II Kähler / projective

The Kodaira problem.

Characterization Theorem: (Kodaira) A compact complex manifold $X$ is projective iff

1. Kähler form $\omega$ on $X$ with $[\omega] \in H^2(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$

$[\omega] = c_h$ class of the closed 2-form $\omega$. 
Given $X = \text{Kähler}$ cplc

then the set $\{w\}$, $w$ a Kähler form of Kähler classes is an open convex cone (the Kähler cone)

$K(X) \subset H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$ sub var.space
Here $\mathcal{U}^{1,2}(X)_{\text{IR}} := H^1(X, \Omega^2_{\text{IR}})$

is the set of classes representable
by a closed real $1$-$2$-form.

The set of rational cohomology classes
$H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R})$
is dense, but

$K(X) \subset H^{1,1}(X)_{\text{IR}} \subsetneq H^2(X, \mathbb{R})$

(in general).

and the vector space $H^{1,1}(X)_{\text{IR}}$
may well not contain any
rational cohomology class.

On the other hand,
when $X = X_0$ deforms, say

$(X(t))_{t \in \mathbb{R}} = \text{family of}$

structures.
on fixed differ. mfd $X$, $H^{1,1}(X_t)_{\mathbb{R}} = H^2(X_t, \mathbb{R}) = H^2(X, \mathbb{R})$
deforms differentially with (c.f.),
(As long as $X_t$ remains Kähler)
Assume it deforms enough
so that
$U \subset K(X_t)$ contains $t \in \mathbb{R}$
an open set in $H^2(X, \mathbb{R})$
then this open set contains a rational coh. class $\eta$
$\Rightarrow$ some deformation $X_t$ is projective.
Examples:

1. $X = \text{complex torus}$
2. $X = \text{hyperkähler mfd}$

$\Rightarrow$ projective deformations $X_t$ of $X$ are dense in the local universal family of delas of $X$.

In fact, there is the following infinitesimal criterion for density of proj. delas:

Prop. Assume $X$ is unobstr.

and for some Kähler class $\langle \omega \rangle \in H^1(S^2 X)$,
The map:
\[ \gamma : H^1(\mathcal{O}_X) \to H^2(\mathcal{O}_X) \]
is surjective. (Here \(\gamma\) is contraction + product.)

Then proj. degs of \(X\) are dense in the local univ. family of degs of \(X\).

Use Griffiths' description of inf. var. of Hodge structure to see that assumption (\(\Leftarrow\)) is submersive at \((0, [\omega])\).
\[ H^{m+n}(X_t)_{IR} = H^2(X_t, IR) \cap H^{m+n}(X_t) \]
\[ = H^2(X_t, IR) \cap F^*H^2(X_t), \]
where \( F^*H^2(X_t) = H^{2\ast}(X_t) \oplus H^1(X_t) \).

Thus
\[ U_{\gamma} H^{m+n}(X_t)_{IR} \rightarrow H^2(X_t, IR) \]
is submersive.

If
\[ U_{\gamma} F^*H^2(X_t) \rightarrow H^2(X_t, \mathbb{C}) \]
is submersive.

Griffiths computes the lift of this last map.
Here $0 \in B \iff x_0 = x$ (8)

$T_{B,0} = H^1(x, Tx)$

$(x_t)_{t \in \mathbb{B}}$ universal.

$\sqrt{B}$ is smooth

Now submersive $\Rightarrow$ open and the previous criterion applies.

Budzilov uses this criterion to reprove Kodaira's theorem in case of unobstructed surfaces.

Thm (Kodaira): A compact Kähler surface admits arbitrarily small deformations which are projective.
Kodaira's proof is by classification.

Budhath's proof is for unobstructed surfaces:

He proves that if \( S \) is such that \((w)\) Kähler and \( \mathfrak{g} \in H^0(K_S) \) then \( \mathfrak{g} (w) = 0 \) in \( H^1(R_S(K_S)) \)

Then \( S \) is projective.

(by duality this \( \Rightarrow \) to \((w)\) non surj)

The Kodaira problem:

What about higher dimension?
Q: Let $X$ be a compact Kähler manifold. Does $X$ deform to a projective mfd $Y$?

If we have such a deformation $X \cong \pi \downarrow B$

$\pi$ = smooth proper
$B$ = connected analytic space

$X = X_0$, $Y = X_b$, for some $b$

Then $X \cong Y$

diffeo

because $B$ is path connected.
and over paths the family $X$ is differentiably trivial.

Thus, can ask weaker questions:

Qn: Is a compact Kähler manifold diffeomorphic homeomorphic homotopy equivalent to a projective complex manifold?
The last question is very natural in the sense as the known topological restrictions on projective m.f.d.s come from Hodge theory, hence are satisfied as well by Kähler m.f.d.s: In fact they can be used to distinguish topologically the class of pt Kähler m.f.d.s from that of symplectic ones.
Topological restrictions on Kähler manifolds coming from Hodge $\Theta$:

a) Hodge decomposition

$$\Rightarrow b_{2i+1} \text{ is even}$$

b) Hodge decomposition is compatible with $U : \text{H}^k(X) \otimes \text{H}^l(X) \to \text{H}^{k+l}(X)$ in the sense that

$$\text{H}^{p,1}(X) \otimes \text{H}^{r,5}(X) \to \text{H}^{p+r,6}(X)$$

$$\Rightarrow \text{restrictions on cohomology ring}$$

c) Hard Lefschetz theorem $\Rightarrow$
$b_{2i}(x)$, $2i \leq n$ are increasing with $i$.

