in the Bend and Break Range

When the degree of a Fermat Hypersurface becomes large (meaning outside the bend and break range), much can "go wrong". We give all the examples of this phenomenon of which we know.

- We cannot hope that $\overline{\mathcal{M}}_{0,0}(X, 2)$ has the expected dimension or is irreducible for large values of d. Pick a point $x \in X_{n-1,1}$, this is one of finitely many conical points on X. The space of lines through x has dimension $n-3$ and so the space of reducible conics through x has dimension $2n-6$. This locus will be contained in a separate component (and have dimension larger than expected dimension) whenever $2n-6 \geq 3n-2d-2$; i.e. if $2d \geq n+4$. Of course, one could still hope that $\mathcal{M}_{0,0}(X,2)$ is irreducible and of the expected dimension.

- Even the latter cannot hold always. If $n = 2k+1$ (for simplicity), X contains a linear $\mathbb{P}^k$. The space of (smooth) conics in $\mathbb{P}^k$ has dimension $3k-1$. This locus must also be contained in its own component whenever $3k-1 > 3n-2d-2$. Reworking the algebra, this reads $2d > 3k+2 = (3/2)(n-1)+2$. It is possible that in this range, the moduli space is irreducible.

- The goal, for $d < n$ is to apply Theorem 2.3.6. According to that Theorem, we know that there is a unique component of the space of conics corresponding to conics passing through a general point of X. Unfortunately, it is difficult to control the open set guaranteed by the adjective, "general". The initial hope was that the non-clumping locus on X would be the desired open set. This has the nice property that we have already verified that all lines passing through a non-clumping point are smooth points in the moduli space.

  If it were true that (*): all conics through a non-clumping point were unobstructed, then we would be able to conclude there is a unique component of the space of conics passing through a non-clumping point. Assuming this for a moment, any other component would correspond to conics contained completely in the "clumped" loci on X, and in turn would correspond to conics contained in lower dimensional Fermat hypersurfaces. Via an inductive procedure, in some cases we would be able to conclude that these conics could not contribute an entire component and so the space would be irreducible of the expected dimension as desired. Unfortunately, this (*) is simply not true for the following reason.

  Suppose that $[q_0, \ldots, q_m]$ is an unclumped but obstructed conic on $X(d,m)$ for some $d > m$. This will mean that q_i is a degree two polynomial in two variables,

$\sum q_i^d = 0$, and that no subset of the q_i satisfy this relation. Such an obstructed conic will "propagate" to higher dimensional Fermat hypersurfaces in the following way. Let $(c_0, \ldots, c_k)$ be a tuple such that $\sum c_i^d = 1$ and no subset satisfies $\sum c_{i_j}^d = 0$. Then consider the following conic: $[c_1 q_0, c_2 q_0, \ldots, c_k q_0, q_1, \ldots q_m]$. A quick check shows that this is an unclumped conic on $X(d, n)$ where $n = m + k$.

Lemma 2.4.37. *If $[q_0, \ldots, q_m]$ is an unclumped, obstructed conic on $X(d, m)$, then $[c_1 q_0, c_2 q_0, \ldots, c_k q_0, q_1, \ldots, q_m]$ with the c_i as above is an obstructed conic on $X(d, m+k)$. In particular, we can find a k dimensional family of unclumped, obstructed conics on $X(d, m+k)$.*

Proof. Similar to the case of lines, the conic is obstructed if $H^0(\mathbb{P}^1, f^* T_{\mathbb{P}^m}) \to H^0(\mathbb{P}^1, f^* N_{X/\mathbb{P}^n}) = H^0(\mathbb{P}^1, \mathcal{O}(2d))$ fails to be surjective. The cokernel of this map is identified with the cokernel of the map given by $(p_0, \ldots, p_m) \to \sum q_i^{d-1} p_i$. Then we can check immediately that the map corresponding to the new conic on $X(d, n)$ is not surjective because the original one failed to be □

The question then becomes whether or not there are such unclumped conics when $d > m$. The answer is that there are some, but we cannot effectively bound their dimension. Thanks to beautiful work of Bruce Resnick [Res], we can write down the following examples (for more see the cited location):

Example 2.4.38. Let $q_0 = x^2 + \sqrt{2}xy - y^2$, $q_1 = ix^2 - \sqrt{2}xy + iy^2$, $q_2 = -x^2 + \sqrt{2}xy + y^2$, and $q_3 = -ix^2 - \sqrt{2}xy - iy^2$. Then $\sum q_i^5 = 0$ and this conic is unclumped. Thus, we get an obstructed conic on $X(5, 3)$, which will propagate as above. Resnick shows that $\sum q_i^d = 0$ for $d = \{1, 2, 5\}$, but for no other values of d. In fact, these are the only four quadratic forms up to scaling that satisfy this relation when $d = 5$, (see [Res] Theorem 3.5). (Resnick's results are for the most part concerned with what values of d can occur for r homogeneous polynomials of degree v in w variables).

Example 2.4.39. Let $t = 2c + 1$ be odd, and fix ζ to be a primitive t-th root of unity. Let $q_j = \zeta^j x^2 + \zeta^{-j} y^2$ for $0 \leq j \leq t - 1$ and $q_t = xy$. Up to scalar multiples of the q_i (which depend on d), these polynomials satisfy $\sum q_i^d = 0$ for $d < t$ odd and $d < 2t$ even.

Example 2.4.40. Let t be arbitrary now and again fix ζ to be a primitive t-th root of unity. Fix an integer s so that $2 \leq s \leq t - 1$. Let $q_j = \zeta^j x^2 + \alpha xy \zeta^{-j} y^2$ for $0 \leq j \leq t - 1$ and $q_t = xy$ where $\alpha \in \mathbb{C}$ is a complex parameter. Up to scalar multiples of the q_i (which depend on d) and a careful choice of α, these polynomials satisfy $\sum q_i^d = 0$ for $d < t$ and $d = t + s$. Again, for details consult the location cited. This example gives particularly "bad" examples; that is, unclumped conics on relatively large degree Fermats. In other words, there is an unclumped conic on $X(2t-1, t)$. When $t = 4$ for example, there is an unclumped conics on $X(7, 4)$. For $k > 0$, this will contribute a k dimensional family of conics on $X(7, k + 4)$, but note that this will not be enough to contribute an entire component in the interesting range.

In other words, it is not known that $\mathcal{M}_{0,0}(X, 2)$ is *not* irreducible for any Fermat Hypersurface in the already restricted range $2(n+1) < 4d < 3n$. The examples show that there can be obstructed conics through unclumped points. We hope to show that such conics cannot contribute an entire component (to the space of conics through these points). If so, we would be able to conclude that, in an even further restricted range, the space of conics not contained in the boundary is irreducible. Nevertheless, this seems to be a difficult problem given the sporadic and semi-arithmetic nature of the examples Resnick has produced (and indeed, he makes no claims to have found "all" such examples).

- Though there can be obstructed smooth conics through points on $X_{0,1}$ (here $X = X(d, n)$), if we can bound the dimension of these conics, it is possible to apply a version of Theorem 2.3.6. Thus we can ask the questions (**) Is there a bound on the dimension of obstructed conics through a point of $X_{0,1}$? Also we can ask (***) When $d \geq n$, can we bound the dimension of obstructed, unclumped conics on X? An effective bound would allow us to push the result of Corollary 2.4.23 slightly past the bend and break range.

2.5 Appendix: Some Representable Functors

2.5.1 Well Known Facts

Suppose that S is a scheme and E is a rank r vector bundle on S. In other words, E is a locally free $\mathcal{O}_S$-module of rank r. Define a functor Gr_n from S-schemes to sets

$$Gr_n : \text{S-}Sch \to Set$$

$$U \mapsto \{\text{Set of rank n quotient bundles } H \text{ of } E_U\}.$$

Here and in the following, E_T denotes the pullback of E to T by the structure map $T \to S$. We have the following well known Proposition.

Proposition 2.5.1. *The functor $\mathcal{F}$ is represented by a smooth projective S-scheme, $Grass_n(E)$ which comes with a universal quotient bundle Q of $E_{Grass_n(E)}$. In other words, morphisms $U \to Grass_n(E)$ are in bijective correspondence with rank n quotients G of E_U through the relationship $f \mapsto f^*Q$. The universal n-plane is $\mathbb{P}(Q) \subset \mathbb{P}(E) \times G$.*

Proof. See for example, Section 1.7 of [Kol96]. □

Suppose now that $P = \mathbb{P}(E)$ is a projective bundle over S of rank m. Define another functor $\mathcal{G}$ from the category of S-schemes to sets by:

$$\mathcal{G} : \text{S-}Sch \to Set$$

$$U \mapsto \begin{cases} \text{Closed subschemes } V \subset U \times_S \mathbb{P}(E) \\ \text{such that V is a degree } d \text{ hypersurface in each fiber over } U. \end{cases}$$

More precisely (though equivalently), setting $P_d(t) = \binom{m+t}{m} - \binom{m+t-d}{m}$, this should be reformulated as

$$U \mapsto \begin{cases} \text{Closed subschemes of } \mathbb{P}(E) \times_S U \\ \text{which are proper and flat over } U \\ \text{and have Hilbert polynomial } P(t). \end{cases}$$

This is a Hilbert scheme. An element on the right side will be called a flat family of degree d hypersurfaces of $\mathbb{P}(E)$. Denoting $H = \mathbb{P}(\mathrm{Sym}^d E^*)$, we have:

Lemma 2.5.2. *The above Hilbert functor $\mathcal{G}$ is represented by the scheme H. That is $H = Hilb_{P_d(t)}(\mathbb{P}(E)/S)$. In particular it is smooth.*

Proof. See [Kol96] 1.4.1.4. □

Fix two polynomials $Q_1(t)$ and $Q_2(t)$ and define another functor:

$$\mathcal{G}' : \text{S-}Sch \to Set$$

$$Y \mapsto \begin{cases} \text{Pairs of closed subschemes } V_1 \subset V_2 \subset \mathbb{P}(E) \times_S Y \\ \text{which are proper and flat over } Y \\ \text{such that } V_1 \text{ is flat over } Y \text{ and has Hilbert polynomial } Q_1(t) \\ \text{and } V_2 \text{ is a flat over } Y \text{ with Hilbert polynomial } Q_2(t)\ . \end{cases}$$

The Hilbert scheme which parameterizes subschemes of $\mathbb{P}(E) \times_S Y$ flat over Y with Hilbert polynomial $Q_i(t)$ will be denoted H_i and the universal family will be denoted U_i (for $i = 1, 2$).

Lemma 2.5.3. *The functor $\mathcal{G}'$ is represented by a closed subscheme $Z \subset H_1 \times H_2$.*

Proof. See [Ser06], 4.5.1. □

Remark 2.5.4. The generalization to longer chains is clear.

Remark 2.5.5. In the case where both the varieties $V_1 \subset V_2$ in the preceding Lemma, are families of linear spaces, the result is called a (partial) flag variety. It is well known to be smooth (in fact, it is a projective bundle over the Grassmannian).

2.5.2 Well Known Extensions of Well Known Facts

In the special case where V_1 is a linear subspace and V_2 is a flat family of hypersurfaces, something even more can be said:

Lemma 2.5.6. *With the notation as above, suppose that $Q_2(t) = P_d(T)$ and $Q_1(t) = \binom{n+t}{n}$. In the following, denote $G = Grass(n, E)$. Then the corresponding functor $\mathcal{G}'$, which is representable by the previous Lemma, is in fact represented by $Z = \mathbb{P}(K^*)$. Here K is the kernel of the surjection $Sym^d(V)_G \to Sym^d(Q)$. In particular, Z is smooth.*

Proof. With the notation as in the above, we have that $H_1 = H$ and $H_2 = G$. The ideal sheaf of U_1 is $(p_2^*\mathcal{O}_H(-1))(-d) \subset \mathcal{O}_{\mathbb{P}(E)\times H}$ from the proof of Lemma 2.5.2. In the notation of that Lemma, one checks that $N = d$ works and the map u in Lemma 2.5.3, is the composition:

$$u : p_*(\mathcal{O}_H(-1)_{\mathbb{P}(E)\times G\times H}) \to \mathrm{Sym}^d(E) \otimes \mathcal{O}_{G\times H} \to \mathrm{Sym}^d(Q) \otimes \mathcal{O}_{G\times H}$$

And Z is the scheme of zeroes of u. This map is simply:

$$\mathcal{O}_H(-1)_{G\times H} \to^{a*} \mathrm{Sym}^d(V)_{G\times G} \to \mathrm{Sym}^d(Q_H)_{G\times H}$$

Then the ideal of Z is equal to the image of the composition of the dual of u:

$$\mathrm{Sym}^d(Q^*)_{H\times G}(-1) \to \mathrm{Sym}^d(V^*)_{H\times G}(-1) \to \mathcal{O}_{H\times G}.$$

From this sequence, observe that Z and $\mathbb{P}(K^*)$ have the same ideal sheaf, and so are equal. □

Lemma 2.5.7. *Let $S = Spec\ (k)$. Define another functor:*

$$\mathcal{C} : k - Sch \to Set$$

$$U \mapsto \begin{cases} \textit{Pairs of subschemes } V_1 \subset V_2 \subset \mathbb{P}^n \times_k U \\ \textit{which are proper and flat over } U \\ \textit{such that } V_1 \textit{ is flat over } U \textit{ and has Hilbert polynomial } 2t+1 \\ \textit{and } V_2 \textit{ is a flat over } U \textit{ with Hilbert polynomial } \binom{n+2}{2}. \end{cases}$$

That is, V_1 is a conic in $\mathbb{P}^n$ and V_2 is the plane which contains it. Actually the data of the plane is redundant. Then we have $Hilb_{2t+1}(\mathbb{P}^n) = \mathbb{P}(Sym^2(Q^*))$ where Q is the universal rank 3 bundle over $Grass(3, n+1)$ and this scheme represents the functor above.

Remark 2.5.8. Pointwise this is already clear. That is if $n = 2$, the result is Lemma 2.5.2. This result globalizes that one. The proof is almost identical. See Remark 2.2.7.

Lemma 2.5.9. *Again take $S = Spec\ (k)$. Define another functor:*

$$\mathcal{D} : k - Sch \to Set$$

$$U \mapsto \begin{cases} \textit{Tuples of subschemes } V_0 \subset V_1 \subset V_2 \subset \mathbb{P}^n \times_k U \\ \textit{which are proper and flat over } U \\ \textit{such that: } V_0 \textit{ is flat over } U \textit{ with constant Hilbert polynomial } 1, \\ V_1 \textit{ is flat over } U \textit{ and has Hilbert polynomial } 2t+1, \\ \textit{and } V_2 \textit{ is a flat over } U \textit{ with Hilbert polynomial } \binom{t+2}{2}. \end{cases}$$

That is, fiberwise this says V_0 is a point on V_1 which is a conic in $\mathbb{P}^n$ and V_2 is the plane which contains it. Again the data of the plane is redundant. The functor $\mathcal{D}$ is represented by a closed subscheme $Z \subset \mathbb{P}(Sym^2(Q^*)) \times_G Fl(1,3,n+1) \cong \mathbb{P}(Sym^2(Q^*)) \times \mathbb{P}^n$. Here $F = Fl(1,3,n+1)$ is the flag variety which parameterizes point contained in plane contained in $\mathbb{P}^n$. In particular it is smooth. In fact, Z can be identified canonically with $\mathbb{P}(K^*)$ where K is the kernel of the surjection $Sym^2(Q)_F \to Sym^2(Q')$. Here Q is the universal rank 3 bundle on the Grassmannian and Q' is the universal rank 1 bundle on the Flag variety.

Remark 2.5.10. This is simply a globalization of Lemma 2.5.6. Note that the same proof will work for linear spaces contained in degree d hypersurfaces in $\mathbb{P}^r \subset \mathbb{P}^n$. This special case is simply the one applied in this paper. The proof follows the Lemma 2.5.6 in the same way that Lemma 2.5.7 globalizes Lemma 2.5.2 and so will be left to the reader.

Lemma 2.5.11. *Again take $S = Spec\ (k)$. Define another functor:*

$$\mathcal{E} : k - Sch \to Set$$

$$U \mapsto \begin{cases} \textit{Tuples of subschemes } V_0 \subset V_1 \subset H \subset \mathbb{P}^n \times_k U \\ \textit{which are proper and flat over } U \\ \textit{such that: } V_0 \textit{ is flat over } U \textit{ with constant Hilbert polynomial } 1, \\ V_1 \textit{ is flat over } U \textit{ and has Hilbert polynomial } 2t+1, \\ H \textit{ is flat over } U \textit{ with Hilbert polynomial } P_d(t). \end{cases}$$

The functor $\mathcal{E}$ is represented by a smooth scheme.

Proof. Suppose that H is the scheme which represents $\mathcal{E}$. By Yoneda's Lemma, there is a natural map from H to $\mathbb{P}(K^*)$ (notation as in Lemma 2.5.9) corresponding to the transformation of functors which forgets the data of the degree d hypersurface. We'll compute the fiber over a point which corresponds to a $x \in C$ where C is a conic in $\mathbb{P}^n$. Since C is a complete intersection, we may choose coordinates so that its ideal has the form $(F(x_0, x_1, x_2), x_3, \ldots, x_n)$ where F is homogenous of degree 2. To say that a degree d hypersurface V contains C is to say that the defining equation for V can be written $B \cdot F + A_3 \cdot x_3 + \ldots + A_n \cdot x_n$ where B is homogeneous of degree $d-2$ and A_i is homogeneous of degree $d-1$. As these are the only conditions, we note that the space of hypersurfaces containing C is itself a projective space. As H maps to $\mathbb{P}^{K^*}$ with projective space fibers, it itself is smooth. In fact, as the Proof suggests, it is another projective bundle as in the previous Lemmas. We leave the details to the reader. □

Chapter 3

Complete Families of Rational Curves

3.1 Introduction and Main Theorem

In this chapter we prove the following theorem.

Theorem 3.1.1. *If X is the base of a complete family of linearly non-degenerate degree $e \geq 3$ curves in $\mathbb{P}^n$ with maximal moduli, then* $\dim X \leq n-1$. *If X is the base of such a complete family of non-degenerate degree 2 curves in $\mathbb{P}^n$, then* $\dim X \leq n$.

We first introduce the notation used above. Since $H_2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$, we will use the standard notation $e = e \cdot [line]$ for an element in the Chow ring.

The Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ parameterizes isomorphism classes of pairs (C, f) where C is a proper, connected, at-worst-nodal, arithmetic genus 0 curve, and f is a stable morphism $f : C \to \mathbb{P}^n$ such that $f_*[C] = e[line] \in H_2(Y, \mathbb{Z})$. This is a Deligne-Mumford stack whose coarse moduli space, $\overline{M}_{0,0}(\mathbb{P}^n, e)$, is projective; see, for example, [FP97].

Let $\mathcal{U} \subset \mathcal{M}_{0,0}(\mathbb{P}^n, e)$ be the open substack parameterizing maps $f : \mathbb{P}^1 \to \mathbb{P}^n$ which are isomorphisms onto their image such that the span of each image is a $\mathbb{P}^e$ (in particular, $e \leq n$). Note that no point in $\mathcal{U}$ admits automorphisms and $\mathcal{U}$ is isomorphic to an open subscheme in the appropriate Hilbert and Chow schemes. In particular, $\mathcal{U}$ is a quasi-projective variety over $\mathbf{C}$.

Definition 3.1.2. Suppose X and $\mathcal{C}$ are proper varieties and $\pi : \mathcal{C} \to X$ is a proper surjective morphism. We will consider diagrams of the form:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi} & & \\ X & & \end{array}$$

In the case where each fiber of π is a $\mathbb{P}^1$ and f, restricted to each fiber, corresponds to a point in $\mathcal{U}$, we will call the diagram a *complete family of linearly non-degenerate degree e curves*. Such a family induces a map $\alpha : X \to \mathcal{U}$. If the map is generically finite, that is, if $\dim X = \dim \alpha(X)$, we will call the diagram a *family of maximal moduli*. We will refer to X as the base of the family. Note that $\mathcal{C}$ is the pullback of the universal curve over $\mathcal{U}$, and so we will refer to the map f above as ev. The notation $(\mathcal{C}, X, ev, \pi, n, e)$ will denote a complete family of linearly non-degenerate degree e curves in $\mathbb{P}^n$, as above.

One can ask for the largest number of moduli of such a family, that is, the dimension of the base X of a family of maximal moduli. This is also the largest dimension of a proper subvariety of $\mathcal{U}$. A simple argument shows that the number of moduli of a complete, linearly non-degenerate family of degree e curves in $\mathbb{P}^e$ is in fact 0. The bend and break lemma ([Deb01], Proposition 3.2) gives a strict upper bound on the dimension of complete subvarieties of $X \subset \mathcal{M}_{0,0}(\mathbb{P}^n, e)$, namely $2n - 2$. When the genus of the curves in question is positive, M. Chang and Z. Ran have obtained a similar dimension bound. They prove that if Λ is a closed non-degenerate family of positive genus immersed curves in $\mathbb{P}^n$, then $\dim \Lambda \leq n - 2$ [CR94]. Theorem 3.1.1 addresses the situation where the curves are rational and required to be linearly non-degenerate.

3.1.1 Discussion

Question 3.1.3. What is the best possible result along the lines of Theorem 3.1.1? For any value $e > 1$, there are certainly examples of complete, linearly non-degenerate r dimensional families in $\mathbb{P}^{r+e}$. One way to construct such families is to take the Segre embedding:

$$\mathbb{P}^1 \times \mathbb{P}^r \xrightarrow{(e,1)} \mathbb{P}^N ,$$

where $N = (e+1) \cdot (r+1) - 1$. Project from a point $p \in \mathbb{P}^N$ not in any $\mathbb{P}^e$ spanned by the image of $\mathbb{P}^1 \times \{q\}$ for every point $q \in \mathbb{P}^r$. This gives an r dimensional family of non-degenerate degree e curves in $\mathbb{P}^{N-1}$. Continue projecting in this fashion. We can always find a point p to project from as long as $N > r + e$. So we arrive at an r dimensional family of degree e curves in $\mathbb{P}^{r+e}$.

Question 3.1.4. Does there exist a complete family with maximal moduli of degree e non-degenerate rational curves in $\mathbb{P}^m$ whose base has dimension greater than $m-e$? Does there exist a complete 2 parameter family of smooth conics in $\mathbb{P}^3$? Does there exist a complete 2 parameter family of smooth cubics in $\mathbb{P}^4$?

Question 3.1.5. Does there exist a similar bound if the condition of being linearly non-degenerate is removed?

Question 3.1.6. If the variety swept out by these curves is required to be contained in a smooth hypersurface, does the bound improve? In fact, this question was the original motivation for this work.

3.1.2 Outline of Proof

We give a brief outline of the proof:

Let $e > 2$ and fix X to be the base of a complete family of linearly non-degenerate degree e curves in $\mathbb{P}^n$ with maximal moduli. Assume that $\dim X \geq n$. Using results from section 2, we will reduce the situation to the case where the universal curve $\mathcal{C}$ over X is the projectivization of a rank 2 vector bundle $\mathcal{E}$ on X. The situation will then be further reduced to the case where we have the following maps:

Diagram 3.1.7.

$$\begin{array}{ccc} \mathcal{C} = \mathbb{P}(\mathcal{E}) & \xrightarrow{ev} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{\phi} & \mathrm{Grass}(e+1, n+1) \end{array}$$

Here ϕ is the generically finite map which associates to each map the e-plane it spans. Using the universal curve $\mathcal{C}$, we will form the following commutative diagram.

Diagram 3.1.8.

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{\gamma} & \mathrm{Fl}(1,\ldots,e+1;n+1) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ X & \xrightarrow{\phi} & \mathrm{Grass}(e+1, n+1) \end{array}$$

The map γ associates to point of the universal curve (that is, a map $f : \mathbb{P}^1 \to \mathbb{P}^n$ and a marked point $p \in \mathbb{P}^1$), the sequence of osculating k-planes to $f(\mathbb{P}^1)$ at $f(p)$. The map between the flag variety and the Grassmannian is the obvious projection.

In Section 3, we will construct an ample line bundle $\mathcal{L}$ on $\mathrm{Fl}(1,\ldots,e+1;n+1)$ and give a cohomological argument to show that $c_1(\mathcal{L})^{n+1}$ pulls back to 0 by γ. This will allow us to conclude the Theorem when $e \geq 3$. In the case $e = 2$, a different computation is needed, but similar ideas apply.

Notation 3.1.9. Fix the ambient $\mathbb{P}^n$. We will denote by $\mathrm{Fl}(a_1,\ldots,a_k;n+1)$ with $a_1 < a_2 < \ldots < a_k$ the flag variety parameterizing vector quotient spaces $\mathbb{C}^{n+1} \to A_k \to A_{k-1} \to \ldots \to A_1$ (all arrows surjective) such that $\dim(A_i) = a_i$. In the special case $\mathrm{Fl}(a;n+1)$ we will write $\mathrm{Grass}(a, n+1)$, the Grassmannian of a dimensional quotients of $\mathbb{C}^{n+1}$. We will follow the convention of [Gro61] and denote the set of hyperplanes in the fibers of $\mathcal{E}$ by $\mathbb{P}(\mathcal{E})$.

3.2 Reductions

We first prove some general lemmas. We will apply these to the case of a complete family of linearly non-degenerate degree e curves in the following section.

Proposition 3.2.1. *Suppose that $\pi : \mathcal{C} \to X$ is a proper, surjective morphism of complete varieties where each fiber of π is abstractly isomorphic to $\mathbb{P}^1$. Then there exists a surjective, generically finite map $f : X' \to X$ such that in the fiber square*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & \mathcal{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

π' realizes $\mathcal{C}'$ as the projectivization of a rank 2 vector bundle $\mathcal{E}$ on X'. That is, $\mathcal{C}' = \mathbb{P}(\mathcal{E})$.

Proof. Let $i : \nu \to X$ denote the inclusion of the generic point into X. Let $\mathcal{C}_\nu$ be the generic fiber. That is, there is a fibered square:

$$\begin{array}{ccc} \mathcal{C}_\nu & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \nu & \xrightarrow{i} & X \end{array}$$

Let y be a closed point of $\mathcal{C}_\nu$, and let $X' = \overline{y}$ in $\mathcal{C}$. Note that X' is irreducible, proper, and $\pi(X') = X$. The restricted map $f = \pi|_{X'} : X' \to X$ is proper, and has only one point in the generic fiber, so is generically finite.

