Hodge theory and the topology of compact Kähler and complex projective manifolds

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0 Introduction

The goal of these notes is to introduce to Hodge theory as a powerful tool to understand the geometry and topology of projective complex manifolds. In fact, Hodge theory has developed in the context of compact Kähler manifolds, and many of the notions discussed here make sense in this context, in particular the notion of Hodge structure. We will not explain in these notes the proofs of the main theorems (the existence of Hodge decomposition, the Hodge-Riemann bilinear relations), as this is well-known and presented in [44] I, [22], but rather give a number of applications of the formal notion of Hodge structures, and other objects, like Mumford-Tate groups, Hodge classes, which can be associated to them. So let us in this introduction roughly explain how this works.

For a general Riemannian manifold (X, g), one has the Laplacian Δ_d acting on differential forms, preserving the degree, defined as

$$\Delta_d = dd^* + d^*d,$$

where $d^* = \pm * d*$ is the formal adjoint of d (here $*$ is the Hodge operator defined by the metric induced by g on differential forms and the Riemannian volume form, by the formula

$$\alpha \wedge *\beta = <\alpha, \beta> Vol_g.)$$

Hodge theory states that if X is compact, any de Rham cohomology class has a unique representative which is a $\mathcal{C}^\infty$ form which is both closed and coclosed ($d\alpha = d(*\alpha) = 0$), or equivalently harmonic ($\Delta_d\alpha = 0$).

On a general complex manifold endowed with a Hermitian metric, the d-operator splits as

$$d = \partial + \overline{\partial},$$

where each operator preserves (up to a shift) the bigradation given by decomposition of $\mathcal{C}^\infty$-forms into forms of type (p, q). It is always possible to associate to ∂ and $\overline{\partial}$ corresponding Laplacians Δ_∂ and $\Delta_{\overline{\partial}}$ which for obvious formal reasons will also preserve the bigradation and even the bidegree. However, it is not the case in general that Δ_d preserves this bigradation. It turns out that, as a consequence of the so-called Kähler identities, when the metric h is Kähler, one has the relation

$$\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}},$$

which implies that Δ_d preserves the bidegree.

This has for immediate consequence the fact that each cohomology class can be written as a sum of cohomology classes of type (p, q), where cohomology classes of type (p, q) are defined as those which can be represented by closed forms of type (p, q).

Another important consequence of Hodge decomposition is the fact that the so-called L-operator of a symplectic manifold, which is given by cup-product with the class of the symplectic form, satisfies the hard Lefschetz theorem in the compact Kähler case, where one takes for symplectic form the Kähler form. This leads to the Lefschetz decomposition and to the notion of (real) polarization of a Hodge structure.

So far, the theory sketched above works for general compact Kähler manifolds, and indeed, standard applications of Hodge theory are the fact that there are strong topological restrictions for a complex compact manifold to be Kähler. Until recently, it was however not clear that Hodge theory can also be used to produce supplementary topological restrictions for a compact Kähler manifold to admit a complex projective structure.

Projective complex manifolds are characterized among compact Kähler manifolds by the fact that there exists a Kähler form which has rational cohomology class. As this criterion depends of course of the complex structure, it has been believed for a long time that a small perturbation of the complex structure would allow to deform the Kähler cone to one which contains a rational cohomology class, and thus to deform the complex structure to a projective one. This idea was supported by the fact that in symplectic geometry there is no obstruction to do this, and more importantly by a theorem of Kodaira, which states that this effectively holds true in the case of surfaces.

We recently realized however in [42] that Hodge theory can be used to show that there are indeed non trivial topological restrictions for a compact Kähler manifold to admit a projective complex structure. One of the goals of these notes is to introduce the main notions needed to get the topological obstruction we discovered in this paper. The key notion here is that of polarized rational Hodge structure, which we develop at length before coming to our main point.

We tried in the course of these notes to give a number of geometric examples and other applications of the notions related to Hodge theory.

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1 Hodge structures

1.1 The Hodge decomposition

1.1.1 The Frölicher spectral sequence

Let X be a complex manifold of dimension n. The holomorphic de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{\partial} \Omega_X \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Omega_X^n \to 0$$

is a resolution of the constant sheaf $\mathbb{C}$ on X, by the holomorphic Poincaré lemma. Thus its hypercohomology is equal to the cohomology of X with complex coefficients:

$$\mathbb{H}^k(X, \Omega_X^{\cdot}) = H^k(X, \mathbb{C}). \tag{1.1}$$

It is interesting to note that if X is projective, the holomorphic de Rham complex is the analytic version of the algebraic de Rham complex, and thus by the GAGA principle [38], its hypercohomology is equal to the hypercohomology of the algebraic de Rham complex, although the last one is computed with respect to the Zariski topology, in which the constant sheaf has no cohomology. In fact the algebraic de Rham complex is not at all a resolution of the constant sheaf in the Zariski topology (cf [4] for investigation of this).