$L$ is injective on $H^x(x)$, $x \leq n-1$.

Similarly, $b_{2i+1}(x)$, $2i+1 \leq n$ are increasing with $i$.

d) Hodge index:

Case of surfaces:

$\langle \cdot, \cdot \rangle$ on $H^2(S, \mathbb{R})$.

$q^+_i$ positive on $\langle \mathbf{w} \rangle + (H^{2,0}(S) \otimes H^{0,2})$

even dim.

$q^-_i$ negative on $H^{1,1}(S, \mathbb{R})$

$\Rightarrow b^+_i$ is odd.
Theorem: (V.03) In any dimension $n \geq 4$, if compact Kähler mld's, which do not have the homotopy type (the cohomology ring) of a projective mld.

In dimension $n \geq 6$, if simply connected such examples.
Another version of Kodaira problem:

The examples are very simple geometrically.

1. Start from a complex torus + blow it up along complex submanifolds.

(Simply connected case)

Start from $K = \text{Kummer variety of a complex torus}$ + blow-up complex submanifolds in $K \times K$.

NB: the complex torus is very special.
These examples are bimeromorphically equivalent to either a torus, or a self-product $K \times K$, $K$ = Kummer mfd,
which satisfy themselves the property of deforming to a projective mfd.
Leads to:
An (Campana, Biquard, Yau) Does a compact Kähler mfd
admit a smooth bimeromorphic model, which deforms to a projective mfd?

$\left(\text{A birational version of Kodaira's problem}\right)$

Answer is again NO,

for topological reasons

Thm (V. 04) In dimension $\geq 10$, if compact Kähler mfd's, no smooth birational model of which has the homotopy type of a projective mfd.
Outline of the topological obstruction for $X = \text{compact Kähler}$ to admit a projective complex structure:

The ring $H^*(\mathbb{C}, X)$ is given. If $X$ admits a projective...
(c) If the ring structure on $H^*(X)$ is rich enough, compatibility with $\nu$ may force the existence of deg 2 Hodge classes. (integral deg 2 classes of type $(1, 1)$)

(e) They in turn may force the existence of endos of Hodge structures on $H^1$ or $\Omega^2$

(??) Certain endo’s prevent the existence of polarization
complex structure

Hodge structures of weight $k$ on each $H^k(X, \mathbb{Q})$ compatible with cup product,

AND integral polarization on the Hodge structure on $H^n$ or $H^E$
The starting point of all these constructions is: The existence of certain endomorphisms \( \Phi \) acting on a complex torus \( T \) may prevent \( T \) to be projective (\( \equiv \) illustration of point). 

Construction: \( \Phi \) maps \( \Lambda \subset \mathbb{C}^n \) lattice, \( \Phi: \Lambda \rightarrow \Lambda \) endomorphism.
Assume:
The eigenvalues of $\mathcal{A}$ are all distinct, and none is real.

Choose $\lambda_1, \ldots, \lambda_n$, $n$ eigenvalues pairwise not conjugate.

Consider $P_{1,0} = \text{eigenspace of } \mathcal{A}$ associated to $\lambda_1, \ldots, \lambda_n$. Thus

\[ P_{1,0} \oplus P_{1,0} \neq P_{1,0} \]
\[ T = \frac{T^*}{\operatorname{det}(T)} \phi \]

induces \( \Phi_T : T \to T \).

**Prop.:** Assume \( n \geq 2 \)

and: \( \operatorname{Gal}(Q(\sqrt[n]{1}, \sqrt[n]{2}, \ldots, \sqrt[n]{n} : Q) \)

acts as \( \Gamma_n \) on eigenvalues of \( \phi \).

Then \( T \) is not projective.

**Proof:** One proves the stronger statement

\( \operatorname{NS}(T^*) = 0 \), where
\[ \text{NS}(T) = \sum \mathcal{O}(\ell), \quad \ell = \text{hos} \]

line bundle on \( T \)

\[ = \sum H^2(T, \mathbb{Z}) \cup H^{1,1}(T). \]

Lefschetz

Then on \( \mathcal{A}, \mathcal{M} \)-classes

In fact: \( H^2(T, \mathbb{Q}) = \Lambda^2 \mathfrak{p}^* \mathfrak{q} \)

and \( \varphi_T^* = \Lambda^2 \varphi \mathfrak{c} \rightarrow H^2(T, \mathbb{Q}) \)

Assumption on

\( \text{Gal} (\mathbb{Q}(\lambda_i, \bar{\lambda}_i) : \mathbb{Q}) \rightarrow \)

The action of \( \Lambda^2 \varphi \) is irreducible.

But \( \Lambda^2 \varphi = \varphi^* \) leaves \( \text{NS}(T) \otimes \mathbb{Q} \) stable.
Thus either $NS(T) \subset 0$ \hfill (23)

or $NS(T) \subset H^2(T \cup \partial).$

as $H^2(T) \not\subset 0$ because $n \geq 2.$ and

$NS(T) \subset H^1 \cup \partial,$

the second case is impossible.

$\Rightarrow$ $NS(T) \subset \partial = 0$ \hfill (5)