Consider then, the fibered square which defines $\mathcal{C}'$.

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & \mathcal{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

Note that X' maps to $\mathcal{C}$ by construction, so (by the universal property of fiber products) there is a section of π'. That is, there is a map $\sigma : X' \to \mathcal{C}'$ such that $\pi' \circ \sigma = id_{X'}$.

The existence of the section allows us to conclude that $\mathcal{C}' \cong \mathbb{P}(\mathcal{E})$ by a standard argument. For example, the argument used in [Har77] V.2 Proposition 2.2 applies word for word. □

In the case where a projective bundle over X admits a map to $\mathbb{P}^n$, we are able to adjust the bundle (using another finite base change) to control the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Proposition 3.2.2. *Suppose that $\mathcal{E}$ is a rank 2 vector bundle on a variety X and let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the natural map. Suppose in addition that $\mathbb{P}(\mathcal{E})$ admits a map to $\mathbb{P}^n$ which is degree e on each fiber. Then there exists a finite, surjective map $f : X' \to X$ such that in the fiber product diagram*

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{E}_{X'}) & \xrightarrow{f'} & \mathbb{P}(\mathcal{E}) & \xrightarrow{ev} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \\ X' & \xrightarrow{f} & X & & \end{array}$$

we have that $\pi'_* ev'^* \mathcal{O}(1) = Sym^e(\mathcal{E}_{X'})$ where $ev' = ev \circ f'$.

Proof. First we remark that $ev^*\mathcal{O}(1)$ is a line bundle that is degree e on each fiber of π. Thus $ev^*\mathcal{O}(1) = \mathcal{O}(e) \otimes \pi^*(\mathcal{N})$ for some line bundle $\mathcal{N}$ on X. This follows by the description of the Picard group of a projective bundle [Har77]. Then $\pi_* ev^*\mathcal{O}(1) = \mathrm{Sym}^e(\mathcal{E}) \otimes \mathcal{N}$. If there is a line bundle $\mathcal{L}$ on X such that $\mathcal{L}^e \simeq \mathcal{N}$ then it is an easy exercise to show that $\mathrm{Sym}^e(\mathcal{E}) \otimes \mathcal{N} \simeq \mathrm{Sym}^e(\mathcal{E} \otimes \mathcal{L})$ and it is well known [Har77] that $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$. Finally, Lemma 2.1 of [BG71] implies that there exists a finite, surjective map $\tau : X' \to X$ and a line bundle $\mathcal{L}$ on X' such that $\mathcal{L}^{\otimes e} \simeq \tau^*\mathcal{N}$. □

3.3 Proof of Theorem 3.1.1 in High Degrees

Before looking at the general case, we first prove a stronger (though well-known) result than the main theorem would imply when $n = e$:

Proposition 3.3.1. *If* $n = e$*, and* $(\mathcal{C}, X, ev, \pi, n, n)$ *is a family of maximal moduli as in Definition 3.1.2, then* $\dim X = 0$*. That is, there is no complete curve contained in* $\mathcal{U} \subset \mathcal{M}_{0,0}(\mathbb{P}^n, n)$*.*

Proof. The space of rational normal curves in projective space is well-known to be $\mathbf{PGL}_{n+1}/\mathbf{PGL}_2$. By Matsushima's criterion, the quotient of a reductive affine group scheme by a reductive subgroup is affine [BB63]. As no affine variety contains a positive dimensional complete subvariety, the proposition follows. Note that there has been recent success in determining the effective cone of this moduli space, (see [CHS08]). □

We are now ready to prove the main theorem for $e > 2$.

Proof of Theorem 3.1.1. Fix $(\mathcal{C}, X, ev, \pi, n, e)$ to be a family of maximal moduli as in Definition 3.1.2 with $2 < e < n$. By way of contradiction, assume that $\dim X \geq n$. By taking an irreducible proper subvariety of X, and restricting the family, we may assume that $\dim X = n$.

For any point $x \in X$, denote by $\phi(x)$ the linear e-plane spanned by the image of the map corresponding to x. That is, $\phi(x) = \mathrm{Span}(ev(\pi^{-1}(x)))$. The map $\phi : X \to \mathrm{Grass}(e + 1, n + 1)$ is well-defined because each curve corresponding to a point in X is linearly non-degenerate. This morphism factors through $\alpha : X \to \mathcal{U}$ (notation as in Definition 3.1.2) and so is generically finite by Proposition 3.3.1.

Applying Proposition 3.2.1 and then Proposition 3.2.2 we may assume that there is a generically finite, surjective map $f : X' \to X$ such that we have a fiber product diagram:

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{f'} & \mathcal{C} & \xrightarrow{ev} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \\ X' & \xrightarrow{f} & X & & \end{array}$$

where $\mathcal{E}$ is a rank two vector bundle on X' and $\pi'_*(f' \circ ev)^*\mathcal{O}(1) = \mathrm{Sym}^e(\mathcal{E})$. The collection $(\mathbb{P}(\mathcal{E}), X', f' \circ ev, \pi', n, e)$ is still a family of linearly non-degenerate degree e curves with maximal moduli, and dim $X' = n$. The composed map $f \circ \phi$ is a generically finite map from X' to the Grassmannian. To simplify notation, we rename this new family $(\mathbb{P}(\mathcal{E}), X, ev, \pi, n, e)$.

We construct the universal section. Let $Y = \mathbb{P}(\mathcal{E})$ and consider the fiber product diagram:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_{\mathrm{Y}}) & \longrightarrow & \mathbb{P}(\mathcal{E}) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y & \longrightarrow & X \end{array}$$

We have a natural section $\sigma : Y \to \mathbb{P}(\mathcal{E}_{\mathrm{Y}})$ given by the diagonal map. This section corresponds to a surjection $\mathcal{E}_Y \to \mathcal{L}$ where $\mathcal{L} = \sigma^*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathrm{Y}})}(1)$. Let $\mathcal{L}_1 = \mathcal{L}$ and let $\mathcal{L}_2$ be the line bundle such that

$$0 \to \mathcal{L}_2 \to \mathcal{E}_{\mathrm{Y}} \to \mathcal{L}_1 \to 0.$$

This sequence induces a filtration on $\mathrm{Sym}^e(\mathcal{E})$:

$$\mathrm{Sym}^e(\mathcal{E}_{\mathrm{Y}}) = F^0 \supset F^1 \supset \dots F^e \supset F^{e+1} = 0,$$

such that $F^p/F^{p+1} \simeq \mathcal{L}_2^p \otimes \mathcal{L}_1^{e-p}$ ([Har77] II.5). Note that Y corresponds to curves parameterized by X and a point on that curve. We have a natural map from $Y \to$ $\mathrm{Grass}((, e) + 1, n + 1)$ by composition, and the data of the F^ps induce a map from $\gamma : Y \to \mathrm{Fl}(1, \dots, e + 1; n + 1)$. Informally, the information of "the point" on the curve induces a linear filtration of the $\mathbb{P}^e$ spanned by the curve. The linear spaces between the point and the entire $\mathbb{P}^e$ are the osculating k-planes, $k = 1, \dots, e$. We can see this by working locally where the map is defined by $t \to (1, t, t^2, \dots, t^e, 0, \dots, 0)$. All the maps in Diagram 3.1.7 and Diagram 3.1.8 have been constructed.

On $\mathrm{Fl}(1, \dots, e + 1; n + 1)$ we have the natural sequence of universal quotient bundles

$$\mathcal{O}^{n+1} \to \mathcal{Q}_{e+1} \to \dots \to \mathcal{Q}_1 \to 0.$$

Recall the previously defined map: $\gamma : \mathbb{P}(\mathcal{E}) \to \mathrm{Fl}(1, \ldots, e+1; n+1)$. The proof hinges on the following construction.

Proposition 3.3.2. *There exists an ample line bundle on the flag variety* $Fl(1, \ldots, e+1; n+1)$ *whose first Chern class* $D \in H^2(Fl, \mathbb{Z})$ *satisfies* $\gamma^*(D^{n+1}) = 0$.

Assuming the proposition for the moment, we always have that $D^{\dim Y} \cdot \gamma(Y) > 0$ because γ is generically finite and D is ample (see Lemma 3.3.3 below). Since $\dim Y = n+1$, we can rewrite this as $(D|_{\gamma(Y)})^{n+1} > 0$. Applying Lemma 3.3.3, we see that $\gamma^*(D^{n+1}) > 0$ which contradicts Proposition 3.3.2 above. Hence we can conclude that $\dim \mathbb{P}(\mathcal{E}) < n+1$ and so $\dim X < n$. The theorem follows. □

It remains to prove Proposition 3.3.2.

Proof. For $p = 0, \ldots, e$ let $x_p = c_1(\ker \mathcal{Q}_{p+1} \to \mathcal{Q}_p)$. By construction of γ, we have

$$\gamma^* x_p = c_1(F_p/F_{p+1}) = pc_1(\mathcal{L}_2) + (e-p)c_1(\mathcal{L}_1).$$

Consider the projection map

$$pr : \mathrm{Fl}(1, \ldots, n; n+1) \to \mathrm{Fl}(1, \ldots, e+1; n+1)$$

and the injective map it induces on cohomology (always with rational coefficients):

$$pr^* : H^*(\mathrm{Fl}(1, \ldots, e+1; n+1)) \to H^*(\mathrm{Fl}(1, \ldots, n; n+1))$$

It is well known that $H^*(\mathrm{Fl}(1, \ldots, n; n+1)) = \mathbb{Q}[x_0, \ldots, x_n]/\mathcal{I}$ where $\mathcal{I}$ is the ideal of symmetric polynomials in the x_is [Ful98]. By a slight abuse of notation, denote $pr^*(x_i)$ again by x_i.

In the cohomology ring of full flags, we claim that $x_p^{n+1} = 0$ for each p. To see this, note that in this ring, the following identity holds:

$$T^{n+1} = (T - x_1) \cdot (T - x_2) \cdot \cdots \cdot (T - x_n)$$

since on the right hand side each coefficient of T^k with $k < n+1$ is a symmetric polynomial. Taking $T = x_p$ proves the identity. Then since pr^* is injective, we must also have that $x_p^{n+1} = 0$ in the cohomology ring of partial flags, so $(pc_1(\mathcal{L}_2) + (e-p)c_1(\mathcal{L}_1))^{n+1} = 0$ for each $p = 0, \ldots, e$.

To simplify notation, in what follows we write $z = c_1(\mathcal{L}_1)$ and $y = c_1(\mathcal{L}_2)$. For relevant facts about the cohomology ring of the flag variety, see Appendix 3.5. For any $D = \lambda_0 x_0 + \ldots + \lambda_e x_e$ we have:

$$\begin{aligned}\gamma^*(D) &= \gamma^*(\lambda_0 \cdot x_0 + \ldots + \lambda_e \cdot x_e) \\ &= \sum_{p=0}^{e} \lambda_p \cdot (py + (e-p)z) \\ &= (\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + e\lambda_e)y + (e\lambda_0 + (e-1)\lambda_1 + \ldots + \lambda_{e-1})z\end{aligned}$$

Let A be the coefficient of y and B the coefficient of z. If we can choose $\lambda_0, \ldots, \lambda_e$ so that $\gamma^*(D) = Ay + Bz$ is a $\mathbb{Q}$ multiple of one of the $(py + (e-p)z)$ then for some rational number m we have:

$$\begin{aligned}\gamma^*(D^{n+1}) &= (m(py + (e-p)z))^{n+1} \\ &= 0\end{aligned}$$

It remains to show that D can be chosen with these properties. See Appendix 3.5 for a description of the ample cone of the flag variety. To arrange this choice of D, set

$$\lambda_0 = \frac{1}{e}, \lambda_1 = \frac{1}{e-1}, \ldots, \lambda_i = \frac{1}{e-i}, \ldots, \lambda_{e-1} = 1.$$

Then obviously we have that $B = e$. We will prove that λ_e can be chosen to satisfy:

$$\lambda_e > \lambda_{e-1} = 1 \text{ and } \frac{A}{B} = e - 1$$

This is equivalent to:

$$e\lambda_e = e(e-1) - \sum_{i=1}^{e-1} \frac{i}{e-i}$$
$$\lambda_e = (e-1) - \sum_{i=1}^{e-1} \frac{i}{e(e-i)}$$

Using partial fractions and simplifying, we get

$$\lambda_e = e - \sum_{i=0}^{e-1} \frac{1}{e-i}$$

It is then easy to show this is strictly larger than 1 as long as $e \geq 3$. Therefore, D can be chosen with the required positivity property and the proof is complete when $e \geq 3$. A simple calculation shows this method cannot work when $e = 2$. To show a slightly weaker result in that case, we need another method. □

We include the statement of the projection formula used in the proof above:

Lemma 3.3.3. *[Deb01] Let $\pi : V \to W$ be a surjective morphism between proper varieties. Let $D_1, \ldots, D_r$ be Cartier divisors on W with $r \geq \dim(V)$. Then the projection formula holds, i.e.:*

$$\pi^* D_1 \cdots \pi^* D_r = \deg(\pi)(D_1 \cdots D_r).$$

3.4 The Proof for Conics

In this section we prove a bound for families of smooth conics one dimension weaker than for a family of higher degree curves. Note that for conics (and cubics), being linearly non-degenerate is equivalent to having smooth images.

Theorem 3.4.1. *If* $(\mathcal{C}, X, ev, \pi, 2, n)$ *is a family of linearly non-degenerate conics in* $\mathbb{P}^n$ *with maximal moduli, then* $\dim X \leq n$.

Proof. Exactly as in the case $e > 2$, we apply Proposition 3.2.1 and then Proposition 3.2.2 to reduce to the case where the family has the form:

$$\begin{array}{ccc} \mathcal{C} = \mathbb{P}(\mathcal{E}) & \xrightarrow{ev} & \mathbb{P}^n \\ \downarrow \pi & & \\ X & & \end{array}$$

where $\mathcal{E}$ is a rank two vector bundle on X and $\pi_* ev^* \mathcal{O}(1) = \mathrm{Sym}^2(\mathcal{E})$. As in the higher degree case, we have a generically finite map $\phi : X \to \mathrm{Grass}(3, n+1)$. On the Grassmannian $\mathrm{Grass}(3, n+1)$, we have the tautological exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}^{n+1} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the tautological rank 3 quotient bundle. Applying Lemma 2.1 from [BG71] again, and pulling back the family one more time, we may further assume that $\phi^*(\mathcal{Q}) = \mathrm{Sym}^2(\mathcal{E})$.

Now we proceed with a Chern class computation. First, we compute the Chern polynomial:

$$c_t(\mathrm{Sym}^2(\mathcal{E})) = 1 + 3c_1(\mathcal{E})t + (2c_1(\mathcal{E})^2 + 4c_2(\mathcal{E}))t^2 + 4c_1(\mathcal{E})c_2(\mathcal{E})t^3$$

If we let $A = 3c_1(\mathcal{E})$, $B = 2c_1(\mathcal{E})^2 + 4c_2(\mathcal{E})$, and $C = 4c_1(\mathcal{E})c_2(\mathcal{E})$, an easy computation shows

$$9AB - 27C - 2A^3 = 0$$

Write $\widetilde{A} = c_1(\mathcal{Q})$, $\widetilde{B} = c_2(\mathcal{Q})$, and $\widetilde{C} = c_3(\mathcal{Q})$. These classes pull back under ϕ in the following way:

$$A = c_1(\mathrm{Sym}^2(\mathcal{E})) = c_1(\phi^*(\mathcal{Q})) = \phi^*(c_1(\mathcal{Q})) = \phi^*(\widetilde{A})$$

Here, we have used the properties of ϕ and the functoriality of Chern classes. Similarly $B = \phi^*(\widetilde{B})$ and $C = \phi^*(\widetilde{C})$. By the functoriality of Chern classes and the above relationships, we have

$$\phi^*(9\widetilde{A}\widetilde{B} - 27\widetilde{C} - 2\widetilde{A}^3) = 0$$

Let $\xi = 9\widetilde{A}\widetilde{B} - 27\widetilde{C} - 2\widetilde{A}^3$. It becomes convenient to rewrite ξ in terms of the Chern roots of $\mathcal{Q}$. If $\alpha_1, \alpha_2, \alpha_3$ are the Chern roots of $\mathcal{Q}$, then we calculate:

$$\begin{aligned}
\widetilde{A} &= \alpha_1 + \alpha_2 + \alpha_3 \\
\widetilde{B} &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 \\
\widetilde{C} &= \alpha_1\alpha_2\alpha_3 \\
\xi &= (\alpha_1 + \alpha_2 - 2\alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_1)(\alpha_1 + \alpha_3 - 2\alpha_2)
\end{aligned}$$

Now let $f = \phi_*[X] \in H^*(\mathrm{Grass}((,3), n+1), \mathbb{Q})$ where $[X]$ is the fundamental class of X. The projection formula then gives $\xi \cdot f = 0$.

Since $c_1(\mathcal{Q})$ is positive, $c_1(\phi^*\mathcal{Q})$ is positive by Lemma 3.3.3, and we get the desired bound on $\dim X$ by showing that $c_1(\phi^*\mathcal{Q})^{n+1} = 0$. Since we have already shown that $\phi^*(\xi) = 0$, it would suffice to show that $c_1(\mathcal{Q})^{n+1}$ is divisible by ξ in $H^*(\mathrm{Grass}(3, n+1))$. Instead, we show that this relationship holds in the cohomology ring of full flags, and argue that this is enough to conclude.

Claim: ξ divides $(\alpha_1 + \alpha_2 + \alpha_3)^{n+1}$ in $H^*(\mathrm{Fl}, \mathbb{Q})$, where Fl denotes the space of full flags.

Consider the following fiber square:

$$\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\phi'} & \mathrm{Fl} \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{\phi} & \mathrm{Grass}(3, n+1)
\end{array}$$

We have presentations for the cohomology rings:

$$H^*(\mathrm{Grass}(3, n+1), \mathbb{Q}) = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3]/I$$

$$H^*(\mathrm{Fl}, \mathbb{Q}) = \mathbb{Q}[\alpha_1, \ldots, \alpha_{n+1}]/(Symm)$$

where *Symm* is the ideal generated by the elementary symmetric functions, and the injective map p^* satisfies $p^*(\alpha_i) = \alpha_i$ for $i = 1, 2, 3$. In $H^*(\mathrm{Fl}, \mathbb{Q})$ we have

$$T^{n+1} = (T - \alpha_1) \cdots (T - \alpha_{n+1})$$

as before. Evaluate the two sides of the equation at $T = \frac{\alpha_1+\alpha_2+\alpha_3}{3}$ to find:

$$\begin{aligned}(\alpha_1 + \alpha_2 + \alpha_3)^{n+1} &= (\frac{\alpha_2 + \alpha_3 - 2\alpha_1}{3})(\frac{\alpha_1 + \alpha_3 - 2\alpha_2}{3})(\frac{\alpha_1 + \alpha_2 - 2\alpha_3}{3})g'(\alpha) \\ &= \xi \cdot g(\alpha)\end{aligned}$$

for some polynomials g' and g which proves the claim. To finish the proof, remark that the fibers of p are projective varieties, that is, effective cycles, and so the same is true of p'. By [Ful98], we have

$$(p')^*\phi^*(c_1(\mathcal{Q}))^{n+1} = (\phi')^*p^*(c_1(\mathcal{Q}))^{n+1}$$

The left hand side of the equation gives an effective cycle on $\widetilde{X}$, in particular, a non-zero cohomology class. On the right side, however, we get:

$$\begin{aligned}(\phi')^*p^*(c_1(\mathcal{Q}))^{n+1} &= (\phi')^*(\alpha_1 + \alpha_2 + \alpha_3)^{n+1} \\ &= (\phi')^*(\xi \cdot g(\alpha)) \\ &= (\phi')^*(p^*\xi \cdot g(\alpha)) \\ &= (\phi')^*p^*\xi \cdot (\phi')^*g(\alpha) \\ &= (p')^*\phi^*\xi \cdot (\phi')^*g(\alpha) \\ &= 0 \cdot (\phi')^*g(\alpha) \\ &= 0\end{aligned}$$

This gives a contradiction, so we conclude that $\dim(X) \leq n$. □

3.5 Appendix - Divisors on the Flag Variety

We include some notes on the ample cone of the flag variety $F = \mathrm{Fl}(1, \ldots, e+1; n+1)$. Let w_i be the $\mathbb{P}^1$ constructed by letting the i^{th} flag vary while leaving the others constant. These $e+1$ lines freely generate the homology group $H_2(F)$. They are also generators of the effective cone of curves. The $e+1$ Chern classes $x_p = c_1(ker(\mathcal{Q}_{p+1} \to \mathcal{Q}_p))$ generate $H^2(F)$ and we check that the intersection matrix $\langle x_i, w_j \rangle$ is given by :

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -1 & 1 \end{pmatrix}$$

with 1's on the diagonal and -1's on the lower diagonal. The ample cone of F is given by combinations of the x_i's which evaluate positively. That is, by $\mathbb{Q}$ divisors $\lambda_0 x_0 + \ldots + \lambda_e x_e$ where $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_e$.

In fact, it is well known that for varieties of the type $F = G/B$, the Picard group of F is isomorphic to the character group of F, often denoted $X(T)$ where T is a maximal torus. Any character can be written as a linear combination of the fundamental weights $\lambda = \sum a_i t_i$ and a character is called dominant if all $a_i \geq 0$, regular if all a_i are non-zero. The ample divisors correspond exactly to the dominant and regular characters, (see [LG01]). In our case, the full flag variety corresponds to G/B for $G = SL(n+1)$. The simple roots correspond to $s_i = \alpha_i - \alpha_{i+1}$ for $0 \leq i \leq n$. Suppose $L = \lambda_0 x_0 + \ldots + \lambda_n x_n$ where the x_i are as above. Then L corresponds to the weight $\lambda_0 s_0 + \ldots + \lambda_n s_n$ which is dominant if and only if L is ample, if and only if $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$. The case of the partial flag variety then follows immediately from this one.

Chapter 4

Cubics and Rational Simple Connectedness

4.1 Introduction

The study of rational curves on algebraic varieties has applications in many areas of algebraic geometry. We focus on a generalization of the property of being rationally connected to one involving a "higher" connectivity condition developed by Joe Harris, Johan de Jong, and Jason Starr; see [HS05] and [dS06]. All varieties, schemes, stacks, and maps between them will be assumed to be over Spec $\mathbb{C}$.

A smooth variety X is rationally connected if two general points on X can be connected by a $\mathbb{P}^1$ on X (see [Kol96] Section IV.3). If $M = Hom(\mathbb{P}^1, X)$ and $M_2 = M \times \mathbb{P}^1 \times \mathbb{P}^1$, then there is an evaluation map $\phi : M_2 \to X \times X$. This map sends (ϕ, p, q) to $(\phi(p), \phi(q))$. The variety X is rationally connected if and only if the map ϕ is dominant.

The algebraic property of being rationally connected is analogous to the topological property of being path-connected. Indeed, this similarity stems from the analogy that the projective line in algebraic geometry plays the role of the unit interval in topology. There is also an algebraic property analogous to the topological property of being simply connected. The main observation is that a space is simply connected if the space of paths with specified endpoints is itself path- connected. Carrying through the analogy, a variety is called rationally simply connected if the space of rational curves of sufficiently positive homology class passing through two general points on X is itself rationally connected.

To implement this idea, we need to work with a compactification of the space of rational curves on X. We use the Kontsevich moduli space. Let X be a hypersurface in projective n-space for $n \geq 4$. The stack $\overline{\mathcal{M}}_{0,n}(X,e)$ parameterizes stable, n-pointed, degree e maps from arithmetic genus 0 curves; see [FP97]. When X is a general hypersurface, this stack is irreducible and of the expected dimension as long as $\deg(X)$ is at most $1+\dim(X)/2$; see [HRS04]. For arbitrary smooth X, this need not be the case and verifying strong rational simple connectedness becomes more technical.

Definition 4.1.1. A hypersurface $X \subset \mathbb{P}^n$ is *rationally simply connected* if for each $e \geq 2$, there is a given irreducible component $M_{e,m} \subset \overline{\mathcal{M}}_{0,2}(X,e)$ such that the evaluation map $ev : M_{e,2} \to X \times X$ is dominant and a general fiber is rationally connected.

A theorem of de Jong and Starr [dS06] says that any smooth degree d hypersurface $X \subset \mathbb{P}^n$ is rationally simply connected if $n+1 \geq d^2$ (with the exception of a quadric surface).

The fiber of the map $ev : \overline{\mathcal{M}}_{0,m}(X,e) \to X^m$ over a general m-tuple of points on X plays the role of the space of m-pointed paths on X. In topology, if the space of based-paths is path-connected, then the same is true of m-pointed paths. The corresponding fact for rationally simply connected varieties is not a priori true.

Definition 4.1.2. A hypersurface $X \subset \mathbb{P}^n$ is *strongly rationally simply connected* if for every $m \geq 2$ there is a number e and a given irreducible component of $M_{e,m} \subset \overline{\mathcal{M}}_{0,2}(X, e)$ such that the evaluation map $ev : M_{e,m} \to X^m$ is dominant and a general fiber is rationally connected.

The property of being strongly rationally simply connected is related to the existence of sections. A theorem of Hassett (Theorem 1.4 [dS06]) shows that if X is a K-variety, where K is the function field of a curve, then X satisfies weak-approximation if $X \otimes_K \overline{K}$ is strongly rationally simply connected. There is also the expectation that (strong) rational simple connectedness will sometimes imply the existence of sections for maps from $X \to S$ to a surface; see [dHS08]. In the case of families of degree d hypersurfaces in projective n-space with $d^2 \leq n$ over a surface, the existence of such sections is guaranteed by the Tsen-Lang theorem. One hope is that the methods developed to verify the property of being strongly rationally simply connected can be applied to other varieties to produce sections where they are otherwise not known to exist.

It is also shown in [dS06] that a *general* degree d hypersurface in $\mathbb{P}^n$ is strongly rationally simply connected if $d^2 \leq n$ and that an arbitrary smooth hypersurface satisfies the same property if $2d^2 - d \leq n + 1$. The expectation in [dS06] is that the genericity assumption should be removed and that every smooth degree d hypersurface in the given range is strongly rationally simply connected. Here we settle this matter for $d = 3$.

In [dS06], the property of being strongly rationally simply connected is shown to be implied by the existence of m-twisting surfaces on X for $m = 1, 2$. Let $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi : \Sigma \to \mathbb{P}^1$ be the first projection map. Denote by F the class of a fiber and F' the class of a square zero section on Σ. If $f : \Sigma \to X$ is a morphism, we may consider the associated map $(\pi, f) : \Sigma \to \mathbb{P}^1 \times X$. The normal sheaf of this map will be denoted $\mathcal{N}_f$.

Definition 4.1.3. Suppose $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi : \Sigma \to \mathbb{P}^1$ is the first projection map.

For an integer $m > 0$ a map $f : \Sigma \to X$ is an *m-twisting surface* on X if

1. The sheaf f^*T_X is globally generated.

2. The map (π, f) is finite and $H^1(\Sigma, \mathcal{N}_f(-F' - nF)) = 0$.

The properties of such surfaces are reviewed in Appendix 4.8. Our construction will produce surfaces which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \to X$ such that fibers of the first projection are mapped to lines on X. We will identify these surfaces with maps $\mathbb{P}^1 \to F(X)$ where $F(X)$ denotes the Fano scheme of lines on X. For an m-twisting surface, every infinitesimal deformation of a curve given by the union of a constant section and m fibers extends to an infinitesimal deformation of the surface. Loosely speaking, this is the meaning of m-twisting surfaces.

Theorem. *If X is a smooth degree 3 hypersurface in $\mathbb{P}^n$ with $n \geq 9$, then X contains m-twisting surfaces for $m = 1, 2$. Such an X is strongly rationally simply connected. For $n < 9$ a smooth cubic hypersurface does not contain 2-twisting surfaces.*

Outline of the Argument:

The chapter applies a careful understanding of lines and planes on a smooth cubic hypersurface X along with an algebraic foliation argument to show that curves inside $\overline{\mathcal{M}}_{0,1}(X, 1)$ (corresponding to ruled surfaces on X with fixed section class) can be made appropriately "positive". This positivity implies that deformations of the surfaces (which will be abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) fixing a section and 2 fibers are unobstructed. In Section 4.2 we review some well known properties of the Fano variety of lines, $F(X)$. This space is particularly well behaved when the hypersurface has degree 3. In Section 4.3, we undertake a careful analysis of the variety of planes contained in X and show that it is reasonably well behaved.

Section 4.4 is devoted to studying the deformations theory of maps $\mathbb{P}^1 \to F(X)$, respectively maps $\mathbb{P}^1 \to \mathcal{C}$ ($\mathcal{C}$ is defined to be $\overline{\mathcal{M}}_{0,1}(X, 1)$, it admits an evaluation map $ev : \mathcal{C} \to X$). Such maps are equivalent to giving surfaces on X ruled by lines (respectively such surfaces on X with a section). Indeed, the deformation theory of maps to the space of lines and of the surfaces on X are closely related. The goal is to produce a map $\mathbb{P}^1 \to \mathcal{C}$, such that the relative tangent bundle T_{ev} pulls back to an ample bundle on $\mathbb{P}^1$. For the corresponding surface, every deformation in X of a minimal section curve extends to a deformation in X of the entire surface.

In Section 4.5, we further study cohomology and deformation theory. We first show that there exist maps $\mathbb{P}^1 \to \mathcal{C}$ pulling back T_{ev} to a globally generated bundle, i.e., X contains 1-twisting surfaces. Our main results are in Section 4.6 and Section 4.7. We define an integral foliation $\mathcal{D} \subset T_{ev}$ spanned by the positive directions of the pullback of T_{ev} by all 1-twisting morphisms. We then construct an X-scheme Y and a rational transformation of X-schemes $\tau : \mathcal{C} \dashrightarrow Y$, whose vertical tangent bundle is $\mathcal{D}$. Finally we prove $Y = X$ by a careful study of the curves which are contracted by τ. Thus $\mathcal{D}$ equals T_{ev}. This implies that every union of sufficiently many 1-twisting rational curves in $\mathcal{C}$ deforms to a 2-twisting rational curve in $\mathcal{C}$. We will review a result of [dS06]: existence of a 2-twisting surface and rational simple connectedness by conics implies strong rational simple connectedness.

4.2 The Fano Scheme of Lines

Here we recall some well-known facts about the scheme of lines on a smooth hypersurface (much of the material is contained in or follows from [CG72] Sections 5-7 and [Kol96] Section V.4). Fix X to be a smooth degree 3 hypersurface in $\mathbb{P}^n$. Denote by $F(X)$ the scheme which parameterizes lines lying completely in X. It is a subscheme of the Grassmannian $G = \mathrm{Grass}(2, n+1)$. Denote by Q the universal rank 2 quotient bundle on G. The equation defining X gives a regular section of $\mathrm{Sym}^3(Q)$, and the zero locus of this section is exactly $F(X)$ (see e.g. [Kol96] V.4.7). The scheme $F(X)$, called the Fano scheme, equals the Hilbert scheme $Hilb^{t+1}(X)$, the Chow Variety $Chow_{1,1}(X)$ and the Kontsevich space $\overline{\mathcal{M}}_{0,0}(X, 1)$.

Lemma 4.2.1. *For every smooth cubic hypersurface $X \subset \mathbb{P}^n$ of dimension at least 2, $F(X)$ is smooth of dimension $2n-6$.*

Proof. This is well known (see Exercise V.4.4.1 of [Kol96]) but we include an outline of the proof.

Given a degree 1 map $f : \mathbb{P}^1 \to X$, the space of first order deformations of f are parameterized by $H^0(\mathbb{P}^1, f^*T_X)$; obstructions to lifting infinitesimal deformations to a small extension are contained in $H^1(\mathbb{P}^1, f^*T_X)$. The sheaf f^*T_X is locally free of rank $n-1$ and has degree $n-2$. This can be seen by analyzing the tangent bundle sequence. Because $N_{X/\mathbb{P}^n}$ is isomorphic to $\mathcal{O}_X(3)$, this is the exact sequence,

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to \mathcal{O}_X(3) \to 0.$$

We next consider the same sequence for the map f. Because $T_{\mathbb{P}^1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$, this is the sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^1}(2) \to f^*T_X \to N_{\mathbb{P}^1/X} \to 0. \tag{4.1}$$

Because all locally free sheaves on $\mathbb{P}^1$ decompose as a direct sum of line bundles (see [Har77] Exercise V.2.6), we may write,

$$f^*T_X \cong \mathcal{O}(a_1) \oplus \bigoplus_{i=2}^{n-1} \mathcal{O}(a_i)$$

with $a_1 \geq 2$ and $a_2 \geq a_3 \geq \ldots \geq a_{n-1}$. Using the fact that $N_{\mathbb{P}^1/\mathbb{P}^n}$ is isomorphic to $\bigoplus_{i=1}^{n-1} \mathcal{O}(1)$ we have the normal bundle sequence,

$$0 \to N_{\mathbb{P}^1/X} \to \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(3) \to 0.$$

This implies that $N_{\mathbb{P}^1/X} \cong \bigoplus \mathcal{O}(b_i)$ where each $b_i \leq 1$. It follows that the sequence (4.1) is split exact. Since $\sum_{i=2}^{n-1} a_i = n-4$, it follows that each $a_i \geq -1$ and $H^1(\mathbb{P}^1, f^*T_X) = 0$. Thus, $F(X)$ is smooth at $f(\mathbb{P}^1)$ and has dimension $H^0(\mathbb{P}^1, N_{\mathbb{P}^1/X}) = 2n-6$ (see [Kol96], proof of V.4.3.7). □

The argument above shows that there are only two possible splitting types for the normal bundle of a line on a smooth degree three hypersurface. Following [CG72], we make the following definition:

Definition 4.2.2. Given a line $L \subseteq X$ (with $deg(X) = 3$), we say L is a *type I*, respectively a *type II*, line if $N_{\mathbb{P}^1/X} \cong \mathcal{O}(1)^{n-4} \oplus \mathcal{O}^2$, respectively $N_{\mathbb{P}^1/X} \cong \mathcal{O}(1)^{n-3} \oplus \mathcal{O}(-1)$.

Suppose X is given as the zero locus of a degree 3 homogeneous equation $F(x_0, \ldots, x_n)$. The Gauss map is the morphism

$$D_X : X \to \mathbb{P}^{n*}$$
$$[x_0, \ldots, x_n] \longmapsto [\frac{\partial f}{\partial x_0}(\vec{x}), \ldots, \frac{\partial f}{\partial x_n}(\vec{x})].$$

In coordinate free terms, it associates to each point $x \in X$, the projective hyperplane T_x which is tangent to X at x. Because X is smooth, this is a well-defined morphism. We sum up some well-known results on the geometry of lines on X as described in [CG72]:

Proposition 4.2.3. *The map D_X is finite-to-one and generically injective. For a line $L \subset X$, L is type I if and only if the map $D_X|_L : L \to D_X(L)$ is an isomorphism to a smooth plane conic. These are both equivalent to the condition that the base locus of the linear system Span($D_X(L)$) has dimension $n-3$. Similarly, L is type II if and only if the map $D_X|_L : L \to D_X(L)$ is a 2-to-1 cover of a line. These are both equivalent to the condition that the base locus of the linear system Span($D_X(L)$) has dimension $n-2$.*

Proof. See [CG72], 5.14, 5.15, 6.6, 6.7, and 6.19. □

From this, one concludes:

Proposition 4.2.4. *([CG72], 7.6) Let $F_2(X) \subseteq F(X)$ be the subscheme of type II lines on X. Then* $\dim F_2(X) \leq n-3$

Remark 4.2.5. This estimate is sharp, as there are smooth degree three hypersurfaces X such that $\dim F_2(X) = n-3$; for example, the Fermat hypersurface given by $x_0^3 + x_1^3 + \ldots + x_n^3 = 0$. There is an $n-3$ dimensional family of lines on the Fermat connecting the "conical point" $[1, -1, 0, \ldots, 0]$ to points of the form $[0, 0, a_2, \ldots, a_n]$, and each of these lines is type II.

Proposition 4.2.6. *([Kol96] V. 4.7) The canonical bundle of $F(X)$ is isomorphic to $\mathcal{O}_{F(X)}(5-n)$. In particular, when $n \geq 6$, the variety $F(X)$ is a Fano manifold in the sense that its anticanonical bundle is ample.*

From here on, we will consider $F(X)$ as being embedded in projective space by the Plücker embedding. The restriction to $F(X)$ of the universal rank 2 quotient bundle on the Grassmannian will be denoted again by Q. When $n \geq 5$, we have $\mathrm{Pic}(F(X)) = \mathbb{Z}$ (see [DM98]). By a line, respectively a conic, on $F(X)$, we will mean a $\mathbb{P}^1 \to F(X) \subseteq \mathbb{P}^N$ of degree one, respectively embedded of degree 2.

Lemma 4.2.7. *For any line L on $F(X)$, $Q|_L = \mathcal{O}_L \oplus \mathcal{O}_L(1)$.*

The rank 2 universal quotient bundle Q on the Grassmannian $G = \mathrm{Grass}(2, n+1)$ is globally generated and has degree 1, so the same is true when it is restricted to any line (see [CC] for a much stronger result).

In fact, all lines on the Grassmannian may be constructed as follows. Let V be a vector space of dimension $n+1$. Let V_3 be a 3-dimensional quotient of V and let V_1 be a 1-dimensional quotient of V_3. The set of 2-dimensional quotients V_2 of V_3 such that V_1 is quotient of V_2 determine a line on the Grassmannian G and all lines are determined this way. As is well known (Borel-Weil Theorem), this $\mathbb{P}^1$ is a generator for $H_2(G, \mathbb{Z})$.

Another way to observe this phenomenon is to consider the dual picture. That is, given subvector spaces $W_{n-2} \subseteq W_n \subseteq V$ we can again consider all $n-1$ dimensional subspaces of V containing W_{n-2} and contained in W_n. Consider the bundle whose fiber over $W_{n-2} \subseteq W_{n-1} \subseteq W_n$ is $(W_{n-1}/W_{n-2}, W_n/W_{n-1})$. As W_{n-1} varies, this bundle is $\mathcal{O} \oplus \mathcal{O}(-1)$ (on that $\mathbb{P}^1$) and of course, this is the dual of $Q|_L$.

Geometrically, a line on $\mathrm{Grass}(2, n+1)$ corresponds to a "line of lines" in projective space, which sweep out a plane. One obtains such a line by choosing a $\mathbb{P}^2 \subseteq \mathbb{P}^n$, fixing a point on the plane, and considering the union of all the lines on the plane passing through that fixed point. Often this is referred to as a degenerate scroll or a $(1,0)$-scroll. The line L on the Grassmannian is contained in $F(X)$ if and only if the plane that L sweeps out is contained in X.

Suppose as above that $X \subseteq \mathbb{P}(V)$, where $\dim V = n+1$, and $n \geq 9$. Pick a general point $[l] \in F(X)$ and denote the line on X it represents by l. By a dimension count, we expect every line on X to be contained in at least an $(n-8)$-dimensional family of planes contained in X. This is true.

Lemma 4.2.8. *(Théorém 5.1 [DM98]), each line $l \subset X$ is contained in at least an $(n-8)$-dimensional family of 2-planes contained in X.*

(We will see in the following that this is the actual dimension for a general line, see Proposition 4.3.16). So we may pick two distinct planes on X which contain l and which correspond to two distinct three dimensional quotients of V, V_3 and W_3. Choose distinct points on the line l corresponding to distinct one dimensional quotients, call them V_1 and W_1. With notation as above, the pair of spaces (V_3, V_1) and (W_3, W_1) correspond to two lines on $F(X)$ which intersect at $[l]$.

We now study how nodal curves $L_1 \cup L_2 \subset F(X)$ can be deformed to smooth curves.

Definition 4.2.9. A vector bundle E on $\mathbb{P}^1$ is *almost balanced* if $E \cong \mathcal{O}(r)^{\oplus a} \oplus \mathcal{O}(r-1)^{\oplus b}$ for some integer r. The subbundle $\bigoplus \mathcal{O}(r) \subset E$ will be denoted $\mathrm{Pos}(E)$. The rank of $\mathrm{Pos}(E)$ will be called the *positive rank* of E, and the number r the *positive degree.*

Definition 4.2.10. Given two vector subspaces U, V of a fixed finite dimensional vector space W, we say that U, V are *as linearly independent as possible* if $\dim(U \cap V) = max(0, \dim(U) + \dim(V) - \dim(W))$.

The following result is certainly well known to experts.

Theorem 4.2.11. *Let $\mathcal{C} \to B \ni 0$ be a flat family of genus 0 curves over a one-dimensional, irreducible, smooth pointed base. Assume that the central fiber $C = C_1 \cup C_2$ is reducible with a single node p, and that the general fiber is a smooth rational curve. If E is a vector bundle on $\mathcal{C}$ of rank n such that $E|_{C_i}$ is almost balanced for $i = 1, 2$ and if $Pos(E|_{C_1})|_p$ and $Pos(E|_{C_2})|_p$ are as linearly independent as possible inside $E|_p$, then $E|_{C_b}$ is almost balanced for a general $b \in B$.*

Proof. Suppose E has positive rank r_1, respectively r_2, and positive degree d_1, respectively d_2, on C_1, respectively C_2. We may assume that the map $\mathcal{C} \to B$ has two sections σ_1, σ_2 such that $\sigma_1 \cdot C_i = \delta_{1i}$ and $\sigma_2 \cdot C_i = \delta_{2i}$ (Kroenecker delta symbols). Indeed, because either component of the central fiber may be blown down over B, this is a special case of the fact that weak approximation is satisfied for rationally connected varieties over a curve (see Theorem 3 in [HT06]). Denote the divisor $\sigma_i(B)$ by D_i. Define E' to be $E(-d_1 D_1 - d_2 D_2)$. Restricted to a general fiber, the bundle $E'|_{C_b} \cong E|_{C_b}(-d_1 - d_2)$ has degree $r_1 + r_2 - 2n$. Because C_b is a smooth rational curve, we may write $E'|_{C_b} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{C_b}(b_i)$ for integers b_i.

Let $k_i : C_i \to C_0$ be the inclusion. Then we have the short exact sequence

$$0 \to E'|_C \to k_{1*}(E'|_{C_1}) \oplus k_{2*}(E'|_{C_2}) \to E'_p \to 0.$$

The relevant part of the associated long exact sequence reads

$$0 \to H^0(C, E'|_C) \to H^0(C_1, E'|_{C_1}) \oplus H^0(C_2, E'|_{C_2}) \xrightarrow{s} E'|_p$$

$$\to H^1(C, E'|_C) \to H^1(C_1, E'|_{C_1}) \oplus H^1(C_2, E'|_{C_2}) = 0.$$

Suppose first that $r_1 + r_2 > n$. Then the linear independence assumption implies that s is surjective and so $H^1(C, E'|_C) = 0$. By the semicontinuity theorem ([Har77], III.12.8), $H^1(C_b, E'|_{C_b}) = 0$ for general $b \in B$. We conclude that $b_i \geq -1$ for each i. The same argument applies to $E'(-D_1)$; in this case the corresponding map s is injective and we conclude that $b_i \leq 0$, i.e., that $E|_{C_b}$ is almost balanced for a general $b \in B$.

The case $r_1+r_2 \leq n$ is similar. In this case, the independence assumption implies that the map s is injective, and so $H^0(C, E'|_C) = 0$. Thus $H^0(C_b, E'|_{C_b}) = 0$ for a general $b \in B$ and $b_i < 0$ for each i. On the other hand, $E'(D_1)|_{C_1}$ is globally generated and thus $H^1(C, E'(D_1)) = 0$. It follows that $H^1(C_b, E'(D_1)|_{C-b}) = 0$ for a general $b \in B$ and that $b_i + 1 > -1$ for each i. Then $E|_{C_b}$ is almost balanced for general $b \in B$. □

Definition 4.2.12. (See [Kol96], Section II.3) A map $f : \mathbb{P}^1 \to X$ is called *unobstructed* if it is a smooth point of $Hom(\mathbb{P}^1, X)$. The map f is called *free* if f^*T_X is globally generated.

Proposition 4.2.13. *With the notation as above, let $L_1 = (V_3, V_1)$ and $L_2 = (W_3, W_1)$ be two lines on $F(X)$ intersecting at a general point $[l] \in F(X)$. Suppose that $V_1 \neq W_1$ and let L be the union $L_1 \cup L_2$; L has a single node at $[l]$. The curve L is unobstructed and a general deformation of L parameterizes a smooth conic C on $F(X)$ such that $Q|_C$ is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Such a C parameterizes the lines in one of the rulings of a smooth quadric surface contained in X.*

Proof. Because the lines L_1 and L_2 contain a general point of $F(X)$, they are free (see [Kol96] Theorem II.3.11), so unobstructed (see [Kol96] II.1.7). Denote by $k : L \to F(X)$ and $k_i : L_i \to F(X)$ the inclusions. Consider the sequence

$$0 \to \mathcal{O}_{L_2}(-[l]) \to \mathcal{O}_L \to \mathcal{O}_{L_1} \to 0.$$

Tensoring with $k^*T_{F(X)}$ yields

$$0 \to k_2^*T_{F(X)}(-[l]) \to k^*T_{F(X)} \to k_1^*T_{F(X)} \to 0.$$

Since the L_i are free lines, we conclude from the long exact sequence in cohomology that $H^1(L, k^*T_{F(X)}) = 0$; i.e., that L is unobstructed. Since the restriction of Q

to any $\mathbb{P}^1 \subset F(X)$ is globally generated, the only question is whether $Q|_C$ splits as $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or $\mathcal{O}(2) \oplus \mathcal{O}$ when restricted to a general deformation C of L. The former case implies that C sweeps out a smooth quadric surface on X, the latter that C sweeps out a conical quadric surface on X.

At the point $[l]$, write $Q_{[l]} = Y$ where Y is a two dimensional quotient of V and $l = \mathbb{P}(Y)$. By Lemma 4.2.7 we have $Q|_{L_i} \cong \mathcal{O}(1) \oplus \mathcal{O}$. Since the quotients V_1 and W_1 are distinct, the subspaces $\mathcal{O}_{L_V}(1)|_{[l]}$ and $\mathcal{O}_{L_W}(1)|_{[l]}$ of Y are distinct. The proposition now follows from Theorem 4.2.11. $\square$

We also have the following result on the subvariety of $F(X)$ consisting of lines through a general point.

Lemma 4.2.14. *If X is a smooth hypersurface in $\mathbb{P}^n$ with $n \geq 5$, the space of lines through a general point $x \in X$ is a smooth $(2,3)$ complete intersection in $\mathbb{P}^{n-2}$.*

Proof. Because lines through a general point are free ([Kol96] Theorem II.3.11), the space of lines through a general $x \in X$ is smooth (see [Kol96] II.1.7). The fact that this space is a $(2,3)$ complete intersection can be verified by a computation in coordinates; see, for example, [CS09] Lemma 2.1. $\square$

4.3 Planes on Cubic Hypersurfaces

Let X be any smooth degree 3 hypersurface in $\mathbb{P}^n$. In this section, assume that $n \geq 6$.

Let $\mathcal{P}$ be the variety of planes contained in X. It is a subscheme of the Grassmannian $Gr = \text{Grass}(3, n+1)$.

Proposition 4.3.1. *([DM98] Théorème 2.1) The variety $\mathcal{P}$ is connected, and every irreducible component has dimension at least $3n - 16$ whenever $n > 5$.*

Suppose X is defined by the degree 3 homogeneous equation $f(x_0, \dots x_n) = 0$. As in Section 4.2, we have the Gauss map $D_X : X \to \mathbb{P}^{n*}$. It is quasifinite and generically injective by Proposition 4.2.3.

Remark 4.3.2. There is some confusing notation when one considers the meaning of the "type" of a line on X and the "type" of a plane on X which we are about to introduce. We feel compelled to remain consistent with the notation in the literature.

Definition 4.3.3. For $P \in \mathcal{P}$, we say D_X has *rank* r_P on P if the span of $D_X(P)$ has dimension r_P. We will refer to such a plane P as having *type* r_P.

The Gauss map is given by quadrics; there is a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{P}^5 & & \\ & \overset{v_2}{\nearrow} & \downarrow pr & & \\ P & \xrightarrow{D_P} & \mathbb{P}^r & \hookrightarrow & \mathbb{P}^{n*}. \end{array} \tag{4.2}$$

Here r is the rank of P, v_2 is the Veronese 2-uple embedding, D_P is $D_X|_P$ and pr is projection from a linear subspace in $\mathbb{P}^5$. For any $P \in \mathcal{P}$, we automatically have the inequality $2 \leq r_P \leq 5$.

Remark 4.3.4. For each $P \in \mathcal{P}$, there is the twisted normal bundle sequence

$$0 \to N_{P/X}(-1) \to \mathcal{O}_P^{n-2} \to \mathcal{O}_P(2) \to 0.$$

Considering the associated cohomology sequence, the map on cohomology

$$\begin{array}{ccc} \mathbb{C}^{n-2} & \longrightarrow & H^0(P, \mathcal{O}_P(2)) \\ \uparrow & \overset{\partial}{\nearrow} & \\ \mathbb{C}^{n+1} & & \end{array}$$

is given by linear combinations of partial derivatives of f. This is the map ∂. With the notation from above, this map has rank $r_P + 1$.

Denote by $\mathcal{P}_j$ the locus $\{P \in \mathcal{P} | r_P = j\}$. This is locally closed in $\mathcal{P}$.

Proposition 4.3.5. *The dimension of* $\mathcal{P}_2$ *is at most* $n-4$*.*

Remark 4.3.6. This is claimed in Lemma 3.3 of [Iza99] as following by reasoning similar to [CG72] Section 7. Here we fill in some of the details.

Proof. If P is a plane of type 2, D_P is degree 4.

Consider the incidence correspondence $I = \{(x,P)\} \subset X \times \mathcal{P}_2$ such that $x \in P$ and such that x does not map to a branch point of $D_P : P \to D(P)$. We denote by π_X and π_P the two projections. The map π_P is surjective and of relative dimension 2. Indeed by generic smoothness, the map D_P is étale over a dense open subset of $D_P(P)$.

We claim that $\pi_X : I \to X$ is quasi-finite. Let $(y_0, P) \in I$ be a point. Because y_0 does not map to a ramification point of D_P, there are four distinct points $(y_0, y_1, y_2, y_3) \in P$ in the fiber of D_P over $D_P(y_0)$. We claim that the four points, (y_0, y_1, y_2, y_3) cannot be collinear. Assume to the contrary that they are all contained in a line $L \subset P$. As the map D_X is given by quadrics, it has degree 2 when restricted to L. Because L contains four points in the fiber of D_X, the entire line is contracted to the point $D_X(y_0)$. This is a contradiction to the fact that D_X is quasi-finite, see Proposition 4.2.3. Therefore, the plane P is the unique 2-plane containing the four points y_i. These four points are contained in the fiber $D_X^{-1}(D_X(y_0))$. As the Gauss map D_X is quasi-finite, the fiber $D_X^{-1}(D_X(y_0))$ is a finite set. Thus the set of non-collinear 4-tuples of points in $D_X^{-1}(D_X(y_0))$ is finite as well. Therefore, the set of 2-planes P of type 2 containing y_0 is also a finite set. This proves that the projection π_X is quasi-finite.

As the map D_X is generically injective but fails to be injective on planes of type 2, the dimension of $\pi_X(I)$ is at most $n-2$. Since π_X is quasi-finite, the dimension of I is also bounded from above by $n-2$. This implies that the dimension of $\mathcal{P}_2$ is at most $n-4$. □

Remark 4.3.7. In [Iza99], it is claimed that $\dim \mathcal{P}_2 \leq 5$ independently of n. It is also claimed that $\dim \mathcal{P}_3 \leq n-2$ (Lemmas 3.3 and 3.6). We give counterexamples to these statements.

Example 4.3.8. Suppose $X = \{x_0^2 x_1 + x_1^3 + K = 0\}$, where K is a generic degree three homogeneous polynomial in $x_2, \dots x_n$. One checks immediately that X is smooth. The point $p = [1, 0, \dots, 0] \in X$ is a "conical" point in the following sense. Let Y be the zero locus of K in the embedded $\mathbb{P}^{n-2}$ defined by $x_0 = x_1 = 0$. If $q \in Y$, the entire line $\overline{pq}$ is contained in X. Thus, X contains a cone over a variety of dimension $n-3$. For any line L contained in Y, the lines $\overline{pq}$ are contained in X as q ranges over all the points in L. Thus the 2-plane $\overline{pL}$ is contained in X. Choose $x_0, x_1, z_2, \dots, z_n$ to be coordinates on $\mathbb{P}^n$ so that the line L contained in Y is given by $x_0 = x_1 = z_4 = \dots z_n = 0$ and the plane $\overline{pL}$ is given by $x_1 = z_4 = \dots z_n = 0$. The equation defining Y in $\mathbb{P}^{n-2}$ can be written $z_4 Q_4 + \dots + z_n Q_n$ where Q_i is a degree 2 polynomial in the z variables. The cubic equation defining X can be written $F = x_0^2 x_1 + x_1^3 + z_4 Q_4 + \dots + z_n Q_n$. Note that

$$\frac{\partial F}{\partial x_0}|_P = 0,$$

$$\frac{\partial F}{\partial x_1}|_P = x_0^2,$$

$$\text{and } \frac{\partial F}{\partial z_i}|_P = Q_i(z_2, z_3, 0, \dots, 0) = \widetilde{Q_i}.$$

Here $2 \leq i \leq n$ and $\widetilde{Q_i}$ is a degree 2 polynomial in the variables z_2, z_3. Since the space of such polynomial is 3-dimensional, $r_P \leq 3$. The case $r_P = 3$ occurs exactly when L is a type I line on Y and $r_P = 2$ occurs when L is a type II line on Y.

If Y is a smooth degree 3 hypersurface in $\mathbb{P}^{n-2}$, the variety of lines on Y is smooth of dimension $2n-10$. The variety of type II lines on Y has dimension at most $n-5$ and this upper bound can be reached (on the Fermat hypersurface for example). Thus, there exist cubic hypersurfaces such that the dimension of the families of planes $\mathcal{P}_2$ and $\mathcal{P}_3$ are $n-5$ and $2n-10$ respectively.

Example 4.3.9. The second example deals exclusively with the Fermat hypersurface

$$X = \{x_0^3 + \dots x_n^3 = 0\} \subset \mathbb{P}^n.$$

Assume $n \geq 5$. Choose a partition of $\{0, \dots, n\}$ into subsets I_1, I_2, I_3 of cardinality at least 2. Let X_k be defined by the equation $\Sigma_{i \in I_k} x_i^3 = 0$ in $\mathbb{P}^{|I_k|-1}$ for $k = 1, 2, 3$. Note that $\dim X_k = |I_k| - 2$. For each triple of points $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$, we embed a linear 2-plane into X by

$$\mathbb{P}^2 \to X, [y_1, y_2, y_3] \mapsto [\dots y_i \cdot x_{i,k} \dots].$$

If P is the image of this embedding, $r_P = 2$. The 3 global sections giving the map $D_X|_P$ are y_1^2, y_2^2, y_3^2. Each partition I_1, I_2, I_3 as above gives a family of planes in $\mathcal{P}_2$ of dimension $n - 5$, providing another counterexample to Lemma 3.3 in [Iza99].

Note that the more subdivisions there are, the larger the dimension of the projective space we will find on our Fermat variety. With $r + 1$ mutually disjoint sets I_k we will see a $\mathbb{P}^r \subseteq X$. Choosing a general plane contained in this projective space however, it will no longer satisfy $r_P = 2$. The same argument shows that we can form an $n + r - 8$ dimensional space of planes on X in this way where r is the number of subdivisions (notice that we can have at most $n/2$ subdivisions).

We can at least show that

Proposition 4.3.10. *For any smooth cubic hypersurface $X \subset \mathbb{P}^n$ with $n \geq 6$,* $\dim \mathcal{P}_3 \leq 2n - 9$.

Remark 4.3.11. Given Example 4.3.8, we cannot hope for a much better bound.

Proof. If $P \in \mathcal{P}_3$, then the map *pr* in diagram 4.2 is projection from a line. This line must intersect the secant variety of $v_2(\mathbb{P}^2)$, which is a degree three hypersurface, in at least one point. Such a point of intersection must actually lie on a 1-parameter family of secant lines. So the image under the projection contains at least one double line; i.e., P contains a line of type II. (See [Har92], page 144).

In fact, up to the action of $\mathbf{PGL_3}$, every such projection (that is, the restriction of D_X to the plane) is conjugate to one of the following (listed with the corresponding type II lines).

$$[X_0^2 + X_1^2 + X_2^2, X_0X_1, X_0X_2, X_1X_2]; \mathbf{V}(X_0), \mathbf{V}(X_1), \text{ and } \mathbf{V}(X_2)$$

$$[X_0^2, X_1^2, X_2^2, (X_0 + X_2)X_1]; \mathbf{V}(X_1) \text{ and } \mathbf{V}(X_0 + X_2)$$

$$[X_0^2 + X_1X_2, X_1^2, X_2^2, X_0X_2]; \mathbf{V}(X_2)$$

$$[X_0^2, X_1^2, X_2^2, X_0X_2] \text{ all lines of the form } \mathbf{V}(aX_0 + bX_2)$$

That these are the only possibilities up to the action of $\mathbf{PGL_3}$ can be verified by identifying such projections with pencils of conics in $\mathbb{P}^2$ which contain no double line as a member. Given $P \in \mathcal{P}_3$, consider the exact sequence

$$0 \to N_{P/X}(-1) \to \mathcal{O}_P^{n-2} \to \mathcal{O}_P(2) \to 0.$$

Here we think of the -1 that we twisted down by as a type II line L contained in P. The hypothesis that P has rank equal to 3 says that the rank of the map $H^0(\mathcal{O}_P^{n-2}) \to H^0(\mathcal{O}_P(2))$ equals 4. Thus the kernel, $H^0(N_{P/X}(-1))$, has dimension $n-6$. As this vector space is the Zariski tangent space to the variety of 2-planes containing L, we conclude that this variety has dimension at most $n-6$. Since the dimension of the variety of type II lines on X is at most $n-3$, and since the variety of 2-planes containing each such line has dimension at most $n-6$, the variety of rank 3 planes on X has dimension at most $(n-3)+(n-6)$, i.e., $2n-9$. □

Using the same argument one can show

Proposition 4.3.12. *The dimension of the subvariety $\mathcal{P}_4'$ of $\mathcal{P}^4$ consisting of the planes containing a type II line is at most $2n-10$.*

Question Is it true that $\mathcal{P}_4' \subset \mathcal{P}_4$ is codimension 1 and an ample divisor? If so, we can conclude that $\dim \mathcal{P}_4 \leq 2n-9$ as well.

Proposition 4.3.13. *If $n \geq 6$ and $P \in \mathcal{P}$ with $r_P \geq 3$ then P is a smooth point of $\mathcal{P}$. Also if $n \geq 7$ then $\mathcal{P}$ is irreducible and has the expected dimension $3n - 16$.*

Proof. For a plane $P \subset X$, we have that $N_{X/\mathbb{P}^n}|_P = \mathcal{O}_P(3)$. The normal bundle sequence then reads

$$0 \to N_{P/X} \to \mathcal{O}_P(1)^{n-2} \to \mathcal{O}_P(3) \to 0.$$

The variety $\mathcal{P}$ will be smooth at point $[P]$ corresponding to $P \subset X$ if $H^1(P, N_{P/X}) = 0$. By considering the long exact sequence of cohomology associated to the sequence above, this is equivalent to the surjectivity of the map on global sections,

$$H^0(P, \mathcal{O}_P(1)^{n-2}) \to H^0(\mathcal{O}_P(3)).$$

We may choose coordinates so that the plane P is given by the vanishing of $x_3 = \ldots = x_n = 0$. Let $F(x_0, \ldots, x_n)$ be the equation defining X in $\mathbb{P}^n$. The above map is identified with multiplication by the partial derivatives $\frac{\partial F}{\partial x_i}|_P$ for $3 \leq i \leq n$. By assumption on the type of P, we have that $V = \mathrm{Span}(\frac{\partial F}{\partial x_i}|_P) \subset H^0(P, \mathcal{O}_P(2))$ is a base point free linear system of dimension at least 3.

The first part of the Proposition will follow from the claim that $V \cdot H^0(P, \mathcal{O}_P(1)) = H^0(P, \mathcal{O}_P(3))$. It suffices to prove this claim when V is a base point free linear system of dimension exactly 3. As the claim is invariant under the action of $\mathbf{PGL_3}$, it is enough to check that it holds for each of the four different linear systems listed in Proposition 4.3.10. This is a straightforward computation in each case. Thus, $\mathcal{P}$ is smooth at each point corresponding to a plane of type at least 3.

By Proposition 4.3.5, the locus where $\mathcal{P}$ is either not of dimension $3n - 16$ or not smooth has dimension at most $n - 4$. We conclude that $\mathcal{P}$ is a local complete intersection when n is at least 6. This follows because the defining equations of $\mathcal{P}$ in $\mathrm{Grass}(3, n+1)$ cut out a variety of the expected dimension (see [Ful98], Theorem 14.4). When $n \geq 6$, the variety $\mathcal{P}$ is connected. Since the codimension of $\mathcal{P}_2$ is at least 2 when n is at least 7, it follows from Hartshorne's connectedness theorem ([Eis95] Theorem 18.12) that $\mathcal{P}$ is irreducible for all n at least 7. □

Proposition 4.3.14. *When $n \geq 8$, a general plane $P \in \mathcal{P}$ is type* 5.

Proof. By Lemma 4.2.14, if $x \in X$ is a general point then the space of lines through x is a smooth $(2,3)$ complete intersection $V \subset \mathbb{P}^{n-2}$. A plane containing x corresponds to a line in V.

Choose coordinates so that $x = [1, 0, \ldots, 0]$ and the projective tangent plane to X at x is given by $x_1 = 0$. Write the equation defining X in the form $F = x_0^2 x_1 + x_0 x_1 L' + x_0 Q + x_1 Q' + K$ where the degree of L', resp. Q and Q', resp. K, is 1, resp. 2, resp. 3, and each is a function of $(x_2, \ldots, x_n)$. The equations Q and K cut out V in $\mathbb{P}^{n-2}$. By [Kol96] Exercise V.4.10.5, every point of V is contained in a line contained in V. By [Kol96] Theorem II.3.11, for a general point of V, every line on V containing this point is a free line. Denote the plane it corresponds to by P. We need to show that $\frac{\partial F}{\partial x_i}|_P$ give 6 independent quadrics as i ranges from 0 to n. When $i = 0$, $\frac{\partial F}{\partial x_0}|_P = (2x_0 x_1 + Q)|_P$ which will be identically 0. When $i = 1$, we have $\frac{\partial F}{\partial x_1}|_P = x_0^2 + G$ will give one quadric. To see the five others, look at the restricted tangent bundle sequence:

$$0 \to T_V|_l \to T_{\mathbb{P}^{n-2}}|_l \to N_{V/\mathbb{P}^{n-2}}|_l \to 0.$$

We have an isomorphism $N_{V/\mathbb{P}^{n-2}}|_l \cong \mathcal{O}(2) \oplus \mathcal{O}(3)$. Twisting down by 1, we know that $H^1(l, T_V|_l(-1)) = 0$ because l is free. Then the map $H^0(l, \mathcal{O}^{n-3}) \to H^0(l, \mathcal{O}(1) \oplus \mathcal{O}(2))$ is surjective and is given by the partial derivatives, as in Remark 4.3.4. This implies that the span of $\frac{\partial F}{\partial x_i}|_P$ for $2 \leq i \leq n$ in $H^0(P, \mathcal{O}_P(2))$ is 5 dimensional and does not contain $\frac{\partial F}{\partial x_1}|_P$.

Thus, a general 2-plane on X containing a general point of X is type 5. □

We also discuss the subvariety of planes containing a given line.

Proposition 4.3.15. *If $n \geq 8$, and if l is a line contained in a type r_P plane on X, then the dimension of the subvariety of $\mathcal{P}$ consisting of planes containing l is at most $n-3-r_P$.*

Proof. This follows from the same long exact sequence as in Proposition 4.3.10. □

For a general line though, we have:

Proposition 4.3.16. *For every integer $n \geq 3$, for every line l contained in X, the variety in $\mathbb{P}^{n-2}$ parameterizing 2-planes in X containing l is an intersection of hypersurfaces of type $(1,1,1,2,2,3)$. For $3 \leq n \leq 7$, a general line is contained in no 2-plane contained in X. For $n \geq 8$, the variety parameterizing 2-planes in X and containing l is smooth of dimension $n-8$.*

Proof. We leave it to the reader to compute that the equations defining planes containing a given line $l \subset X$ are of the specified type. By Proposition 4.3.5 and Proposition 4.3.13, the dimension of every irreducible component of the incidence correspondence

$$I := \{([l],[P]) | l \subset P \subset X\}$$

is the maximum of $3n-14$ and $n-2$. Thus the projection onto $F(X)$ cannot be dominant for $n \leq 7$. For $n \geq 8$, every fiber of the projection is nonempty because the vanishing set in $\mathbb{P}^{n-2}$ of 6 positive degree homogeneous equations is nonempty. We denote $F_{\text{type I}}$ the dense open subset of $F(X)$ parameterizing type I lines. Denote by $I_{\text{type I}}$ the inverse image of $F_{\text{type I}}$ in I. By Proposition 4.3.5 and Proposition 4.3.13 again, $I_{\text{type I}}$ is generically smooth and the singular locus (given by type 2 planes) cannot dominate $F(X)$. By generic smoothness, a general fiber of $I_{\text{type I}}$ is smooth of dimension $\dim(I) - \dim(F(X))$, which is $n-8$. Thus, a general fiber is a smooth complete intersection in $\mathbb{P}^{n-2}$ of type $(1,1,1,2,2,3)$. □

Corollary 4.3.17. *If $n \geq 8$ and $l \subset X$ is a general line, then every 2-plane on X containing l is type 5.*

Proof. By Proposition 4.3.16, the variety of planes on X containing l is smooth of dimension $n-8$. For such a plane P then, $H^1(P, N_{P/X}(-1)) = 0$. Thus the map

$$H^0(P, \mathcal{O}_P^{n-2}) \to H^0(P, \mathcal{O}_P(2))$$

is surjective (as in Proposition 4.3.10). It follows that $r_P = 5$. □

Remark 4.3.18. Suppose $Y \subset X \subset \mathbb{P}^n$ is a reduced subscheme. We have that $\mathrm{Span}(Y) \cap X$ contains Y as a codimension 0 subscheme. Then the "other" subscheme of this intersection is the residual to Y in $\mathrm{Span}(Y) \cap X$. This has a rigorous algebraic definition, see [Ful98] Section 9.2. Since our argument is mainly a geometric one, we forego making the discussion of the residual more precise.

Lemma 4.3.19. *Suppose X is a smooth degree 3 hypersurface in $\mathbb{P}^n$ with $n \geq 8$. If $l, l' \subset X$ are general lines such that $l \cap l' \neq \emptyset$, then Span$(l, l') \cap X = l \cup l' \cup m$ where m is a third line on X not equal to l or l'.*

Proof. By Lemma 4.2.14, the space of lines intersecting l is at least $n-3$ dimensional. By Proposition 4.3.16, the dimension of planes on X containing l is $(n-8)$-dimensional, and so the variety of lines which meet l and are contained in a plane on X containing l is $(n-6)$-dimensional. Thus, if $l \cap l'$ is generic, $\mathrm{Span}(l, l') \not\subseteq X$. We will argue that generically, $\mathrm{Span}(l, l') \cap X$ consists of three distinct lines.

Consider a projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-2}$ from l. Blowing up along l and letting $\widetilde{X}$ be the strict transform of X, we obtain

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & \tilde{\mathbb{P}^n} & & \\ & \searrow^{\pi_X} & \downarrow \pi & & \\ & & \mathbb{P}^{n-2} & \longleftarrow Q_l \longleftarrow & T_l. \end{array}$$

Generically, π_X is a conic bundle. The discriminant locus Q_l of π parameterizes planes P containing l such that $P \cap X$ is reducible, non-reduced, or has dimension 2. This last locus, which we denote by T_l, has dimension $n-8$. A general line in

$\mathbb{P}^{n-2}$ corresponds to a line of planes in $\mathbb{P}^n$ containing l which sweep out a $\mathbb{P}^3 \subseteq \mathbb{P}^n$ containing l. Since $\mathbb{P}^3 \subset \mathbb{P}^n$ is general, the resulting surface $S = X \cap \mathbb{P}^3$ is a smooth cubic surface containing l. The line l meets 10 other lines on S which are divided up into 5 pairs $\{m, m'\}$ so that $\mathrm{Span}(l, m) \cap S = l \cup m \cup m'$. Thus, a generic line in $\mathbb{P}^{n-2}$ meets Q_l in five points and at such a point of intersection, the residual to l in $\mathbb{P}^2 \cap X$ is the union of two distinct lines, neither of which is l. □

In fact, more is true for the residual to a generic intersecting pair (l, l') in X:

Lemma 4.3.20. *With the notation and assumptions as above, for a generic pair (l, l') of intersecting lines on X, the residual line m to $l \cup l'$ in Span$(l, l') \cap X$ lies on a smooth, $(n-8)$-dimensional family of planes on X.*

Proof. Denote by $D_l \subseteq F(X)$ the closure of the locus of lines contained in X which meet l. By Lemma 4.3.19, there is a rational map $\iota : D_l \dashrightarrow D_l$ sending a line l' to $m \neq l, l'$ so that $\mathrm{Span}(l, l') \cap X = l \cup l' \cup m$). This map is injective on the open set U where it is defined. By Proposition 4.3.16, there is an open set $V \subseteq F(X)$ such that for each $[p] \in V$ there is a smooth $(n-8)$-dimensional family of planes on X containing the corresponding line p. Since $l \in D_l \cap V$, $D_l \cap V \neq \emptyset$ and thus $D_l \cap V \cap U \cap \iota(U) \neq \emptyset$. The map is a bijection on $D_l \cap V \cap U$ and the lemma follows. □

We collect facts here about the interaction between lines and planes on a cubic hypersurface, specialized to the case of $X \subset \mathbb{P}^9$.

Proposition 4.3.21. *Suppose X is a smooth cubic hypersurface in $\mathbb{P}^9$. Let $\mathcal{J}$ be the incidence correspondence in $X \times F(X) \times F(X)$ which is the closure of the locus (x, l, l') such that $l \cap l' = \{x\}$. Let p_1 be the projection from $\mathcal{J}$ to X. There is an open set $U \subseteq X$ and an open set $V \subseteq p_1^{-1}(U)$ such that for each $(x, l, l') \in V$ the following holds.*

(i) For each $x \in U$, no line of type II or plane of type 2 on X contains x. The variety of lines on X containing x is a smooth type $(2, 3)$ complete intersection in $\mathbb{P}^7$, and

the variety of planes containing x is a smooth five dimensional variety. Moreover, x is contained in no $\mathbb{P}^4 \subset X$.

(ii) The variety of planes on X containing l or l' is smooth and one dimensional 1. *Moreover, no $\mathbb{P}^3 \subset X$ contains l or l'.*

(iii) A general point $q \in l$ satisfies the condition (i).

(iv) The plane Span(l, l') is not contained in X and Span$(l, l') \cap X$ is the union of three distinct lines, $l \cup l' \cup m$.

(v) The line m from (iv) is contained in a smooth, one-dimensional family of planes.

Proof. Condition (i) follows from the fact that $F(X)$ and $\mathcal{P}$ are generically smooth and from Lemma 4.2.14. Note that neither type II lines nor type 2 planes can sweep out an open subset of X. Denote this open set in X by U_1. The last statement in (1) follows from the Appendix in [BHB06] where it is shown that there can be at most finitely many $\mathbb{P}^m$'s on a smooth degree $d \geq 3$ hypersurface in $\mathbb{P}^{2m+1}$.

By Proposition 4.3.16, a general line on X satisfies the first condition in (ii). We show that a general line through a general point also satisfies this property. Consider the incidence correspondence $\mathcal{I} \subseteq X \times F(X) \times \mathcal{P}$ consisting of $x \in l \subseteq P$ with the projection to X. By generic smoothness, this projection has smooth 6 dimensional fibers over an open set U_2 in X. Set $U_3 = U_2 \cap U_1$. The fiber over $x \in U_3$ maps to the space of lines on X through x which is smooth of dimension 5. The fiber of this map is the space of planes on X containing l and by generic smoothness, a general fiber is smooth and one-dimensional. A $\mathbb{P}^3$ on X containing l would correspond to a line in the space of planes containing l. By Proposition 4.3.16, the space of planes containing l is smooth, irreducible, 1 dimensional, and not a line, and so no such $\mathbb{P}^3$ can exist. This shows (ii).

Fix a point $x \in U_3$. Let D_x be the variety of lines on X through x, and let W be the open set of D_x corresponding to lines which satisfy (ii). Denote by $Y \subseteq X$ the union of the lines corresponding to points in W. Since $Y \cap U_3$ is not empty (it contains x), a general point of Y also lies in U_3. In other words, a general point on a general line through $x \in U_3$ is also in U_3. This shows (iii).

The first statement of (iv) follows by the same proof of (ii). The second statement is that of Lemma 4.3.19.

Condition (v) follows by Lemma 4.3.20. $\square$

4.3.1 Planes and Quadric Surfaces

We continue to assume that $X \subset \mathbb{P}^9$ is a smooth cubic hypersurface. Suppose that $x \in X$ is a general point, σ, l are two general lines containing x, and m is the residual line to $\sigma \cup l$ in $\mathrm{Span}(\sigma, l) \cap X$. For any irreducible (but possibly singular) quadric surface $\Sigma \subset X$ containing σ, l, we have $\mathrm{Span}(\Sigma) = \mathbb{P}^3$. By Proposition 4.3.21(ii), this $\mathbb{P}^3$ cannot be contained on X. Then $\mathbb{P}^3 \cap X$ is a degree 3 surface on X and so must be the union of Σ and a 2-plane $P \subset X$. Since this intersection must also contain the line m, and m does not lie on Σ, we must have $m \subseteq P$. Conversely, for any plane $P \subseteq X$ which contains m, $\mathrm{Span}(P, \sigma, l) = \mathbb{P}^3$ cannot be contained in X and this $\mathbb{P}^3 \cap X = P \cup \Sigma_P$ where $\Sigma_P \subset X$ is a degree 2 surface containing σ and l.

Thus we get a bijection $(\Sigma, \sigma, l) \leftrightarrow (P, m)$ between quadric surfaces on X containing σ, l and 2-planes on X containing m. In the correspondence from plane on X containing m to quadric surface, a priori, it is possible that a degenerate union of two planes is produced. However for the generic setup, this does not occur.

Lemma 4.3.22. *Suppose $X \subset \mathbb{P}^9$ is a smooth cubic hypersurface. Let (σ, l) be two general lines containing a general point of X, and let m be the residual line as above. There is no plane P on X containing m such that Span$(P, \sigma, l) \cap X$ is the union of P and a reducible quadric surface.*

Proof. The intersection point $q = \sigma \cap l$ is a general point in X. Thus q is not contained in a plane P on X of type $r_P = 2$. Let $D_q \subset F(X)$ be the variety of lines on X passing through q. By Lemma 4.2.14, it is a smooth complete intersection of type $(2, 3)$ in $\mathbb{P}^7$. Suppose that there is a plane $P \subset X$ containing m such that $\mathrm{Span}(P, q) \cap X$ is the union of three planes, P, P_σ, and P_l where $P_\sigma \supset \sigma$ and $P_l \supset l$. The planes P_σ and P_l must intersect in a line (which of course passes through q). The lemma follows from the claim that two general points in D_q cannot be connected by a chain of two lines in D_q.

By Proposition 4.3.16, the space of planes on X containing σ is one dimensional, and so the space of lines in D_q containing $[\sigma]$ is also one dimensional. These lines sweep out a 2-dimensional subvariety W of D_q. As the line w corresponding to $[w] \in W$ contains q, it cannot be contained in any plane on X of type 2. Thus, w is only contained in planes in X of type $r_P \geq 3$ and so can be contained in at most a 3 dimensional family of planes in X by Proposition 4.3.15. The point $[w]$ then is contained in at most a 3 dimensional family of lines on D_q. Because a general point of W is contained in a one-dimensional family of lines on D_q, we compute that the variety $\{v \in D_q | v \text{ is connected to a point in W by a line}\}$ can have dimension at most 4. This variety will not contain a general point of D_q (which has dimension 5). This proves the claim. □

Lemma 4.3.23. *The above bijection between irreducible quadric surfaces containing σ, l and planes containing m is an isomorphism of the appropriate components of the Hilbert scheme. In particular, if the space of planes containing m is smooth and one dimensional, then the space of irreducible quadric surfaces containing (σ, l) is smooth and one dimensional as well.*

Proof. There is a map ϕ from the smooth, one dimensional variety of planes containing m to $\mathrm{Grass}(4, 10)$ given by $P \mapsto \mathrm{Span}(P, \sigma, l)$. The derivative of this map cannot vanish at any point, because the plane is must be contained in the intersection with the $\mathbb{P}^3$ it defines and X. Thus this map is a closed immersion.

Similarly, there is also a map from the space of irreducible quadric surfaces containing (σ, l) to Grass$(4, 10)$ given by $\Sigma \mapsto \mathrm{Span}(\Sigma)$. Again, the derivative of this map cannot vanish. Thus the map is also a closed immersion, and so an isomorphism with the image of ϕ. $\square$

4.4 Deformation Theory

In this section we assume that $X \subset \mathbb{P}^n$ is a smooth degree 3 hypersurface and that $n \geq 9$.

We will focus on properties of rational curves in the universal line $\mathcal{C} \cong \overline{\mathcal{M}}_{0,1}(X, 1)$. We have the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{ev} & X \\ \downarrow{\scriptstyle \pi} & & \\ F(X) & & \end{array} \tag{4.3}$$

where ev is the map which sends a pointed line on X to the corresponding point.

A map $f : \mathbb{P}^1 \to F(X)$ corresponds to a diagram

$$\begin{array}{ccccc} & & \overset{h}{\frown\!\!\!\longrightarrow} & & \\ \Sigma & \xrightarrow{f'} & \mathcal{C} & \xrightarrow{ev} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^1 & \xrightarrow{f} & F(X) & & \end{array} \tag{4.4}$$

where $\Sigma \cong \mathbb{P}^1 \times_{F(X)} \mathcal{C}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and h is the composition of the two horizontal maps. Conversely, the data of

$$\begin{array}{ccc} \Sigma & \xrightarrow{h} & X \\ \downarrow{\scriptstyle \pi'} & & \\ \mathbb{P}^1 & & \end{array}$$

such that each fiber of π' is a $\mathbb{P}^1$ and is mapped to a line on X by h determines a map $f : \mathbb{P}^1 \to F(X)$. Analogously, the data of a map $f : \mathbb{P}^1 \to \mathcal{C}$ is equivalent to a diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{h} & X \\ \sigma \uparrow \downarrow \pi' & & \\ \mathbb{P}^1 & & . \end{array}$$

Here, each fiber of π' is a $\mathbb{P}^1$ mapped to a line on X, and σ is a section of π'. From now on we notate π' by simply π and trust no confusion will result.

Definition 4.4.1. We will call $f : \mathbb{P}^1 \to \mathcal{C}$ a *family of pointed lines.* Such a map determines and is determined by the tuple (Σ, π, σ) as above.

Remark 4.4.2. The obstruction space to a map $g : \mathbb{P}^1 \to X$ is given by $\mathbb{E}xt^2(g^*\Omega_X \to \Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1})$. Here $g^*\Omega_X \to \Omega_{\mathbb{P}^1}$ is the relative cotangent complex for g, which will be denoted by L_g. There is a long exact sequence

$$0 \to \mathbb{E}xt^0(L_g, \mathcal{O}_{\mathbb{P}^1}) \to Ext^0(\Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \to Ext^0(g^*\Omega_X, \mathcal{O}_{\mathbb{P}^1}) \to$$

$$\to \mathbb{E}xt^1(L_g, \mathcal{O}_{\mathbb{P}^1}) \to Ext^1(\Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \to Ext^1(g^*\Omega_X, \mathcal{O}_{\mathbb{P}^1}) \to$$

$$\to \mathbb{E}xt^2(L_g, \mathcal{O}_{\mathbb{P}^1}) \to Ext^2(\Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \to 0.$$

We have $Ext^2(\Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \cong H^2(\mathbb{P}^1, \mathcal{O}(2)) = 0$. Then because $Ext^1(\Omega_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) = 0$, we have

$$\mathbb{E}xt^2(L_g, \mathcal{O}_{\mathbb{P}^1}) \cong Ext^1(g^*\Omega_X, \mathcal{O}_{\mathbb{P}^1}) \cong H^1(\mathbb{P}^1, g^*T_X).$$

Definition 4.4.3. A family of pointed lines $f : \mathbb{P}^1 \to \mathcal{C}$ will be called

1. *Section Unobstructed (S.U.)* if $\mathbb{E}xt^2(L_{ev\circ f}, \mathcal{O}_{\mathbb{P}^1}) = 0$.
2. *Fiberwise Unobstructed (F.U.)* if $\mathbb{E}xt^2(L_{f_t}, \mathcal{O}_{\mathbb{P}^1}) = 0$ for each $t \in \mathbb{P}^1$.

Remark 4.4.4. By the remark above, the condition S.U. is satisfied if $(ev \circ f)^*T_X$ splits as $\bigoplus \mathcal{O}(a_i)$ with each $a_i \geq -1$. Similarly, the condition F.U. is satisfied if each fiber of π corresponds to a smooth point of $F(X)$.

Definition 4.4.5. A family of pointed lines $f : \mathbb{P}^1 \to \mathcal{C}$ will be called

1. *Strongly Section Unobstructed (S.S.U.)* if $(ev \circ f)^* T_X$ splits as $\bigoplus \mathcal{O}(a_i)$ with each $a_i \geq 0$.
2. *Strongly Fiberwise Unobstructed (S.F.U.)* if $h_t : \Sigma_t \to W$ is free for each $t \in \mathbb{P}^1$.

Notation 4.4.6. If $f : \mathbb{P}^1 \to \mathcal{C}$ is a family of pointed lines and the map (π, h) is finite, let $\mathcal{N}_f$ be the normal sheaf for the map

$$(\pi, h) : \Sigma \to \mathbb{P}^1 \times X.$$

When (π, h) is an embedding, this is a vector bundle. The sheaf $\mathcal{N}_f$ is π-flat; see [dS06] Lemma 7.1.

Lemma 4.4.7. *If $f : \mathbb{P}^1 \to \mathcal{C}$ is a family of pointed lines satisfying S.F.U., then $R^1\pi_*\mathcal{N}_f(-\sigma) = 0$.*

Proof. As π is a submersion,

$$\mathcal{N}_f|_{\Sigma_t} \cong h_t^* T_X / T_{\Sigma_t} \cong N_{\Sigma_t/X}$$

for each $t \in \mathbb{P}^1$. Since $h_t(\Sigma_t)$ is a free line on X, $H^1(\Sigma_t, \mathcal{N}_f(-\sigma)|_{\Sigma_t}) = 0$. This implies the claim. □

Lemma 4.4.8. *If $[g] \in \overline{\mathcal{M}}_{0,0}(F(X), 1)$ is a general element and $f : \mathbb{P}^1 \to \mathcal{C}$ is a family of pointed lines such that $[g] = [\pi \circ f]$ then the family $[f]$ is S.F.U. If in addition, σ has self intersection number 1, then the family $[f]$ is S.S.U.*

Proof. An element of $\overline{\mathcal{M}}_{0,0}(F(X), 1)$ corresponds to a 2-plane $P \subset X$ with a marked point. By Proposition 4.3.14, a general plane on X is type 5, and so contains only type I lines. As these lines are free, the first claim follows. If σ has self intersection number 1, then $ev \circ f$ is also a line on P, so must also be free. □

Notation 4.4.9. Denote by $\mathcal{C}_{ev}$ the inverse image under $\pi : \mathcal{C} \to F(X)$ of all points in $F(X)$ corresponding to type I lines. Let T_{ev} be the dual of the sheaf of relative differentials for the map ev.

Lemma 4.4.10. *If* $f : \mathbb{P}^1 \to \mathcal{C}$ *is a family of pointed lines satisfying S.F.U., then* $f^*T_{ev} = \pi_*(\mathcal{N}_f(-\sigma))$.

Proof. We will continue to denote the map $\Sigma \to X$ by h.

For any $[p \in l] \in Im(f) \subseteq \mathcal{C}$ we have that

$$T_{\mathcal{C}}|_{[p,l]} = \text{Coker } \left(H^0(l, T_l(-p)) \to H^0(l, ev^*T_X|_l)\right).$$

This follows from the corresponding sequence for hyper-Ext discussed above and the fact that $H^1(l, T_l(-p)) = 0$. Thus,

$$f^*T_{\mathcal{C}} = \text{Coker } \left(\pi_*T_\pi(-\sigma) \to \pi_*h^*T_X\right).$$

Twisting down the sequence

$$0 \to T_\pi \to T_\Sigma \to \pi^*T_{\mathbb{P}^1} \to 0$$

by σ and pushing it forward by π, implies that

$$\pi_*T_\pi(-\sigma) \cong \pi_*T_\Sigma(-\sigma)$$

because $\pi_*\pi^*(T_{\mathbb{P}^1}(-\sigma)) = 0$. Using the exact sequence

$$0 \to T_\pi \to h^*T_X \to \mathcal{N}_f \to 0$$

and computing that $R^1\pi_*(T_\pi(-\sigma)) = 0$, we conclude that

$$0 \to \pi_*T_\pi(-\sigma) \to \pi_*h^*T_X(-\sigma) \to \pi_*\mathcal{N}_f(-\sigma) \to 0$$

is a short exact sequence. Putting this all together and using $R^1\pi_*h^*T_X(\sigma) = 0$ gives

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & h^*T_X|_\sigma & = & f^*ev^*T_X & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & \pi_*(T_\pi(-\sigma)) & \to & \pi_*h^*T_X & \to & f^*T_{\mathcal{C}} & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & \pi_*(T_\pi(-\sigma)) & \to & \pi_*h^*T_X(-\sigma) & \to & \pi_*\mathcal{N}_f(-\sigma) & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & & .
\end{array}
$$

Since there is also the short exact sequence

$$0 \to f^*T_{ev} \to f^*T_{\mathcal{C}} \to f^*ev^*T_X \to 0,$$

we conclude that $f^*T_{ev} \cong \pi_*\mathcal{N}_f(-\sigma)$. □

4.5 Cohomological Arguments and 1-Twisting Surfaces

In this section we assume that X is a smooth degree 3 hypersurface in $\mathbb{P}^9$. We will begin by concerning ourselves with diagrams, as in the previous section, of the form 4.4 where the $f(\mathbb{P}^1)$ is a line on $F(X)$. The image $h(\Sigma)$ is a plane on X, and $h : \Sigma \to h(\Sigma)$ is the blow-up of the plane at some point $p \in \mathbb{P}^2$. A general plane on X is unobstructed by Proposition 4.3.14. Further, for a general line on X, every plane on X containing it is type 5 by Corollary 4.3.17. For such a 2-plane $P \subset X$, we have $H^1(P, N_{P/X}(-1)) = 0$.

The Picard group of Σ has rank 2, and is generated by the class of the exceptional divisor C of h and the class of a fiber F of $\pi : \Sigma \to \mathbb{P}^1$. We have $C^2 = -1$, $F^2 = 0$,

$C \cdot F = 1$, and $h^*(\mathcal{O}(1)) = C + F$. Cohomology groups on Σ are closely related to cohomology groups on $h(\Sigma) = \mathbb{P}^2$.

Lemma 4.5.1. *Let $P = h(\Sigma)$ be a 2-plane on X as above. If $H^1(P, N_{P/X}(-1)) = 0$, then $H^1(\Sigma, N_{\Sigma/X}(-C - F)) = 0$.*

Proof. Factor the map h as:

$$\overbrace{\Sigma \xrightarrow{\;b\;} P \xrightarrow{\;i\;} X}^{h}$$

where b is the blowup at a point $p \in P$ and i is the inclusion map. There is a commutative diagram

$$\begin{array}{ccccccccc}
 & & G & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & b^*T_{\mathbb{P}^2} & \longrightarrow & b^*i^*T_X & \longrightarrow & b^*N_{\mathbb{P}^2/X} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & T_\Sigma & \longrightarrow & f^*T_x & \longrightarrow & N_{\Sigma/X} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & K & &
\end{array}$$

where G is the cokernel and K is the kernel of the obvious sheaf maps. By the snake lemma, $K \cong G$. The support of G is the exceptional curve, and so the cohomology of G is the same as the cohomology of $G \otimes b^*\mathcal{O}(-1)$. Since $R^0 b_* b^* G = G$ (this holds for any locally free sheaf G on P by the projection formula), the Leray spectral sequence implies

$$H^1(\Sigma, b^*N_{P/X}(-1)) = H^1(P, N_{P/X}(-1))$$

which is 0 by assumption. Applying the Leray spectral sequence again, we find that

$$H^1(\Sigma, b^*T_{\mathbb{P}^2}(-1)) = H^1(\mathbb{P}^2, T_{\mathbb{P}^2}(-1)) = 0.$$

This follows, for example, from the long exact sequence in cohomology determined by the Euler sequence. If $\pi : \Sigma \to \mathbb{P}^1$ is the projection map, the sequence

$$0 \to T_\pi \to T_\Sigma \to \pi^*T_{\mathbb{P}^1} \to 0$$

is exact. We claim that $H^2(\Sigma, T_\Sigma(-1)) = 0$. Indeed, because $\pi^* T_{\mathbb{P}^1} = 2b^*(\mathcal{O}(1))$, $T_\pi = 2F$, and $K_\Sigma = -2b^*\mathcal{O}(1) - F$, Serre duality implies that both $H^2(\Sigma, \pi^* T_{\mathbb{P}^1}(-1))$ and $H^2(\Sigma, T_\pi(-1))$ are zero. The claim then follows from twisting down the sequence by $\mathcal{O}(-1)$ and considering the associated long exact sequence in cohomology.

From the long exact sequence in cohomology associated to the left vertical short exact sequence above, we conclude that $H^1(\Sigma, C(-1)) = 0$. Using this and the long exact sequence in cohomology associated to the twisted right vertical short exact sequence, we conclude that $H^1(\Sigma, N_{\Sigma/X}(-1)) = 0$ as claimed. □

Lemma 4.5.2. *Let $f : \mathbb{P}^1 \to \mathcal{C}$ be a family of pointed lines such that $P = h(\Sigma)$ is a 2-plane on X as above. If $H^1(P, N_{P/X}(-1)) = 0$ and if $\sigma = [C], [C+F]$, then $H^1(\Sigma, N_{\Sigma/X}(-\sigma)) = 0$.*

Proof. The latter case is a restatement of Lemma 4.5.1, and so we may assume $\sigma = C$. There is a commutative square

$$\begin{array}{ccc} H^0(\Sigma, \mathcal{O}(F)) \otimes H^0(\Sigma, N_{\Sigma/\mathbb{P}^n}(-C-F)) & \longrightarrow & H^0(\Sigma, N_{\Sigma/\mathbb{P}^n}(-C)) \\ \downarrow & & \downarrow \\ H^0(\Sigma, \mathcal{O}(F)) \otimes H^0(\Sigma, h^* N_{X/\mathbb{P}^n}(-C-F)) & \longrightarrow & H^0(\Sigma, N_{X/\mathbb{P}^n}(-C)). \end{array}$$

The left vertical arrow is surjective because $H^1(\Sigma, N_{\Sigma/X}(-C-F)) = 0$. The bottom horizontal arrow is surjective because this is multiplication on global sections $H^0(\Sigma, \mathcal{O}(F)) \otimes H^0(\Sigma, \mathcal{O}(2F + 2C)) \to H^0(\Sigma, \mathcal{O}(3F + 2C))$. Thus, the right vertical arrow surjects and we conclude that $H^1(\Sigma, N_{\Sigma/X}(-C)) = 0$. □

We have the following commutative diagram of short exact sequences.

Diagram 4.5.3.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \pi^*T_{\mathbb{P}^1} & \longrightarrow & \mathcal{N}_f & \longrightarrow & N_{\Sigma/X} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \pi^*T_{\mathbb{P}^1} & \longrightarrow & \pi^*T_{\mathbb{P}^1} \oplus h^*T_X & \longrightarrow & h^*T_X & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & T_\Sigma & = & T_\Sigma & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Lemma 4.5.4. *Under the assumptions of Lemma 4.5.2, both groups* $H^1(\Sigma, \mathcal{N}_f(-\sigma))$ *and* $H^1(\mathbb{P}^1, \pi_*\mathcal{N}_f(-\sigma))$ *are zero.*

Proof. By the long exact sequence in cohomology associated to the top horizontal row in diagram 4.5.3 twisted down by σ, to prove the first part of the lemma it suffices to show that $H^1(\Sigma, \pi^*T_{\mathbb{P}^1}(-\sigma))$ and $H^1(\Sigma, N_{\Sigma/X}(-\sigma))$ are both zero. The latter group is 0 by Lemma 4.5.2. In either of the cases $\sigma = C$ or $\sigma = C + F$, we compute that $R^1\pi_*\pi^*T_{\mathbb{P}^1}(-\sigma) = 0$ and that $H^1(\mathbb{P}^1, \pi_*\pi^*T_{\mathbb{P}^1}(-\sigma)) = 0$. The desired vanishing then follows from the Leray spectral sequence.

The second part of the lemma follows from another application of the Leray spectral sequence. Since $H^1(\mathbb{P}^1, \pi_*\mathcal{N}(-\sigma)) \to H^1(\Sigma, \mathcal{N}(-\sigma))$ is injective, the claim follows. □

Suppose now that $[l] \in F(X)$ is a general point and L_1 and L_2 are two general (distinct) lines in $F(X)$ containing $[l]$. Each L_i ($i = 1, 2$) sweeps out a plane, P_i on X which may be assumed to be type 5 by Corollary 4.3.17. Because the L_i are general, the distinguished points $v_i \in P_i$ are distinct points on the line $l \subseteq X$. Denote by Σ_i the corresponding projective bundle over L_i and $b_i : \Sigma_i \to P_i$ the blowup at the point $v_i \in P_i$. On Σ_i, C_i will denote the negative curve class and F_i the class of the fiber. If

$l' \neq l$ is a line on P_1 passing through v_2, then $\sigma_1 = b_1^{-1}(l)$ is a section of $\pi_1 : \Sigma_1 \to L_1$ in the curve class $|C_1 + F_1|$. We define a section σ_2 of $\pi_2 : \Sigma_2 \to L_2$ by choosing the negative section C_2, identified with $b_2^{-1}(v_2)$.

The construction described above gives two maps $f_i : \mathbb{P}^1 \to \mathcal{C}$ (where $\mathcal{C} = \overline{\mathcal{M}}_{0,1}(X,1)$ as before) such that their respective composition with $\pi : \mathcal{C} \to F(X)$ corresponds to the line L_i. The images of f_i intersect at the point $(v_2, [l])$. Let $L_0 = L_1 \amalg_{[l]} L_2$ and $\Sigma_0 = \Sigma_1 \amalg_l \Sigma_2$ be the obvious gluings. Let $\sigma_0 : L_0 \to \Sigma_0$ be the unique section which agrees with σ_1 on L_1 and σ_2 on L_2. This induces a map $f_0 : L_0 \to \mathcal{C}$. Pulling back the tangent bundle sequence for the map $ev : \mathcal{C} \to X$ we obtain

$$0 \to f_0^* T_{ev} \to f_0^* T_{\mathcal{C}} \to f_0^* ev^* T_X \to 0.$$

If $j_i : L_i \to L_0$ is the inclusion map, then for each $i = (1, 2)$ the diagram

$$\begin{array}{ccccccc} \Sigma_i & \xrightarrow{j_i'} & \Sigma_0 & & & & \\ \sigma_i \uparrow\downarrow \pi_i & & \sigma_0 \uparrow\downarrow \pi_0 & & & & \\ L_i & \xrightarrow{j_i} & L_0 & \xrightarrow{f_0} & \mathcal{C} & \xrightarrow{ev} & X \end{array} \qquad f_i : L_i \to \mathcal{C}$$

commutes. By Lemma 4.4.8, both families of pointed lines are S.F.U. By Lemma 4.4.10, $f_0^* T_{ev}|_{L_i} = f_i^* T_{ev} = \pi_{i*}\mathcal{N}_{f_i}(-\sigma_i)$ by Lemma 4.4.10.

By Lemma 4.5.2, for each $i = (1, 2)$, we have that $H^1(L_i, \pi_{i*}\mathcal{N}_{f_i}(-\sigma_i)) = 0$. There is also an exact sequence

$$0 \to \mathcal{O}_{L_2}(-[l]) \to \mathcal{O}_{L_0} \to \mathcal{O}_{L_1} \to 0.$$

Tensoring this sequence with $f_0^* T_{ev}$ gives the relevant piece of the long exact sequence in cohomology

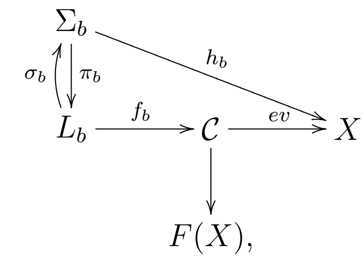
$$H^1(L_2, f_2^* T_{ev}(-[l])) \to H^1(L_0, f_0^* T_{ev}) \to H^1(L_1, f_1^* T_{ev}).$$

We have $H^1(L_1, k_1^* T_{ev}) = H^1(L_1, \pi_{1*}\mathcal{N}_1(-\sigma_1)) = 0$. We also have that

$$H^1(L_2, f_2^* T_{ev}(-[l])) = H^1(L_2, \pi_{2*}\mathcal{N}_{f_2}(-\sigma_2)(-[l]) = H^1(L_2, \pi_{2*}\mathcal{N}_{f_2}(-\sigma_2 - F_2))$$

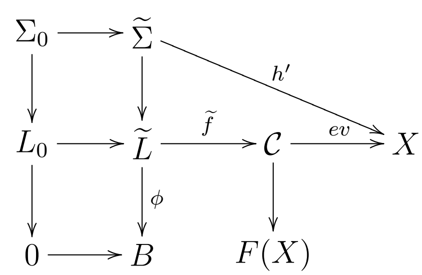
by the projection formula. This last group is 0 by Lemma 4.5.4. We conclude that $H^1(L_0, f_0^* T_{ev}) = H^1(L_0, \pi_0 \mathcal{N}_{f_0}(-\sigma_0)) = 0$ and so $H^1(\Sigma_0, \mathcal{N}_{f_0}(-\sigma_0)) = 0$ as well.

Proposition 4.5.5. *The map $f_0 : L_0 \to \mathcal{C}$ is unobstructed. A general deformation of f_0 corresponds to a map $f_b : L_b \to \mathcal{C}$ where $L_b \cong \mathbb{P}^1$. Further, in the diagram:*



the map $L_b \to F(X)$ has degree 2, $\Sigma_b \cong \mathbb{P}^1 \times \mathbb{P}^1$ (actually to $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1))$), σ_b is a ruling of the quadric surface, and $H^1(L_b, \pi_{b*}\mathcal{N}_{f_b}(-\sigma_b)) = 0$.

Proof. Because L_1 and L_2 are lines through a general point on $F(X)$, they can be smoothed into a conic; see Lemma 4.2.13. By the same Lemma, the corresponding ruled surface over a general deformation is $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1))$. The maps f_i are unobstructed by Lemma 4.4.8. The self intersection of σ_b must be constant in a flat family. Since $\sigma_0^2 = \sigma_1^2 + \sigma_2^2 = 0$, $\sigma_b^2 = 0$ and so σ_b is a ruling of the corresponding quadric surface Σ_b. For a deformation over a 1 dimensional base B, consider the diagram:



By the preceding discussion, a general deformation of L_0 is a pointed family of lines satisfying S.F.U. By Lemma 4.4.10, $\widetilde{f}^* T_{ev}$ restricts to $\mathcal{N}_{f_b}(-\sigma_b)$ on a general fiber $\phi^{-1}(b)$. Since $H^1(L_0, \pi_* \mathcal{N}_{f_0}(-\sigma_0)) = 0$ by the preceding discussion, the semicontinuity theorem ([Har77] III.12.8) implies that $H^1(L_b, \mathcal{N}_{f_b}(-\sigma_b)) = 0$ for a general $b \in B$. □

The remainder of the section is devoted to proving the following.

Theorem 4.5.6. *Let $n = 9$. A general deformation f_b of $f_0 : L_0 \to \mathcal{C}$ as in Proposition 4.5.5 is a smooth quadric surface $\pi : \Sigma \to \mathbb{P}^1$ on X satisfying $H^1(\mathbb{P}^1, \pi_*\mathcal{N}_{f_b}(-\sigma - F)) = 0$ where σ is a ruling not equal to F and F is a fiber of π. This implies that these surfaces are 1-twisting (see Appendix 4.8).*

On a ruled surface $\pi : \Sigma \to \mathbb{P}^1$ with $\Sigma \cong \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1))$ we denote by C the class of a section and F the class of a fiber. We have the numerical data $C^2 = F^2 = 0$, $C \cdot F = 1$, $p_a(\Sigma) = 0$, $c_2(\Sigma) = 4$ and $K_\Sigma = -2C - 2F$ (see [Har77] Chapter V).

We list the steps in a (standard) Grothendieck Riemann Roch calculation:

Calculation 4.5.7. We suppose that X is a smooth degree d hypersurface in $\mathbb{P}^n$, that $f : \Sigma \to X$ is a smooth quadric surface, that $\mathcal{N}$ is the twisted normal sheaf on Σ and that $R^1\pi_*(\mathcal{N}(-C)) = 0$.

Since $T_\pi = 2C$, $ch(T_\pi) = 1 + 2C$ and $Todd(T_\pi) = 1 + C$. Also we have

$$ch(f^*T_X) = (n-1)[id] + (n+1-d)(C+F) + (n+1-d^2)[pt].$$

From $0 \to T_\pi \to f^*T_X \to \mathcal{N}$, we compute

$$ch(\mathcal{N} = (n-2)[id] + (n+1-d)(C+F) - 2C + (n+1-d^2)[pt]$$

$$ch(\mathcal{N}(-C) = (n-2)[id] + (1-d)C + (n+1-d)F + (d-d^2)[pt].$$

Applying Grothendieck-Riemann-Roch,

$$\begin{aligned} ch(\pi_*\mathcal{N}(-C)) &= \pi_*(ch(\mathcal{N}(-C) \cdot Todd(T_\pi)) \\ &= \pi_*((n-2)[id] + (n-1-d)C + (n+1-d)F + (n+1-d^2)[pt]) \\ &= (n-1-d)[\mathbb{P}^1] + (n+1-d^2)[pt]. \end{aligned}$$

If $d = 3$ and $n = 9$, we see that $\pi_*\mathcal{N}(-C)$ has rank 5 and degree 1. In the situation of Proposition 4.5.5, we have $H^1(\mathbb{P}^1, \pi_*\mathcal{N}_{f_b}(-\sigma_b)) = 0$. Since $\pi_*\mathcal{N}_{f_b}(-\sigma_b)$ is locally free on $\mathbb{P}^1$, so we may write $\pi_* \mathcal{N}_{f_b}(-\sigma_b) = \bigoplus_{i=1}^5 \mathcal{O}(a_i)$ with each $a_i > -2$ and $\Sigma a_i = 1$. We will argue that the generic quadric surface must have the most positive splitting type, $(1, 0, 0, 0, 0)$.

Discussion 4.5.8. The variety of lines on $F(X)$ can be identified with a $\mathbb{P}^2$ bundle over $\mathcal{P}$, the variety of planes on X. Since $n \geq 9$ the variety $\mathcal{P}$ is irreducible and has the expected dimension by Proposition 4.3.13. This implies that $\overline{\mathcal{M}}_{0,0}(F(X), 1)$ is irreducible as well. By Proposition 4.3.16, a general line $l \subseteq X$ is contained in a smooth $n-8$ dimensional family of planes. Thus, a general fiber of

$$ev : \overline{\mathcal{M}}_{0,1}(F(X), 1) \to F(X)$$

is also irreducible.

This setup allows us to talk about the good component of $\overline{\mathcal{M}}_{0,0}(F(X), e)$ which we will denote M_e, see [dS06] Section 3. The good component is the unique component whose points parametrize (among others) "smoothed out" configurations of free lines on $F(X)$. By Lemma 4.2.13), a general point of M_2 is a "balanced" conic.

A general point of M_2 (indeed M_e) corresponds to a free curve, and so the good component is a generically smooth and reduced stack, but the discussion also applies to the coarse moduli space. We now lift this discussion to maps to $\mathcal{C}$.

It is well known that there is a cohomology class $\psi \in H^2(\mathcal{C}, \mathbb{Z})$ satisfying the following property: A map $f : B = \mathbb{P}^1 \to \mathcal{C}$ corresponds to a $\mathbb{P}^1$ bundle Σ with a section $\sigma : B \to \Sigma$. Then ψ satisfies $\psi \cdot f_*[B] = \sigma^2$ (see [HM98] Section 6D, this is the inverse of the relative dualizing sheaf for π). Let β_2 be the unique homology class in $H_2(\mathcal{C}, \mathbb{Z})$ such that $\beta_2 \cap \psi = 0$ and $\pi_*(\beta_2) = 2$.

Lemma 4.5.9. *There is a unique component* $\widetilde{M}_2$ *of* $\overline{\mathcal{M}}_{0,0}(\mathcal{C}, \beta_2)$ *dominating* M_2*. This component is generically smooth (and so reduced).*

Proof. Consider a general point $f : \mathbb{P}^1 \to F(X)$ in M_2. We "lift" $[f]$ to a point $[g]$ in $\overline{\mathcal{M}}_{0,0}(\mathcal{C}, \beta_2)$ by choosing a section of $\Sigma \to \mathbb{P}^1$ which has square 0, that is, a line in the ruling. There is a short exact sequence

$$0 \to g^*T_\pi \to g^*T_{\mathcal{C}} \to g^*\pi^*T_{F(X)} \to 0.$$

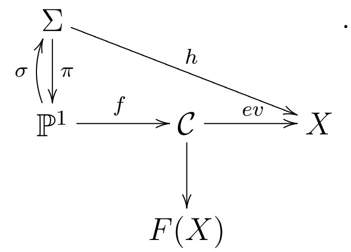
The sheaf $g^*\pi^*T_{F(X)} = f^*T_{F(X)}$ has no H^1 because $[f]$ is a smooth point of M_2. The sheaf $g^*T_\pi = \mathcal{O}(\sigma^2) = \mathcal{O}$ also has no H^1. Thus $[g]$ is a smooth point of $\overline{\mathcal{M}}_{0,0}(\mathcal{C}, \beta_2)$. These points are contained in at most (so exactly) one irreducible component. □

Let $\widetilde{M}_{2,1}$ be the unique component of $\overline{\mathcal{M}}_{0,1}(\mathcal{C}, \beta_2)$ dominating $\widetilde{M}_2$. It is generically a $\mathbb{P}^1$ bundle over $\widetilde{M}_2$ and so is also generically smooth and reduced. Note that a general closed point of $\widetilde{M}_{2,1}$ corresponds to a tuple $(\pi : \Sigma \to \mathbb{P}^1, h : \Sigma \to X, \sigma, l)$ such that $\Sigma \cong \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1))$, h embeds Σ as a quadric surface on X, σ is a section of π with $\sigma^2 = 0$, l is in the class of a fiber, and so both are lines on X.

Let the incidence correspondence $\mathcal{I} \subseteq F(X) \times F(X)$ denote the closure of the locus (l, l') with $l \cap l' \neq \emptyset$. The scheme $\mathcal{I}$ is generically smooth. This can be shown by noting that a general fiber of the first projection $p_1 : \mathcal{I} \to F(X)$ and is generically smooth. There is a map $\widetilde{M}_{2,1} \to \mathcal{I}$ sending a quadric surface with two lines (the section σ and the second line l) on it to the two lines. Again, the map has generically smooth generic fiber. So for a generic fiber, the dimension of the fiber is the dimension of the tangent space to the fiber (at a general point in that fiber). We will argue that a general fiber is actually smooth and one dimensional so that the tangent space to a general fiber is also one dimensional.

Proof of Theorem 4.5.6. The map $pr : \widetilde{M}_{2,1} \to \mathcal{I}$ sends (Σ, σ, l) to (σ, l). By Proposition 4.3.21, $(\sigma, l) \in \mathcal{I}$ is general, then $\text{Span}(l, \sigma) \cap X = l \cup \sigma \cup m$ where m is a line on X not equal to σ or l and is contained in a smooth $(n-8) = 1$ dimensional family of planes on X. Further, we have the correspondence between planes on X containing m and irreducible quadric surfaces Σ containing σ, l, see Lemma 4.3.23. A general fiber of pr is thus one-dimensional.

If $f \in \widetilde{M}_2$ is general, there is a commutative diagram



where Σ is a quadric surface, and σ is a ruling. A global section of $f^*T_{\mathcal{C}}$ corresponds to an infinitesimal deformation of the map f. A global section of $f^*T_{ev} \hookrightarrow f^*T_{\mathcal{C}}$ corresponds to an infinitesimal deformation of f which fixes $ev \circ f(\mathbb{P}^1)$; this is a deformation of the map so that the corresponding deformed quadric surface on X contains the line σ. Similarly, a global section of $f^*T_{ev}(-p)$ with $p \in \mathbb{P}^1$ corresponds to an infinitesimal deformation of f such that σ is fixed and $\pi^{-1}(p)$ is also fixed. By Lemma 4.4.10, $f^*T_{ev}(-p) \cong (\pi_*\mathcal{N}(-\sigma))(-p)$. A global section of $f^*T_{ev}(-p)$ then corresponds to a deformation of the closed embedding $h : \Sigma \to X$ passing through σ and l where l is the image of a fixed fiber F. By Lemma 4.3.23, there is a smooth one dimensional family of quadric surfaces containing the pair (σ, l). So if f is general we compute

$$h^0(\mathbb{P}^1, (\pi_*\mathcal{N}_f(-\sigma - F))) = h^0(\mathbb{P}^1, (\pi_*\mathcal{N}_f(-\sigma))(-p)) = h^0(\mathbb{P}^1, f^*T_{ev}(-p)) = 1.$$

Indeed, because the general fibers of pr is smooth, infinitesimal deformations are unobstructed and will lift to actual deformations. Considering the possible splitting types for $\pi_*\mathcal{N}(-\sigma)$, the only with this property is $\pi_*\mathcal{N}(-\sigma) \cong \mathcal{O}(1) \oplus \mathcal{O}^4$. This implies that $H^1(\mathbb{P}^1, \pi_*\mathcal{N}(-\sigma - F)) = 0$ as claimed. □

Remark 4.5.10. There is a much less round about way to this result. The stack $\overline{\mathcal{M}}_{0,0}(F(X), 1)$ has dimension $3n - 16 + 2 = 13$ as expected and so the dimension of M_2 is $14 + 14 - 12 + 1 = 17$. The dimension of $\widetilde{M}_{2,1}$ then is 19, while the dimension of $\mathcal{I}$ is 18. By virtue of being the good component, the map $pr : \widetilde{M}_{2,1} \to \mathcal{I}$ is dominant; as both are reduced, the generic fiber is generically smooth and one-dimensional; see the proof of Corollary 4.7.1. The rest of the proof is as above, as the computation goes through a general point of the fiber. Nevertheless, the geometric construction of fixing σ, l and considering planes through the residual line m will become important in what follows.

Remark 4.5.11. Deforming two lines on $F(X)$ into a conic gives a quadric surface on X with a choice of ruling. The other ruling gives a different conic on $F(X)$. Suppose two lines L_1 and L_2 on $F(X)$ meet at a point $[l]$. Each line corresponds to a surface Σ_i along with a map to X with image a 2-plane on X containing l. There are two distinguished points on the line $l \subseteq X$, the images of the exceptional divisors E_i, call them x_1, x_2. Suppose the union $\Sigma_1 \cup \Sigma_2$ deforms into a quadric surface Q with rulings A and B. Reducible curve classes on $\Sigma_1 \cup \Sigma_2$ which are in $E_1 \cup (E_2 + F)$ (lines meeting at x_1) or are in $(E_1 + F) \cup E_2$ (lines meeting at x_2) both deform into the same section class, suppose it is A. The surface Q with the other ruling B degenerates to $\Sigma_1 \cup \Sigma_2$ with the distinguished points x_1 and x_2 swapped; the exceptional divisor on Σ_i is now contracted to x_{3-i}. Thus, Q_A and Q_B, the quadric surface considered with its two rulings, correspond to conics in the same component of maps to $\mathcal{C}$ over $F(X)$, namely the good component $\tilde{M}_2$, because they are both "smoothed" out lines (with corresponding section classes) on $F(X)$.

4.6 The Foliation

As in Lemma 4.5.9, we consider the component $\widetilde{M}_r$ consisting of maps to $\mathcal{C}$ which dominate the component M_r from discussion 4.5.8 and have section class of square 0. For a general point $[f]$ of $\widetilde{M}_2$, corresponding to a map $f : \mathbb{P}^1 \to \mathcal{C}$ passing through a general point $y = [p \in l]$ of $\mathcal{C}$, we have $f^*T_{ev} = \mathcal{O}(1) \oplus \mathcal{O}^4$. In particular we have a distinguished one dimensional subspace of $k^*T_{ev}|_y$ corresponding to the positive direction $\mathcal{O}(1)$. If $\mathrm{Span}(\cup \mathcal{O}(1)|_y) \subseteq T_{ev}|_y$ is all of $T_{ev}|_y$ for general y (here the union is taken over all generic maps in $\widetilde{M}_2$ passing through y) then we would be able to smooth out chains of maps $[f] \in \widetilde{M}_2$ to a form a map $[g] \in \widetilde{M}_r$ corresponding to a 2-twisting surface. The details are filled in by the following sequence of lemmas.

Lemma 4.6.1. *Suppose $\phi_1 \in M_{r_1}$ and $\phi_2 \in M_{r_2}$ are general maps $\mathbb{P}^1 \to F(X)$ such that $\phi_1(0) = \phi_2(0)$. There is a unique map $\phi : \mathbb{P}^1 \cup_0 \mathbb{P}^1 \to F(X)$ agreeing with ϕ_1 and ϕ_2 on each component. This map is unobstructed and a general deformation of ϕ is contained in $M_{r_1+r_2}$.*

Proof. This is the property of the good component. See [dS06]. □

Lemma 4.6.2. *Suppose $\phi_1 \in \widetilde{M}_{2r_1}$ and $\phi_2 \in \widetilde{M}_{2r_2}$ are general maps $\mathbb{P}^1 \to \mathcal{C}$ such that $\phi_1(0) = \phi_2(0)$. There is a unique map $\phi : \mathbb{P}^1 \cup_0 \mathbb{P}^1 \to \mathcal{C}$ agreeing with ϕ_1 and ϕ_2 on each component. This map is unobstructed and a general deformation is in $\widetilde{M}_{2(r_1+r_2)}$. Further, the corresponding surface is a $\mathbb{P}^1 \times \mathbb{P}^1$ on X.*

Proof. By Lemma 4.6.1, the underlying curves in $F(X)$ deform into curves which are contained in the good component. By induction, the universal rank 2 bundle Q on $F(X)$ restricts to be balanced on the image of ϕ_i in $F(X)$ (the base case is Proposition 4.2.13). By Theorem 4.2.11, Q restricts to be almost balanced on a general deformation of ϕ. It must be balanced for degree reasons and so the resulting surface on X is a $\mathbb{P}^1 \times \mathbb{P}^1$. Since the self intersection of a section class will be constant in a flat family, and since it is 0 on the special fiber, it is also 0 on a generic fiber. Thus, it is exactly a section of the resulting ruled surface with self square 0. □

For $\phi \in \widetilde{M}_{2r}$ general, $\phi^* T_{ev} = \bigoplus_{i=1}^{5} \mathcal{O}(a_i)$ with each a_i non-negative and at least one of the $a_i > 0$. This follows from the preceding section when $r = 1$. The case $r > 1$ follows from the fact that general $\phi \in \widetilde{M}_{2r}$ is a deformation of chains of lower degree maps with this property. Define $Pos(\phi) := \bigoplus_{i=0}^{r} \mathcal{O}(a_i) \subseteq \phi^* T_{ev}$ to be the subbundle consisting of those summands which have a_i strictly greater than 0. Define $m_\phi = rank(Pos(\phi))$ to be the positive rank of ϕ. Let $m = \max_{\phi \in \widetilde{M}_{2r}} m_\phi$ where the maximum is taken over all positive values of r and all maps ϕ where the generic splitting type occurs.

There is an open set U_{2r} of $\widetilde{M}_{2r}$ where for each map $\phi \in U$, $\phi^* T_{ev}$ has constant splitting type (this follows from the semicontinuity theorem, [Har77] III.12.8) . Fix w to be the smallest integer where $m_\phi = m$ generically on U_{2w}. That is, where the positive rank no longer increases upon gluing together further free curves and smoothing them out. We wish to show that $m = 5$, the maximal possible value. If so, then for a general $\phi \in \widetilde{M}_{2w}$, we will have $H^1(\mathbb{P}^1, \mathcal{N}_\phi(-\sigma - 2F)) = 0$. This implies that the corresponding surface Σ_ϕ is 2-twisting.

The "directions" that the positive summands point in control positive summands of the smoothed out curves in the following sense.

Proposition 4.6.3. *With the notation in Lemma 4.6.2, suppose*

$$Span(Pos(\phi_1)|_0, Pos(\phi_2)|_0)$$

has dimension greater than m_{ϕ_1} *and* m_{ϕ_2}. *Then for a general deformation* ϕ *of* $\phi_1 \cup \phi_2$, $m_\phi > m' = \max(m_{\phi_1}, m_{\phi_2})$

Proof. Denote the domain of ϕ_i by $C_i \cong \mathbb{P}^1$. Let $\pi : C \to B$ be a general deformation of $\phi_1 \cup \phi_2$. This will be a diagram of the form

$$\begin{array}{ccccccc} C_1 \cup C_2 & \longrightarrow & C & \xrightarrow{f} & \mathcal{C} & \xrightarrow{ev} & X . \\ \downarrow & & \downarrow{\scriptstyle \pi} & & & & \\ 0 & \longrightarrow & B & & & & \end{array}$$

We may assume that π has two disjoint sections D_1 and D_2 meeting the central fiber only at a single point of C_1 and C_2 respectively (see the proof of Theorem 4.2.11). Denote the restriction of $f^* T_{ev}(-D_1 - D_2)$ to C, respectively to C_1, C_2, by E, respectively by E_1, E_2. Let $i_1 : C_1 \to C_0$ and $i_2 : C_2 \to C_0$ be the inclusion maps. The short exact sequence

$$0 \to E \to i_{1*}E_1 \oplus i_{2*}E_2 \to E|_0 \to 0$$

gives rise to the long exact sequence in cohomology

$$0 \longrightarrow H^0(C,E) \longrightarrow H^0(C_1,E_1) \oplus H^0(C_2,E_2) \xrightarrow{a} T_{ev}|_0 \longrightarrow H^1(C,E) \longrightarrow 0\,.$$

Here the last 0 follows because for each $\phi_i \in \widetilde{M}_{r_j}$ (with the appropriate r_j) the corresponding bundle $\phi_i^* T_{ev}$ is non-negative. By assumption, the map a has rank greater than m' and so $h^1(C,E) < n - m'$. By the semicontinuity theorem ([Har77] III.12.8), for a general $b \in B$, $h^1(C_b, f^*T_{ev}(-D_1 - D_2)|_{C_b}) < n - m'$ as well. But on a general fiber, $f^*T_{ev}(-D_1 - D_2)|_{C_b} \cong f_b^* T_{ev}(-2)$. As the map $f_b : C_b \to \mathcal{C}$ is free, $f_b^* T_{ev} \cong \oplus \mathcal{O}(a_i)$ with each $a_i \geq 0$. But then we see that the number of i's so that $a_i > 0$ must be greater than m', as claimed. □

We now define a subsheaf of T_{ev} which encodes these positive subspaces. For a general point $p \in \mathcal{C}$, define $\mathcal{D}(p) = \mathrm{Span}_{\phi \in \widetilde{M}_{2r}, \phi(0)=p}(Pos(\phi)|_0)$. Note that this makes sense on an open set U of $\mathcal{C}$ because a curve passing through a general point of $\mathcal{C}$ is free and on a (possibly smaller) open set U', the curves ϕ passing through $p \in U'$ will satisfy that $Pos(\phi)|_p$ has the generic splitting type.

In fact, we can set up the following more general situation.

Situation 4.6.4. Let W be a smooth projective variety, $f : \mathbb{P}^1 \to W$ a free morphism, and E a vector bundle on W such that f^*E is globally generated. The point $[f]$ is a smooth point of the mapping space $Hom(\mathbb{P}^1, X)$, so is contained in a unique irreducible component, M_1. We may define a sequence of components M_i for $i > 1$ in the following way. Let $g : C \to W$ be a map from a nodal curve such that each component C_j of C is parameterized by a smooth point of M_1. Because the C_j are free curves on W, the map g is also a smooth point of the moduli space, may be deformed to a map from an irreducible curve (which is also free), and so defines a unique component M_i. Over an open set $U_i \subset M_i$, for each map $h \in U_i$, the pullback h^*E has constant splitting type (this follows from the semicontinuity theorem). We may define the positive rank for this splitting type as above. We may define $F(p)$ analogously to the definition of $D(p)$ in the preceding discussion. By Proposition 4.6.3, this is well-defined because as we take curves in all M_i through general p, the span of the positive directions must stabilize.

Proposition 4.6.5. *In situation 4.6.4, there is a subsheaf $\mathcal{F}$ of E and an open set $U \subset W$ such that for $p \in U$, $\mathcal{F}|_p = F(p)$. In fact, $\mathcal{F}$ restricts to a vector bundle on some open set of W whose complement has codimension at least 2. In particular, the $\mathcal{D}(p)$ glue together to give a subsheaf $\mathcal{D}$ of the relative tangent bundle T_{ev} having this property.*

Proof. A proof can be found in [She09], Proposition 2.5. Note that the proof given there is not phrased in this generality, but it readily extends. We sketch the steps here. We may find some $M = M_j$ such that for a general $[h] \in M$, the map corresponding to h has maximal positive rank. By shrinking M, we may assume that the splitting type for h^*E is constant over M. The universal map $\pi : M \times \mathbb{P}^1 \to W$ is smooth, denote U the image (which is open). Denote the positive subbundle of π^*E by G and let $Z = M \times \mathbb{P}^1$. Descent data on G from the descent data on π^*E may be constructed. Indeed, for the scheme $Z \times_X Z$ with projections p and q to Z, there is an isomorphism $p^*G \cong q^*G$ because the positive subspaces are well defined. That this isomorphism satisfies the cocycle condition follows from the same fact. By faithfully flat descent, there is a subbundle $\mathcal{F}$ of E over U, that pulls back to G via π.

The bundle $\mathcal{F}$ on U determines a section of $Gr(m, T_{ev})$ over U which extends to an open set with complement of codimension at least 2 because the Grassmannian is projective. Denote again $\mathcal{F}$ the smallest coherent sheaf such that restricts to this bundle over U. This $\mathcal{F}$ satisfies the claim. □

In the situation where $\mathcal{D}$ is a subsheaf of the relative tangent bundle, we may consider the restriction of the Lie bracket.

Lemma 4.6.6. *The sheaf $\mathcal{D}$ above is integrable, that is, $[\mathcal{D}, \mathcal{D}] \subseteq \mathcal{D}$.*

Proof. Again see [She09], Proposition 2.6. Restrict $\mathcal{D}$ to the open set U where it is a vector bundle. Consider the following diagram,

$$\begin{array}{ccccc} \mathcal{D} \otimes_{\mathbf{C}} \mathcal{D} & \xrightarrow{[-,-]} & T_{ev} & \longrightarrow & T_{ev}/\mathcal{D} . \\ & \searrow & & \nearrow^{p} & \\ & & \mathcal{D} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{D} & & \end{array}$$

We form M as in the proof of Proposition 4.6.5. For a general $\phi \in M$, we have $\phi^* T_{ev} = \oplus_{i=1}^{m} \mathcal{O}(a_i) \oplus \mathcal{O}^{5-m}$ with each $a_i > 0$. Let π be the map $M \times \mathbb{P}^1 \to \mathcal{C}$, then we have $\pi^* \mathcal{D} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{D} = \bigoplus \mathcal{O}(a_i + a_j)$ and $\pi^*(T_{ev}/\mathcal{D}) = \mathcal{O}^{5-m}$. The map p pulled back by π corresponds to a map

$$\bigoplus \mathcal{O}(a_i + a_j) \to \mathcal{O}^{5-m}.$$

Since each $a_i > 0$, $\pi^* p = 0$ and by a descent argument, we have $p = 0$. This implies that $\mathcal{D}$ is closed under the Lie bracket, as was to be shown. □

Lemma 4.6.7. *If $\phi \in U_{2w} \subseteq \widetilde{M}_{2w}$ is general, then $\phi(\mathbb{P}^1) \subseteq U$ and $\phi^* \mathcal{D}$ is ample.*

Proof. For the first statement either note that $\bigcup_{\phi \in U_{2w}} \phi(\mathbb{P}^1)$ is open in $\mathcal{C}$ and contained in U, or that the free rational curves can be deformed to miss any fixed codimension two locus; see [Kol96] II.3.7. The last statement is clear, for a bundle to be ample on $\mathbb{P}^1$ simply means it is the direct sum of ample line bundles, which it is by construction here. □

The existence of $\mathcal{D} \subseteq T_{\mathcal{C}}$ gives a holomorphic foliation by the holomorphic Frobenius Theorem (see [Voi07], Section 2.3). The following theorem of Kebekus-Solá-Toma [KSCT07] supplies an algebraic analogue:

Theorem 4.6.8. *(See [KSCT07]) Let X be a complete normal variety and $C \subseteq X$ a complete curve contained entirely in X_{reg}. Suppose $F \subseteq T_X$ is a foliation which is regular and ample on C. Then, for every point $x \in C$, the leaf through x is algebraic.*

Remark 4.6.9. The main result of [KSCT07] is that the leaves of the foliation are rationally connected, a fact we will not use. At the heart of the algebraicity argument is a result due to Hartshorne related to formal neighborhoods of subvarieties.

Theorem 4.6.10. *The leaves of the foliation given by $\mathcal{D}$ on a (possibly different) open set $U \subseteq \mathcal{C}$ are algebraic. Further, there is a projective variety Y fitting into a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} \supseteq U & & \\ \downarrow{\scriptstyle ev} & \searrow^{\tau} & \\ X & \xleftarrow[\psi]{} & Y. \end{array}$$

Over U, the relative tangent bundle T_τ agrees with $\mathcal{D}$.

Proof. Changing notation slightly, denote by $V \subseteq \mathcal{C}$ be the largest open set where $\mathcal{D}|_p = \mathcal{D}(p)$. By Lemma 4.6.7, the open set $U = \bigcup_{\phi \in U_{2w}} \phi(\mathbb{P}^1)$ is contained in V and for each $\phi \in U_{2w}$, $\phi^*\mathcal{D}$ is ample. By Theorem 4.6.8, on the open set U, every leaf of $\mathcal{D}$ is algebraic. We have a map (of sets for the moment),

$$U \to \text{Chow}(\mathcal{C}), u \mapsto \text{ (Leaf through u)}$$

We claim that Y, the image of this map, satisfies the statement of the theorem. Consider the incidence correspondence $\mathcal{I}_U \subseteq U \times \mathcal{C}$ given by $\{(u,x)|x \in \text{ Leaf}(u)\}$. The projection $p_2 : \mathcal{I}_U \to U$ is proper and generically smooth, so gives rise to a map $\tau : U' \to \text{Chow}(\mathcal{C})$ where $U' \subseteq \mathcal{I}_U$ is the smooth locus of p_2; see [Kol96] I.3. It is always the case that $\text{Im}(\tau)$ is constructible, so there is an open subvariety Y_0 which is dense in its closure. Let $U'' = \tau^{-1}(Y_0)$. As $\text{Chow}(\mathcal{C})$ is projective and we are in characteristic 0, Y_0 embeds in a smooth projective variety, Y. The morphism τ is a rational map $\mathcal{C} \dashrightarrow Y$ which is defined on U''. Two points in U'' have the same image in Y if and only if they lie on the same leaf (the leaves of a foliation are disjoint) which implies that $\mathcal{D}|_{U''}$ equals the relative tangent bundle T_τ. Since $\mathcal{C}$ is projective, the map extends over an open set whose complement has codimension at least 2 in $\mathcal{C}$.

There is at least a map of sets $\psi : Y_0 \to X$ sending all points in Leaf(u) to $ev(u)$. Let $f \in \mathbb{C}(X)$ be a rational function on X. The pullback $ev^{-1}(f) \in \mathbb{C}(\mathcal{C})$ is a rational function on $\mathcal{C}$. By construction, this function is constant on the fibers of $U'' \to Y_0$. It then comes from a rational function $f' \in \mathbb{C}(Y)$. In other words, we have an inclusion of fields $\mathbb{C}(X) \subset \mathbb{C}(Y)$ and so a dominant rational map $\psi : Y \dashrightarrow X$. By replacing Y with a blowup, we may as well assume this map is everywhere defined. □

We can describe which points in the fibers of ev are contracted by τ. Let $y \in \mathcal{C}$ and let x_1 and x_2 be two general points in the fiber of ev over $x \in X$. Suppose there exist curves $\alpha_1, \alpha_2 \in \widetilde{M}_{2r}$ such that $\alpha_1(0) = x_1$, $\alpha_2(0) = x_2$, $ev(\alpha_1(\mathbb{P}^1)) = ev(\alpha_2(\mathbb{P}^1))$ and $\alpha_1(\infty) = \alpha_2(\infty) = y$, then the points x_1 and x_2 will have the same image under τ. This is made more precise below.

4.7 2-Twisting Surfaces

We continue to assume that $X \subset \mathbb{P}^9$ is a smooth cubic hypersurface and keep the notation of the previous sections. Pick a general point $y = [q, l] \in \mathcal{C}$, a general line σ through $q \in X$, and a general point x on σ. Denote the residual line to $\sigma \cup l$ in Span$(\sigma, l) \cap X$ by m. By Lemma 4.3.23, the irreducible quadric surfaces on X containing $\sigma \cup l$ are in bijection with the 2-planes on X containing m. For each quadric surface Q which contains $\sigma \cup l$, there is a corresponding line l_Q through x. This l_Q is in the same ruling as l if Q is smooth, or it is σ if Q is a cone over a conic. Because $n = 9$ we get a one parameter family in $ev^{-1}(x)$. Denote this curve by $C_{y,\sigma,x}$. This construction is well-defined by Lemma 4.3.22.

For a fixed general $x \in X$, for each (general) choice of σ, q, l (where q is the point $\sigma \cap l$) we get a complete curve in $ev^{-1}(x)$. Two different choices of the data (σ, q, l) will give rise to distinct curves. There is a 5-dimensional choice for the line $\sigma \subset X$ containing x, a 1-dimensional choice of $q \in \sigma$ and a 5-dimensional choice for the line $l \subset X$ containing q. If everything is chosen generically, then Proposition 4.3.21 describes the behavior.

Lemma 4.7.1. *With the notation as above, there is an* 11*-dimensional family of complete curves in* $ev^{-1}(x) \subset \mathcal{C}$ *for a general point* $x \in X$.

Proof. The dimension follows from the preceding discussion since everything is well defined generically by Proposition 4.3.21. The map $\pi : \widetilde{M_{2,1}} \to \mathcal{I}$ in the proof of Theorem 4.5.6 is dominant. To see this, note that M_2 is the component consisting of free conics on $F(X)$, so that there is a conic passing through a general $[l] \in F(X)$ where the line l contains a general point $x \in X$. Choosing the section l' containing x of the corresponding quadric surface, we obtain a curve in $\tilde{M}_2$. By Remark 4.5.11, we may consider l as the section and l' as the "other" line. Since this surface is 1-twisting, a general deformation of the map $\phi : \mathbb{P}^1 \to \mathcal{C}$ corresponding to this surface is unobstructed even when we impose the condition that the section l is fixed. Thus, our quadric surface can be made to contain a generic line l and a generic line l' meeting l. This shows π is dominant.

Different sets of data (σ, p, l), (σ', p', l') cannot determine the same curve. Suppose the contrary. If $\sigma \neq \sigma'$ then each quadric surface containing σ and l would also have to contain σ'. Since the quadric surface is determined by the $\mathbb{P}^3$ it spans, there cannot be 2 quadric surfaces containing (σ, l, σ') as they would both span the same $\mathbb{P}^3$, and so be equal. If $l \neq l'$, the same argument applies. □

Restricting the commutative diagram in Theorem 4.6.10 to the fiber over a general point $x \in X$, we obtain the diagram

$$\begin{array}{ccc} ev^{-1}(x) & \overset{\tau}{\dashrightarrow} & Y_x \\ {\scriptstyle ev}\downarrow & \swarrow {\scriptstyle \psi} & \\ x & & . \end{array} \tag{4.5}$$

The map τ is still defined on some open set $U_x \subseteq ev^{-1}(x)$ whose compliment has codimension at least 2. The curves $C_{y,\sigma,x}$ form an 11 dimensional family as above and each one is parallel to the foliation $\mathcal{D}$ of Theorem 4.6.10. This property is explained in the following lemma.

Lemma 4.7.2. *With the notation as above, for a general curve $C_{y,\sigma,x}$, the image of the restriction $\tau : U \cap C_{y,\sigma,x} \to Y_x$ is a single point.*

Proof. By Proposition 4.3.21, for a general choice of σ and of l, the variety B of planes through the residual line is smooth and one dimensional. Each $b \in B$ induces a map $\mathbb{P}^1 \to \mathcal{C}$ corresponding to the quadric surface residual to the plane $\mathbb{P}^2_b \subseteq \mathrm{Span}(\mathbb{P}^2_b, x)$. This corresponds to a diagram of the form

$$\begin{array}{ccccc}
\mathfrak{E} & & \xrightarrow{\quad h \quad} & & X \\
{\scriptstyle s}\uparrow\downarrow & & & & \\
E & \xrightarrow{g} & \mathcal{C} & \xrightarrow{ev} & X \\
\downarrow & & & & \\
B & & & & .
\end{array}$$

Here $E \to B$ is a $\mathbb{P}^1$-bundle and $\mathfrak{E} \to B$ is a $(\mathbb{P}^1 \times \mathbb{P}^1)$-bundle. The map $h \circ s = ev \circ g$ is constant on fibers of π, the image is the line σ. This implies that the bundles E and $\mathfrak{E}$ are trivial. The curve $C_{y,\sigma,x}$ is identified with $g^{-1}(ev^{-1}(x))$. The differential $dg : T_{C_{y,\sigma,x}} \to g^*T_{\mathcal{C}}$ factors through g^*T_{ev} as the curve $C_{y,\sigma,x}$ is contracted to a point by ev. Let $U_C = g^{-1}(U \cap C_{y,\sigma,x})$. Then the map $dg : T_{U_C} \to g^*T_{ev}$ and we claim this map factors through $\mathcal{D}|_U$.

Choose $c \in U_C$ and let b be its image in B. There is a Kodaira-Spencer map $T_{B,b} \to H^0(E_b, g^*T_{\mathcal{C}})$ associating to a tangent direction in B the corresponding deformation of the map g (restricted to the fiber). By the previous paragraph, this map factors through $H^0(E_b, g^*T_{ev})$. Further, it actually factors through $H^0(E_b, g^*T_{ev}(-[l]))$ where $[l] \in E_b$ is the point mapping to $[l] \in \mathcal{C}$. This point corresponds to the line $l \subseteq X$ which is contained in every quadric surface in the family. By the proof of Theorem 4.5.6,

$g^*T_{ev}(-1)$ has splitting type $\mathcal{O} \oplus \mathcal{O}(-1)^4$. So the deformation of E_b is given by the positive part of T_{ev} (when restricted to the curve). This is exactly what it means for $dg|_{U_C}$ to factor through $\mathcal{D}|_U$ by definition of the foliation $\mathcal{D}$. As $\mathcal{D}|_U$ is the vertical tangent bundle of the map from $\tau : U \to Y$, the curve $C_{y,\sigma,x}$ is contracted by τ at least where the map is defined. □

With so many (possibly affine) curves on $U \subseteq ev^{-1}(x)$ being contracted, one would hope that there is an actual homology class (i.e. a complete curve) contracted by τ and that this will force the map τ to be equal to the map ev. Indeed, this is how the proof will proceed.

Proposition 4.7.3. *With notation as above, a generic curve $C_{y,\sigma,x}$ is contained in U_x.*

Before proving this statement, we study further the interaction between lines and planes contained in X.

Lemma 4.7.4. *Let $X \subset \mathbb{P}^9$ be a smooth cubic hypersurface and let $x \in X$ be a general point in the sense of Proposition 4.3.21 (i). Suppose that w is any point on any line on X through x. Then the space of lines through w is a 5 dimensional $(2,3)$ complete intersection in $\mathbb{P}^7$.*

Proof. Choosing appropriate coordinates, we may assume that $w = [1, 0, \ldots, 0]$ and that the projective tangent plane to X at x is defined by $x_1 = 0$. Then we may write the equation defining X as

$$x_0^2x_1 + x_0x_1L'(x_1, \ldots, x_9) + x_0Q(x_2, \ldots, x_9) + x_1Q'(x_2, \ldots, x_9) + K(x_2, \ldots, x_9)$$

where the degree of L', respectively Q, Q' and K is 1, respectively 2, 2 and 3. The equations Q and K define D_w scheme theoretically in $\mathbb{P}^7$. The equation K cannot be identically 0, otherwise X would be singular. If Q is identically 0, then w is a conical point; i.e., X contains a cone over a 6 dimensional variety with vertex at w. Each line through a conical point must be type II. Indeed, if some line through w were type I, then the tangent space to D_w at that line would be five- dimensional. By the assumption that w lies on a line through x, there is a type I line through w. Thus D_w contains a component of dimension 5 and so Q is not identically 0.

The variety D_w is, a priori, the union of two varieties $Z_5 \cup Z_6$ where $\dim(Z_i) = i$. We will argue that Z_6 is actually empty. The degree of Z_6 is at most 3, and it cannot be 3 because Q is not identically zero. If the degree were 1, then there would be a linear $\mathbb{P}^6$ worth of lines on X through w which would correspond to a $\mathbb{P}^7$ on X which cannot occur since X is smooth. If the degree were 2, then there would be a linear form L' such that $K = QL'$. The variety Q contains a positive dimensional family of linear $\mathbb{P}^3$'s (see [GH94] Chapter 6.1) and these would sweep out a positive dimensional family of $\mathbb{P}^4$'s on X which is impossible according to the appendix of [BHB06]. Thus, Z_6 is empty. We conclude that D_w is a five dimensional $(2, 3)$ complete intersection in $\mathbb{P}^7$ (but possibly singular). □

Lemma 4.7.5. *Let $D \subset \mathbb{P}^7$ be a $(2, e)$, irreducible complete intersection with $e \geq 2$. If D contains no linear $\mathbb{P}^4$ and no one dimensional family of linear $\mathbb{P}^3$s, then there cannot be a 3 dimensional family of lines through a general point of D.*

Proof. As lines on D are the same as lines on D_{red}, we may assume that D is reduced. Suppose there is a 3 dimensional family of lines through a smooth point of $\xi \in D$. As lines are determined by their tangent directions, the projective tangent plane $P = \mathbb{P}^5$ to D at ξ contains a cone over a threefold. Call this cone C (it is also contained in D).

Consider the quadric Q which cuts out D. Suppose that P is not contained in Q, so that C is contained in $P \cap Q$. Because C is irreducible and four dimensional, it must be a component of this intersection. The intersection is a quadric, so that C is either linear or equal to the quadric. But C cannot be linear because then it would be a $\mathbb{P}^4$ contained in D. The cone C can also not be a quadric in $\mathbb{P}^5$ because such a cone contains at least a 3 dimensional family of $\mathbb{P}^3$s (see [GH94], Chapter 6.1). This implies that P must be contained in Q.

Since P is contained in Q and Q is not reducible, the rank of Q must be either 3, 4, or 5. Let $k \in (3, 4, 5)$. When the rank of Q is k, then Q is the join of a $T = \mathbb{P}^{7-k}$ with a smooth quadric in $\mathbb{P}^{k-1}$ (disjoint from P). In each case, the only linear $\mathbb{P}^5$s on Q are joins of T with the correct dimensional linear spaces contained in the smooth quadric. Let ψ be projection away from T; it is a rational map from D to the smooth quadric. The derivative $d\psi$ at ξ has rank $k - 3$. Indeed, the tangent plane P must contain the linear space $\psi^{-1}(\psi(\xi))$. The image of D then has dimension $k - 3$. Since D is contained in $\psi^{-1}\psi(D)$, it must be equal to it because this inverse image is 5 dimensional. If $k = 3$ then $D = \mathbb{P}^5$. If $k = 4$ then D contains a $\mathbb{P}^4$, the join of $T = \mathbb{P}^3$ with any point in $\psi(D)$. If $k = 5$ then D contains infinitely many $\mathbb{P}^3$s, the join of $T = \mathbb{P}^2$ with any point in the image of $\psi(D)$, which has dimension 2.

Thus each value of k gives a contradiction to our assumptions. As these are the only possible cases, there cannot be a 3 parameter family of lines through a general point of D □

Proposition 4.7.6. *Let X be a smooth cubic hypersurface in $\mathbb{P}^9$. If x is a general point on X and w is an arbitrary point on a an arbitrary line containing x, then there is no 7 dimensional family of 2-planes on X through w.*

Proof. A plane through w corresponds to a line on D_w, the space of lines through w. As above, since lines on D_w are in one to one correspondence with lines on $(D_w)_{red}$, we may suppose D_w is reduced. By Lemma 4.7.4 above, the variety D_w is a 5 dimensional complete intersection in $\mathbb{P}^7$. Suppose by way of contradiction that there is a 7-dimensional family of planes through w and so a 7-dimensional family of lines on D_w. Call this family M, and the universal line over it U. We have the diagram:

$$\begin{array}{ccc} U & \xrightarrow{f} & D_w \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

where the dimension of M is 7, the dimension of U is 8, and the dimension of D_w is 5. If D_w is irreducible, then by Lemma 4.7.5, the fiber over a general point of D_w can have dimension at most 2. Indeed, D_w can contain no $\mathbb{P}^4$ and no one dimensional family of $\mathbb{P}^3$s because X can contain no $\mathbb{P}^5$ and only finitely many $\mathbb{P}^4$'s (see the appendix of [BHB06]). The preimage of the smooth locus, $f^{-1}((D_w)_{sm})$ can have dimension at most 7 then (so is 6 dimensional inside M). Then the 7 dimensional family of lines must be completely contained in the singular locus of D_w. Call $V = ((D_w)_{sing})_{red}$. Note that $\dim(V) \leq 4$ and V is generically smooth. There can be no 3 dimensional family of lines through a general point $v \in V$. If there were, then V would have to contain the entire tangent plane (a $\mathbb{P}^4$) to v at V, which is a contradiction. Then the 7 dimensional family would have to be contained completely in the singular locus of V, which is absurd. This contradiction proves the proposition in this case.

If D_w is reducible, then consider the form of the defining equation for X as in the proof of Lemma 4.7.4. The equation for K can not factor because then X would contain a singular point. Thus the equation for Q must factor as the product of two distinct linear equations; that is, D_w is the union of 2 cubic 5-folds. We leave it to the reader to show that if a (possibly singular) cubic fivefold contains a 3-dimensional family of lines through a general point then it must contain a $\mathbb{P}^4$ or a one dimensional family of $\mathbb{P}^3$'s. The proposition follows. □

Proof of Proposition 4.7.3. If $Z \subseteq ev^{-1}(x)$ denotes the complement of U_x, then $\dim(Z)$ is at most 3. Suppose each curve $C_{y,\sigma,x}$ meets Z. Denote by C_M the universal curve over M, the variety parameterizing the curves $C_{y,\sigma,x}$. We have the diagram

$$\begin{array}{ccc} C_M & \xrightarrow{f} & ev^{-1}(x) \\ \downarrow{\scriptstyle \pi} & & \\ M & & . \end{array}$$

Since every curve meets Z, $f^{-1}(Z)$ maps dominantly onto M. Then $f^{-1}(Z)$ must have dimension at least 11. Suppose that $\dim(Z) = 3$. Then either, i) there is an 8 dimensional family of curves passing through the general point of Z, ii) there is a 9 dimensional family of curves passing through the general point of some surface in Z, iii) there is a 10 dimensional family of curves passing through the general point of a curve on Z, or iv) there is some point on Z through which each curve parameterized by M passes.

Each point on such a curve $C_{y,\sigma,x}$ corresponds to a choice of (σ, l) and a plane containing the residual line m. If the curve $C_{y,\sigma,x}$ passes through $z \in Z$ then the corresponding quadric surface contains the line $l_z \ni x$ (where l_z denotes the line corresponding to the point $z \in Z$). To each curve $C_{y,\sigma,x}$ meeting z, the plane $P \supseteq m$ meets the line l_z since they are contained in the same $\mathbb{P}^3$. The line cannot be contained in the plane because P does not contain x. This describes the map $\{C \in M | C \ni z\} \to \{P | P \cap l_z \neq \emptyset\}$ from curves meeting the point z to planes meeting the line l_z. This map can have at most one dimensional fibers. Otherwise the entire $\mathbb{P}^3 = \mathrm{Span}(x, P)$ would be contained in X and would contain a general line $\sigma \subseteq X$ contradicting Proposition 4.3.21. Note that the fibers will be one dimensional: At a smooth quadric surface containing (σ, l, l_z) the l move in a one dimensional family on the quadric without changing the surface or the corresponding plane (though the line m will “swivel”).

If a point $z \in Z$ is contained in an n dimensional family of curves in M, then the line l_z must intersect an $n-1$ dimensional family of planes contained in X. The claim then follows by a careful analysis of planes and lines contained in X. By part (i) of Proposition 4.3.21, x is contained in a 5 dimensional family of planes in X. By a semi-continuity argument, for a line l containing x, the general point of l must also be contained in a 5 dimensional family of planes. Since l_z contains a general point (namely x) to say that l_z meets an $m > 5$ dimensional family of planes is to say there must be some point on l_z which is contained in an m dimensional family of planes (since by the above discussion, a general point on l_z is contained in a 5 dimensional family).

By Proposition 4.7.6, there is no line containing x which meets a 7 dimensional family of planes. The proposition follows because for $\dim(Z) \leq 3$ and the dimension estimates above, there must be some point $z \in Z$ such that the corresponding line l_z is contained in an 8 dimensional family of curves in M. □

Now that a complete curve is contracted by τ, it is not difficult to show that τ contracts all of $ev^{-1}(x) = D_x$.

Lemma 4.7.7. *For a general $x \in X$, we have that D_x satisfies $H^2(D_x, \mathbb{Z}) = \mathbb{Z}$ and $H_2(D_x, \mathbb{Z}) = \mathbb{Z}$.*

Proof. Since x is generic, we have by Proposition 4.3.21 that D_x is a $(2,3)$ complete intersection in $\mathbb{P}^7$. By the Lefschetz hyperplane theorem, the restriction map on cohomology $H^2(\mathbb{P}^7, \mathbb{Z}) \to H^2(D_x, \mathbb{Z})$ is an isomorphism. As $H^2(\mathbb{P}^7, \mathbb{Z}) = \mathbb{Z}$, the lemma is proved. The same proof works in homology. □

Recall that $U \subseteq ev^{-1}(x) = D_x$ is the largest open subset where the map τ is defined. Call Z the complement of U in X. Recall that $\dim(Z) \leq 3$.

Proposition 4.7.8. *Suppose V is a smooth complete variety and Z is a complete subvariety of codimension at least 2. Denote by $U = V - Z$ the complement. Then $H^2(V, \mathbb{Z}) \to H^2(U, \mathbb{Z})$ is an isomorphism. Similarly $H_2(U, \mathbb{Z}) \to H_2(V, \mathbb{Z})$ is an isomorphism.*

Proof. Define $Z^1 = Z - Z_{sing}$, $V^1 = V - Z_{sing}$. Note that $U = V^1 - Z^1$. We have the

Gysin sequence in cohomology whose relevant portion looks like:

$$H^2_{Z^1}(V^1) \to H^2(V^1) \to H^2(U) \to H^3_{Z^1}(V_1)$$

(all coefficients are understood to be $\mathbb{Z}$). As the codimension of Z^1 in X is $c \geq 2$ we have that $H^2_{Z^1}(V^1) = H^{2-2c}(Z^1) = 0$ and $H^3_{Z^1}(V^1) = H^{3-2c}(Z^1) = 0$ so that $H^2(V^1) \cong H^2(U)$. Now set $Y^2 = Z_{sing}$ and let $Z^2 = Y^2 - Y^2_{sing}$ and $V^2 = V - Y^2_{sing}$. We repeat the same argument above to get that $H^2(V^2) \cong H^2(V^1)$. Repeating the argument implies that $H^2(V) \cong H^2(U)$ (eventually, the set being thrown away has dimension 0).

Thus, $H_2(U)$ has rank 1. By the homology exact sequence of a pair and the Thom isomorphism in homology, the same argument shows that $H_2(U) \cong H_2(V)$. □

Corollary 4.7.9. *For $U \subseteq D_x$ as above, we have that $H^2(U, \mathbb{Z}) = \mathbb{Z}$ and $H_2(U, \mathbb{Z}) = \mathbb{Z}$.*

Proposition 4.7.10. *For a general $x \in X$ as in Proposition 4.3.21, the map τ restricted to $U_x \to Y_x$ coincides with the map ev, that is, Y_x is a point.*

Proof. By Corollary 4.7.9 we know that $H_2(U, \mathbb{Z}) = \mathbb{Z}$. By Proposition 4.7.3, we have that some complete curve C is contracted by τ. This curve determines a homology class which we may write as $d \cdot H$ for some $d > 0$ where H is the positive generator of $H_2(U, \mathbb{Z})$. Of course, every homology class can be written as $e \cdot H$. Clearly then, every multiple of H is also contracted by τ, and τ must map all of U_x to a point. □

Theorem 4.7.11. *Every smooth degree 3 hypersurface $X \subseteq \mathbb{P}^9$ admits 2-twisting surfaces.*

Proof. The vertical tangent bundle for $\tau : \mathcal{C} \dashrightarrow Y$ is the sheaf $\mathcal{D}$ (at least on the open set U where the map τ is defined). By Proposition 4.7.10, the map τ agrees with ev over an open set of X. From this fact, we see that $\mathcal{D}$ has rank 5. The sheaf $\mathcal{D}$ was constructed from positive parts of the pullback of T_{ev} by maps $\phi : \mathbb{P}^1 \to \mathcal{C}$

where $\phi \in \widetilde{M_{2r}}$. Thus, there must be some map $\phi \in \tilde{M}_{2r}$ where $\phi^* T_{ev} = \pi_*(\mathcal{N}_\phi(-\sigma))$ is ample. In other words, we can write $\pi_* \mathcal{N}_\phi(-\sigma) = \bigoplus \mathcal{O}(a_i)$ with each $a_i > 0$. From this we conclude that $H^1(\mathbb{P}^1, \pi_*(\mathcal{N}_\phi(-\sigma))(-2)) = H^1(\mathbb{P}^1, \pi_* \mathcal{N}_\phi(-\sigma - 2F)) = 0$. By the Leray spectral sequence, $H^1(\Sigma, \mathcal{N}_\phi(-\sigma - 2F)) = 0$. The theorem follows. □

Corollary 4.7.12. *Every smooth degree* 3 *hypersurface in* $\mathbb{P}^n$ *where* $n \geq 9$ *admits* 1 *and* 2 *twisting surfaces.*

Proof. We proceed by induction on n where the base case is Theorem 4.5.6 and Theorem 4.7.11. Suppose $Y \subseteq \mathbb{P}^n$ with $n > 9$ is a smooth degree 3 hypersurface. Let X be a smooth hyperplane section of Y. By induction, X admits 1-twisting and 2-twisting surfaces. By Lemma 4.8.4, X admits m-twisting surfaces then for all values of m. Choose m large so that $\Sigma \cong \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(a))$ where $a > 2$ and Σ is m-twisting. From the sequence of inclusions $\Sigma \subseteq X \subseteq Y$ we have the usual normal bundle sequence which fits into the following diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & N_{\Sigma/X} & \longrightarrow & N_{\Sigma/Y} & \longrightarrow & N_{X/Y}|_\Sigma & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{N}_X & \longrightarrow & \mathcal{N}_Y & \longrightarrow & N_{X/Y}|_\Sigma & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \pi^* T_{\mathbb{P}^1} & = & \pi^* T_{\mathbb{P}^1} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}.$$

Here $\mathcal{N}_X$ and $\mathcal{N}_Y$ denote the corresponding relative normal bundles for $\Sigma \subseteq X$ and $\Sigma \subseteq Y$. Since $\Sigma \subseteq X$ is m-twisting, it is also 1 and 2 twisting (Lemma 4.8.4). Note that $N_{X/Y}|_\Sigma \cong \mathcal{O}_\Sigma(1) \cong \mathcal{O}(C + aF)$ where C is the section of minimal degree (i.e. $C^2 = 0$). We have that $H^1(\mathbb{P}^1, \pi_*(\mathcal{N}_X(-C - kF)) = 0$ when $k = 1, 2$ because Σ is k-twisting on X. Also one checks immediately that $H^1(\mathbb{P}^1, \pi_* \mathcal{O}((C + aF) - C - kF)) = 0$

for $k = 1, 2$ and so $H^1(\mathbb{P}^1, \pi_*(\mathcal{N}_X(-C - kF)) = 0$ for $k = 1, 2$ as well. Note also that $R^1\pi_*(\mathcal{N}_X(-C - kF)) = 0$. From the long exact sequence in cohomology associated to the push forward of the middle row, we conclude that Y also admits 1 and 2 twisting surfaces. □

Theorem 4.7.13. *Main Theorem*
Every smooth degree 3 *hypersurface is* $\mathbb{P}^n$ *for* $n \geq 9$ *is strongly rationally simply connected.*

Proof. Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface. We will explain how the existence of 1 and 2 twisting surfaces on X implies that X is strongly rationally simply connected. The methods are taken from the paper [dS06], but the relevant results in their manuscript are scattered throughout the paper making it difficult here to site just one result.

Remark 4.7.14. The moduli space $\overline{\mathcal{M}}_{0,0}(X, e)$ of rational curves on a cubic hypersurface is actually irreducible, see [CS]. However, these spaces of rational curves on degree d hypersurfaces for $d > 3$ can be reducible, but there is always a canonically defined component which in some sense behaves functorially. Because the space of lines through a general point on X (at least if $X \subset \mathbb{P}^n$ is a smooth degree d hypersurface satisfying $n - 2 \geq d$) is irreducible, there is a canonically defined component $M_e \subset \overline{\mathcal{M}}_{0,0}(X, e)$ as discussed in [dS06] Section 3. This component will be referred to as the good component; for its properties, see the location cited. Informally, it is the unique component which parameterizes smoothed out configurations of free curves on X, as well as multiple covers of free curves. Denote by $M_{e,n} \subset \overline{\mathcal{M}}_{0,n}(X, e)$ the unique component dominating M_e. We will continue with the outline of the proof assuming only the existence of the good component, to remain as general as possible.

To remain consistent with the notation of [dS06], a ruled surface Σ on X will be said to have M-class $(e_1 \cdot \alpha, e_2 \cdot \alpha)$ if the fibers are parameterized by points of the good component M_{e_1} and the section of minimal self intersection (classically, the directrix) is parameterized by a point of the good component M_{e_2}. By Theorem 4.5.6 there is a one twisting surface on X and since both fibers and minimal section are mapped to lines (the space of which is irreducible), then these surfaces have M-class $(1\cdot\alpha, 1\cdot\alpha)$. By Lemma 4.8.4, then, there are 1-twisting surfaces on X of M-class $(e_1\cdot\alpha, e_2\cdot\alpha)$ for all $e_1, e_2 \geq 1$. By Theorem 4.7.11, there is a number $e_0 = e_0(n)$ (possibly large) such that there is a 2-twisting surface $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$ of M-class $(1\cdot\alpha, e_0\cdot\alpha)$. To see that this is the case, return to the construction of these surfaces. They were proved to exist by considering smoothed out configurations of glued together conics in the Fano scheme of lines. So fibers of the resulting surface are certainly mapped to lines on X. The minimal section class of Σ is the "smoothed out" section class of the conics which correspond to quadric surfaces. That is, the section class is constructed by smoothing out configurations of free lines on X, and so by definition, is in good component M_{e_0}. By Lemma 4.8.4 again, there is a number e_m, such that there exist m-twisting surfaces of M-class $(1\cdot\alpha, e_m\cdot\alpha)$ on X. From now on we will abbreviate the M type $(c_1\cdot\alpha, c_2\cdot\alpha)$ by (c_1, c_2).

In other words, we are going to prove the following theorem and immediately deduce the main theorem above as a corollary.

Theorem 4.7.15. *Suppose X is a smooth degree d hypersurface in $\mathbb{P}^n$ with $d^2 \leq n$ which satisfies that the space of lines through a general point of X is irreducible. Suppose further that X admits one-twisting surfaces of M-class $(1,1)$ and 2 twisting surfaces of M class $(1, e_0)$ for some e_0. Then X is strongly rationally simply connected.*

Remark 4.7.16. To reiterate, the conditions of the theorem imply that we can speak of the good component $M_e \subset \overline{\mathcal{M}}_{0,0}(X, e)$, that 1 twisting surfaces on X of type (e_1, e_2) exist for all $e_1, e_2 \geq 1$, and that there is a number e_m such that m-twisting surfaces of type $(1, e_m)$ can be found on X.

We prove the theorem in a sequence of steps.

Step 1: For each number $e \geq 2$, consider the good component $M_{e\cdot\alpha,2}$ and the restricted evaluation map: $ev : M_{e\cdot\alpha,2} \to X \times X$. Then a general fiber is rationally connected.

Proof: By the above remark, for each integer k, there is a 1-twisting surface of type $(1, k)$ on X. The proof that the fiber of $ev_e : M_{e\cdot\alpha,2} \to X \times X$ over a general point is rationally connected proceeds by induction on e. The base case is $e = 2$ where we can verify directly that the space of lines through two general points on X is cut out by equations of degrees $(1, \ldots, d, 1, \ldots, d-1)$ in $\mathbb{P}^n$. For a generic choice of a point in $X \times X$, this locus will be smooth, non-empty, Fano, and so rationally connected. We know there exist 1-twisting surfaces of type $(1, 1)$. By Lemma 4.7.19, the MRC quotient of a strong resolution of a fiber of $M_{2,2} \to X^2$ is dominated by $\Delta_{1,1}$. By Lemma 4.7.20 then, because a general fiber of $\Delta_{1,1} \to X^2$ is rationally connected, so is a general fiber of ev_2.

By way of induction, assume $e > 2$ and that the result is known for $e-1$. We will use the existence of a one twisting surface of type $(1, (e-1))$. Using the 1-twisting surface machine, we get again by Lemma 4.7.19 that the image of $\Delta_{1,(e-1)}$ intersects the domain of definition of the MRC fibration of a strong resolution of the fiber. By Lemma 4.7.20, this implies that a general fiber of ev_e is geometrically rationally connected if the fiber of $ev_\Delta : M_\Delta = M_{\alpha,2} \times M_{(e-1)\alpha,2} \to X \times X$ is geometrically rationally connected.

To see the geometric rational connectivity of a general fiber of ev_Δ, consider the projection $p : M_\Delta \to M_{\alpha,2}$. Over the fiber of a general point $(x_1, x_2) \in X^2$, this is exactly the space of pointed lines on X through x_1. As the space of lines through x_1 is rationally connected, so too is this space, F. By [GHS03], to see that the fiber $ev_\Delta^{-1}(x_1, x_2)$ is rationally connected, we can prove that the general fiber of the projection to F is rationally connected. But this is exactly the space of degree $(e-1)$ curves (parameterized by the good component $M_{e-1,2}$) passing through (x_1, x_2) and so is rationally connected by induction. This completes the outline of the proof of rational simple connectedness.

Step 2: We now proceed by induction on m. Let $m > 2$ and fix an integer $e \geq e_m + 2$. Set $e' = e + m$, then a general point of a general fiber of $ev|_M : M_{e'\alpha,m} \subset \overline{\mathcal{M}}_{0,m}(X, e') \to X^m$ is contained in a rational curve which intersects the boundary $\Delta_{1,(e'-1)}$ in a smooth point.

Proof: This follows directly from Lemma 4.7.19 because X admits m twisting surfaces of type $(1, e)$.

Step 3: In fact, the general point of a general fiber of $ev|_M$ is contained in a rational curve intersecting the boundary divisor $\Delta_{2,(e'-2)}$ in a smooth point.

Proof: The proof is similar to Lemma 4.7.17 and Lemma 4.7.19, and in fact follows from these Lemmas using the more degenerate boundary $\Delta_{1,1,(e'-2)}$. To be more precise, consider the three evaluation maps $ev_a : M_{e'\cdot\alpha,m} \to X^m$, $ev_b : M_{1\cdot\alpha,2} \times_X M_{(e'-1)\cdot\alpha,m} \to X^m$, and $ev_c : M_{1\cdot\alpha,2} \times_X M_{1\cdot\alpha,2} \times_X M_{(e'-2)\cdot\alpha,m} \to X^m$. The fibers of these evaluation maps over a common general point form a nested triple of varieties. Lemma 4.7.19 implies that we can connect a general point of a fiber of $ev : M_{e'\cdot\alpha,m+1} \to X^{m+1}$ to a point of $\Delta_{1,(e'-1)}$ along a rational curve (and similarly for $e'-1$). Moreover these boundary points may be taken to be smooth points of the moduli space. The first fact implies that the MRC quotient of a strong desingularization of fiber of ev_a (resp. ev_b) is dominated by the MRC quotient of a strong desingularization of the corresponding fiber of ev_b (resp. ev_c). This is true using the exact same argument found in Lemma 4.7.20. By transitivity, the MRC quotient strict transform of the fiber of ev_c dominates the strong desingularization of the fiber of ev_a. Since the general point of the fiber of ev_c is a smooth point of $M_{e'\cdot\alpha,m+1}$, this MRC quotient is dominated by the transform of $\Delta_{2,e'-2}$. Less formally, given a reduced curve C in class $|F' + (e' - 2)F|$ attached to two additional fibers, we may first smooth out C and one of the fibers (keeping the attachment point with the other fiber fixed) along a $\mathbb{P}^1$; then the resulting curve may also be smoothed out along a $\mathbb{P}^1$ (and all resulting curves may be considered general points of their moduli spaces). The argument above implies that we may in fact do this along a single rational curve.

Step 4: The general fiber of $ev_\Delta : M_{2\cdot\alpha,2} \times_X M_{(e'-2)\cdot\alpha,m} \to X \times X^{m-1}$ is rationally connected.

Proof: Fix a general point $(p, (p_1, \ldots, p_{m-1})) \in X \times X^{m-1})$. By the induction hypothesis, the fiber of $e^{-1}(p_1, \ldots, p_{m-1})$ of $e : M_{(e'-2)\cdot\alpha,(m-1)} \to X^{m-1}$ is rationally connected. We may consider the composition $E : M_{(e'-2)\cdot\alpha,m} \to M_{(e'-2)\cdot\alpha,(m-1)} \to X^{m-1}$ where the first map forgets the last marked point. The general fiber of this composition is generically smooth over $e^{-1}((p_1, \ldots, p_{m-1})$. A conic bundle over a rationally connected variety is rationally connected, so a general fiber of E is also rationally connected. We then conclude that fiber over $E^{-1}(p_1, \ldots, p_{m-1})$ is rationally connected as well. The space $ev_\Delta^{-1}(p, (p_1, \ldots, p_{m-1}))$ projects onto $E^{-1}(p_1, \ldots, p_{m-1})$. The fiber of this projection is the space of conics through p and what may be taken to be a general point on X, p'. By Step 1, this space is rationally connected.

Step 5: A general point of a general fiber of $ev|_M$ is rationally connected.

Proof: By Step 4, a general point of $ev_\Delta : M_{2\cdot\alpha,2} \times_X M_{(e'-2)\cdot\alpha,m} \to X \times X^{m-1}$ is rationally connected. By Step 3, a general point of the fiber of ev_Δ can be connected to a general point of $ev|_M$ along a $\mathbb{P}^1$ (in the fiber). The proof then follows from another application of Lemma 4.7.20 where we take V to be a general fiber of ev_Δ and W to be the corresponding fiber of $ev|_M$.

□

Lemma 4.7.17. *Suppose $f : \Sigma \to X$ is an m twisting surface of type (e_1, e_2) and write $e = e_2 + me_1$. The map f induces a morphism (of stacks):*

$$\overline{\mathcal{M}}_{0,n}(\Sigma, F' + mF) \to \overline{\mathcal{M}}_{0,n}(X, e).$$

Points corresponding to reduced divisors $D \in |\mathcal{O}_\Sigma(F' + mF)|$ with n distinct smooth *marked points are smooth points in $\overline{\mathcal{M}}_{0,n}(X,e)$. Call $U_{m,m+1}$ the open subset of $\overline{\mathcal{M}}_{0,m+1}(\Sigma, F'+mF)$ parameterizing smooth divisors in the corresponding curve class with $m+1$ distinct marked points. Let M_{m+1} be the component of $\overline{\mathcal{M}}_{0,m+1}(X,e)$ containing the image of $\overline{U_{m,m+1}}$. A general point of M_{m+1} is contained in a map $g : \mathbb{P}^1 \to M_{m+1}$ contained in a fiber of $ev : \overline{\mathcal{M}}_{0,m+1}(X,e) \to X^{m+1}$ and intersecting a general point of the image of $\overline{\mathcal{M}}(\Sigma, \tau_{F,F'})$ where τ_{e_1,e_2} corresponds to "combs" consisting of a handle of curve class F', m teeth of class F and one marked point on each tooth and a marked point on the handle.*

Proof. We outline the proof, see [dS06] Lemma 8.3. For a reduced divisor D as above, the normal bundle $N_{D/\Sigma}$ is globally generated. Because $N_{\Sigma/X}$ is globally generated, so is $N_{D/X}$, from which the smoothness statement follows. Because the surface is m-twisting, a general deformation of D is followed by a deformation of Σ (on X), which remains an m twisting surface. By a parameter count, a divisor C which is the union of F' and m distinct fibers (and may be taken general in its moduli space) deforms to a smooth divisor in $|F'+mF|$ while fixing $(m+1)$ points. Since it is moving in its linear system, we may assume it is doing so along a $\mathbb{P}^1$. What's more, the deformation of D may be taken to be a general point of M_{m+1} and we may further assume that a $\mathbb{P}^1$ connects it to a smooth point of the boundary. □

Lemma 4.7.18. *In the Lemma above, we have that M_{m+1} is actually the good component $M_{e\cdot\alpha}$ (here $e = e_2 + me_1$). Moreover, the image of $\overline{\mathcal{M}}(\Sigma, \tau_{e_1,e_2})$ is contained in the boundary of this good component.*

Proof. This follows because all curves on Σ are free. Then there is a curve in M_{m+1} whose irreducible components are free, smooth curves parameterized by the good component $M_{e_i\cdot\alpha,0}$ for various e_i. This characterizes the good component. □

Lemma 4.7.19. *Assume there exist m-twisting surfaces on X of type (e_1, e_2). Writ-*

ing $e = e_2 + me_1$*, a general point of a general fiber of the evaluation map,*

$$ev : M_{e\cdot\alpha,m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X,e) \to X^{m+1}$$

is contained in a rational curve intersecting the image of the boundary

$$M_{e_1\cdot\alpha,2} \times_X M_{(e-e_1)\cdot\alpha,m+1} \to M_{e\cdot\alpha,m+1},$$

in a smooth point of the fiber. Thus, the image $\Delta_{e_1,(e-e_1)}$ *of the boundary map intersects the domain of definition of the MRC fibration of a strong resolution of the fiber and in particular, dominates the MRC quotient of this resolution.*

Proof. This follows almost directly from Lemma4.7.17. Indeed, we already know a general point of the fiber of ev is contained in a rational curve intersecting $\mathcal{M}(\Sigma, \tau_{e_1,e_2})$ in a smooth point. Recall that this locus parameterizes combs C whose handle C_0 has degree e_2 and whose m teeth have degree e_1 (with the appropriate markings). We let B_0 be one of the teeth, marked additionally at the point of attachment, and we let B_1 be the union of the handle and all the other teeth, also marked additionally at this point of attachment. Then the pair (B_0, B_1) is parameterized by a point of $M_{e_1\cdot\alpha,2} \times_X M_{(e-e_1)\cdot\alpha,m+1}$ and the image in $M_{e\cdot\alpha,m+1}$ is exactly the original comb C. In other words, $\mathcal{M}(\Sigma, \tau_{e_1,e_2})$ is contained in $\Delta_{e_1,(e-e_1)}$ and so the rational curve from Lemma 4.7.17 does indeed intersect $\Delta_{e_1,(e-e_1)}$, and it does so in a smooth point. □

Lemma 4.7.20. *Suppose that* $V \subset W$ *are projective varieties satisfying i)* $V \cap W^{nonsing} \neq \emptyset$*. ii)* V *is rationally connected. iii) codim*$(V, W) = 1$*. iv) For a general point* $v \in V$*, there is a* $f : \mathbb{P}^1 \to W$ *such that* $f(0) = v$ *but* $f(\mathbb{P}^1) \not\supseteq V$*. Then* W *is also rationally connected.*

Proof. To prove the Lemma, use the existence of the MRC quotient for a strong resolution $\tilde{W}$ of W which exists by [Kol96]. This is a rational map $\phi : \tilde{W} \dashrightarrow Q$ such that a general fiber of the map is an equivalence class for the relation "being connected by a rational curve on $\tilde{W}$". By definition, there is some open set U of

$\tilde{W}$ such that the restriction of ϕ to U is regular, proper, and every rational curve in $\tilde{W}$ intersecting U is contained in U. Since the resolution is an isomorphism over the smooth locus of W, the strict transform of a rational curve through a generic point of W_{smooth} meeting V now meets $\tilde{V}$. In other words, a general point of $\tilde{W}$ is contained in a rational curve meeting $\tilde{V}$. By the preceding remarks then, $\tilde{V}$ meets the generic fiber of ϕ_U. That is, $\tilde{V}$ meets U and $\phi_U(U \cap \tilde{V})$ is dense in Q. However, since V is rationally connected, so is $\tilde{V}$ so that Q must be a point. This implies that $\tilde{W}$ is also rationally connected, so that W is as well.. For a slightly more formal proof, refer to [dS06] Lemmas 8.5 and 8.6. □

Remark 4.7.21. A Different Method?

Having created this family of curves M on $ev^{-1}(x)$ for a general $x \in X$, one could apply the method of forming an algebraic quotient as follows: Take the closure of M in the appropriate Hilbert Scheme, call it $\overline{M}$. Then we have the following diagram:

$$\begin{array}{ccc} \overline{C} & \xrightarrow{f} & ev^{-1}(x) \\ \downarrow{\scriptstyle \pi} & & \\ \overline{M} & & \end{array}$$

Here $\overline{C}$ is the restriction of the universal object over the Hilbert Scheme. Since both the maps π and f are proper, the method explained in Kollár's book [Kol96] gives an open set $U \subset ev^{-1}(x)$, a variety Y and a proper map $g : U \to Y$. This map has the property that its fibers are equivalence classes under the relation of "being connected by curves in $\overline{M}$". In other words, if two points can be connected by a chain of curves from $\overline{M}$ then they map to the same point of Y. The advantage of this method is that the map g is proper, so that all of the curves $C_{y,\sigma,x}$ must be completely contained in U (since they are contracted by g by construction). The same argument applied above will show that Y must be a point. Now however, it is not clear how to proceed. Simply because two general points are connected by chains of curves from M, we are not able to (immediately) conclude that given a point $y \in \mathcal{C}$, then the positive parts of $\phi \in \tilde{M}_{2r}$ passing through y point in "all directions" at y.

We give an example to show what could go wrong. Suppose $X = \mathbb{P}^2$ and let M be the set of all lines through a fixed point $p \in X$. The algebraic method will give that the quotient is a point, because any two points in X can be connected by a chain of curves in M (at most 2 clearly). Applying the distribution method though, we will see that the quotient is a $\mathbb{P}^1$ because at a general point of X the curves in M only point in 1 direction. The point x is a singular point for the associated distribution.

It is not clear how to avoid such a situation in the case in which we are interested. A solution to this problem though, would imply the result for all degree d smooth hypersurfaces in $\mathbb{P}^{d^2}$.

4.8 Appendix : Twisting Surfaces

The information in this section is all contained in [dS06]. Due to the central nature it plays in the work above, the main definitions and results are recorded here for convenience/completeness.

A ruled surface is a map $\pi : \Sigma \to \mathbb{P}^1$ such that each fiber is $\mathbb{P}^1$. Ruled surfaces are well known to be isomorphic to projective bundles $\Sigma \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-h))$ (see [Har77] V.2). The integer h will be called the h-type of the ruled surface. Denote by F the class of a fiber and E the curve class with minimal self intersection, $E^2 = -h$. Denote by F' the divisor class $E + hF$. It is the unique curve class such that $F' \cdot E = 0$. If $X \subset \mathbb{P}^n$ is a hypersurface, a map $h : \Sigma \to X$ induces a morphism $(\pi, h) : \Sigma \to \mathbb{P}^1 \times X$. When (π, h) is finite, the vertical normal sheaf $\mathcal{N}_h$ is defined to be Coker $(T_\Sigma \to (\pi, f)^* T_{\mathbb{P}^1 \times X})$.

Suppose there is a ruled surface on X and a curve class in the ruled surface. Given a deformation of the curve in X, when is there a deformation of the surface which contains the deformation of the curve? The following answer motivates the definition of twisting surfaces.

Lemma 4.8.1. *([dS06] 7.4) Let $h : \Sigma \to X$ be a ruled surface as above such that (π, h)*

is finite. Suppose that $\mathcal{N}_h$ is globally generated and that $H^1(\Sigma, \mathcal{N}_h(-F' - nF)) = 0$ for some positive integer n. Free curves on Σ map to free curves on X. If D is a reduced curve in $|\mathcal{O}(F' + nF)|$ then for every infinitesimal deformation of D in X there is an infinitesimal deformation of Σ in X containing the given deformation of D.

Definition 4.8.2. For an integer $n > 0$, a ruled surface $\pi : \Sigma \to \mathbb{P}^1$ with a map $h : \Sigma \to X$ is a n-twisting surface in X if
1) h^*T_X is globally generated.
2) The map (π, h) is finite and $H^1(\Sigma, \mathcal{N}_h(-F' - nF)) = 0$.

A twisting surface $h : \Sigma \to X$ is said to be of class (β_1, β_2) if $\beta_1 = h_*F$ and $\beta_2 = h_*F'$. On a hypersurface $X \subset \mathbb{P}^n$, these curve classes are identified with integers.

Remark 4.8.3. The work above concerns itself with the existence of twisting surfaces of h-type 0. These are surfaces which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, the minimal curve class is identified with sections of the map π and $F' = E$.

The existence of 1-twisting surfaces of a given class implies the existence of 1-twisting surfaces of "larger" classes. Similarly, the existence of 1 and 2 twisting surfaces implies the existence of m-twisting surfaces for all integers $m > 0$.

Lemma 4.8.4. *([dS06] Lemma 7.6 and Corollary 7.7)*

- *Suppose that $f : \Sigma \to X$ is a 1-twisting of h-type 0 and class (a, b). Then for every pair of positive integers (d_1, d_2) there is a 1-twisting surface of h-type 0 and class $(d_1 a, d_2 b)$.*

- *Suppose that $f_1 : \Sigma_1 \to X$ is a 1-twisting surface of h-type 0 and class (a, b) and $f_2 : \Sigma_2 \to X$ is a 2-twisting surface of h-type 0 and class (a, c). Further, suppose that f_1 and f_2 map their respective fiber classes to points parameterized by the same irreducible component of $\overline{\mathcal{M}}_{0,0}(X, a)$. Then for every positive integer m*

and every non-negative integer r *there exists an* m*-twisting surface* $f : \Sigma \to X$ *of* h*-type 0 and class* $(a, rb + (m-1)c)$*. Moreover, the restriction of* f *to the fiber* F *parameterizes curves in the same components as* f_1 *and* f_2 *above.*

- *For every* $1 \leq l \leq n$*, every* n *twisting surface is also* l *twisting.*

Remark 4.8.5. The 1-twisting surfaces produced in Section 4.5 are quadric surfaces with h-type 0 and class $(1, 1)$.

Bibliography

[BB63] A. Białynicki-Birula, *On homogeneous affine spaces of linear algebraic groups*, Amer. J. Math. **85** (1963), 577–582. MR MR0186674 (32 #4132)

[Beh06] Roya Beheshti, *Lines on projective hypersurfaces*, J. Reine Angew. Math. **592** (2006), 1–21. MR MR2222727 (2007a:14009)

[BG71] Spencer Bloch and David Gieseker, *The positivity of the Chern classes of an ample vector bundle*, Invent. Math. **12** (1971), 112–117. MR MR0297773 (45 #6825)

[BHB06] T. D. Browning and D. R. Heath-Brown, *The density of rational points on non-singular hypersurfaces. II*, Proc. London Math. Soc. (3) **93** (2006), no. 2, 273–303, With an appendix by J. M. Starr. MR MR2251154 (2007h:11042)

[BM96] K. Behrend and Yu. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), no. 1, 1–60.

[CG72] C. Herbert Clemens and Phillip A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356. MR MR0302652 (46 #1796)

[CHS08] Izzet Coskun, Joe Harris, and Jason Starr, *The effective cone of the Kontsevich moduli space*, Canad. Math. Bull. **51** (2008), no. 4, 519–534. MR MR2462457

[CR94] M. C. Chang and Z. Ran, *Dimension of families of space curves*, Compositio Math. **90** (1994), no. 1, 53–57. MR MR1266494 (95b:14020)

[CS09] Izzet Coskun and Jason Starr, *Rational curves on smooth cubic hypersurfaces*, electronic preprint: http://www.math.uic.edu/ coskun/cubicsjasonb.pdf, 2009.

[Deb01] Olivier Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001.

[dHS08] Johan de Jong, Xuhua He, and Jason Starr, *Families of rationally simply connected varieties over surfaces and torsors for semisimple groups*, arXiv, math.AG/0809.5224, 2008.

[dJO97] A. J. de Jong and F. Oort, *On extending families of curves*, J. Algebraic Geom. **6** (1997), no. 3, 545–562. MR MR1487226 (99f:14028)

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109. MR MR0262240 (41 #6850)

[DM98] Olivier Debarre and Laurent Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. **312** (1998), no. 3, 549–574. MR MR1654757 (99j:14048)

[dS06] Johan de Jong and Jason Starr, *Low degree complete intersections are rationally simply connected*, electronic preprint : http://www.math.sunysb.edu/∼jstarr/papers/, 2006.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR MR1322960 (97a:13001)

[FP97] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96.

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR MR1644323 (99d:14003)

[GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original. MR MR1288523 (95d:14001)

[GHS03] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic).

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 222. MR MR0217084 (36 #177b)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[Har92] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992, A first course. MR MR1182558 (93j:14001)

[HM98] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR MR1631825 (99g:14031)

[HRS04] Joe Harris, Mike Roth, and Jason Starr, *Rational curves on hypersurfaces of low degree*, J. Reine Angew. Math. **571** (2004), 73–106. MR MR2070144 (2005e:14067)

[HS05] Joe Harris and Jason Starr, *Rational curves on hypersurfaces of low degree. II*, Compos. Math. **141** (2005), no. 1, 35–92. MR MR2099769 (2006c:14039)

[HT06] Brendan Hassett and Yuri Tschinkel, *Weak approximation over function fields*, Invent. Math. **163** (2006), no. 1, 171–190. MR MR2208420 (2007b:14109)

[HT09] ———, *Weak approximation for hypersurfaces of low degree*, Proceedings of Symposia in Pure Mathematics **80** (2009), 937–955.

[Iza99] Elham Izadi, *A Prym construction for the cohomology of a cubic hypersurface*, Proc. London Math. Soc. (3) **79** (1999), no. 3, 535–568. MR MR1710164 (2000i:14064)

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR MR1440180 (98c:14001)

[KSCT07] Stefan Kebekus, Luis Solá Conde, and Matei Toma, *Rationally connected foliations after Bogomolov and McQuillan*, J. Algebraic Geom. **16** (2007), no. 1, 65–81. MR MR2257320 (2007m:14047)

[LG01] V. Lakshmibai and N. Gonciulea, *Flag varieties*, expanded ed., Editeurs Des Sciences et Des Arts, Travaux en Cours, vol. 63, Hermann, 2001, Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[Res] Bruce Resnick, *Patterns of dependence among powers of polynomials*, electronic preprint: http://www.math.uiuc.edu/ reznick/paper41.pdf.

[Ser06] Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006. MR MR2247603 (2008e:14011)

[She09] Mingmin Shen, *Foliations and rational connectedness in positive characteristic*, Submitted., 2009.

[Tse36] C. Tsen, *Quasi-algebraische-abgeschlossene funktionenkorper*, J. Chinese Math. **1** (1936), 81–92.

[Voi07] Claire Voisin, *Hodge theory and complex algebraic geometry. I*, english ed., Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2007, Translated from the French by Leila Schneps. MR MR2451566