In any case, the holomorphic de Rham complex admits the naïve filtration

$$F^p\Omega^\cdot_X = \Omega_X^{\geq p} := 0 \to \Omega_X^p \to \ldots \to \Omega_X^n \to 0,$$

and there is an induced spectral sequence (called the Frölicher spectral sequence)

$$E_i^{p,q} \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\cdot_X) = H^{p+q}(X, \mathbb{C}),$$

satisfying the following properties:

1. $E_1^{p,q} = H^q(X, \Omega_X^p)$, $d_1 = \partial : H^q(X, \Omega_X^p) \to H^q(X, \Omega_X^{p+1})$.
2. $E_\infty^{p,q} = Gr_F^p H^{p+q}(X, \mathbb{C})$, where the filtration F is defined on $H^k(X, \mathbb{C})$ by

$$F^p H^k(X, \mathbb{C}) := Im\,(\mathbb{H}^k(X, \Omega_X^{\geq p}) \to \mathbb{H}^k(X, \Omega^\cdot_X) = H^k(X, \mathbb{C})). \tag{1.2}$$

This filtration will be called the Hodge filtration only for projective or Kähler compact manifolds, because of the following example 1:

Example 1 If X is affine, then all cohomology groups $H^q(X, \Omega_X^p)$ vanish for $q \geq 1$. Thus the spectral sequence degenerates at E_2, and the filtration is trivial:

$$F^p H^k(X, \mathbb{C}) = 0,\ p > k,\ F^k H^k(X, \mathbb{C}) = H^k(X, \mathbb{C}).$$

In the case of a quasi-projective complex manifold, this filtration is not the Hodge filtration defined by Deligne [13], which necessitates the introduction of the logarithmic de Rham complex on a projective compactification of X with normal crossing divisor at infinity.

Example 2 If X is a compact complex surface, the spectral sequence above degenerates at E_1. This implies for example that holomorphic 1-forms are closed (easy: if $\partial\alpha \neq 0$, then $\int_X \partial\alpha \wedge \overline{\partial\alpha} \neq 0$, contradicting the fact that the integrand is exact.)

Furthermore, we have an exact sequence:

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0. \tag{1.3}$$

In higher dimensions, it is possible to construct examples of compact complex manifolds with non closed holomorphic 1-forms (see [3]).

A complex manifold is Kähler if it admits a Hermitian metric, written in local holomorphic coordinates as

$$h = \sum h_{ij} dz_i \otimes d\overline{z_j}$$

satisfying the property that the corresponding real $(1,1)$-form

$$\omega := \frac{i}{2} \sum h_{ij} dz_i \wedge d\overline{z_j}$$

is closed. There are a number of other local characterizations of such metrics. The most useful one is the fact that at each point, there are holomorphic local coordinates centered at this point, such that the metric can be written as

$$h = \sum_i dz_i \otimes d\overline{z_i} + O(\mid z \mid^2).$$

The main consequence of Hodge theory can be stated as follows:

Theorem 1 *If X is a compact Kähler manifold, then the Frölicher spectral sequence degenerates at E_1.*

For a long time, the only known proof of this statement, even in the case of projective complex manifolds, was transcendent, making use of the theory of harmonic forms. In the paper [16], Deligne and Illusie give a beautiful proof of this statement by reduction to positive characteristic.

The theory of spectral sequences allows to rephrase the theorem above by saying that for X a projective complex of compact Kähler manifold the Hodge filtration F on cohomology has the property that

$$Gr_F^p H^{p+q}(X, \mathbb{C}) \cong H^q(X, \Omega_X^p).$$

In particular, it implies the following:

Proposition 1 *For X a projective complex or compact Kähler manifold, and $\alpha \in H^q(X\Omega_X^p)$, there exists a closed k-form, $k = p + q$*

$$\eta = \eta^{k,0} + \eta^{k-1,1} + \ldots + \eta^{p,q}$$

such that the Dolbeault class of the $\overline{\partial}$-closed form $\eta^{p,q}$ is equal to α.

Hodge theory however provides the following stronger statement:

Theorem 2 *For X a projective complex of compact Kähler manifold, and $\alpha \in H^q(X, \Omega_X^p)$, there exists a closed (p,q)-form $\eta^{p,q}$ such that the Dolbeault class of the $\overline{\partial}$-closed form $\eta^{p,q}$ is equal to α. Furthermore, the de Rham cohomology class of $\eta^{p,q}$ is uniquely determined by α. In other words, denoting $H^{p,q}(X)$ the subspace of $H^{p+q}(X, \mathbb{C})$ consisting of de Rham cohomology classes of closed (p,q)-forms, the natural map*

$$H^{p,q}(X) \to H^q(X, \Omega_X^p)$$

is an isomorphism.

This statement is not implied by the degeneracy at E_1 of the Frölicher spectral sequence. Indeed, consider the case of Hopf surfaces $S = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$, where $k \in \mathbb{Z}$ acts by scalar multiplication by λ^k, for some given $\lambda \in \mathbb{C}^*$, $\mid \lambda \mid \neq 1$. These surfaces have $H^1(S, \mathbb{C}) = \mathbb{C}$ and $H^0(S, \Omega_S) = 0$. Indeed, if $0 \neq \alpha$ is holomorphic 1-form, it is closed, and $\overline{\alpha}$ is not $\overline{\partial}$-exact, because there are no non-constant pluriharmonic functions on S. Thus we would also have $H^1(S, \mathcal{O}_S) \neq 0$ and the exact sequence (1.3) would imply $b_1(X) \geq 2$.

As we know that $H^0(S, \Omega_S) = 0$, the exact sequence (1.3) implies now that $H^1(S, \mathcal{O}_S) \neq 0$. On the other hand, there are no non zero closed form of type $(0,1)$, because if η is such a form, $\overline{\eta}$ is holomorphic. Thus we see that Theorem 2 is stronger than the degeneracy at E_1 of the Frölicher spectral sequence.

A consequence of Theorem 2 is known as Hodge symmetry:

Theorem 3 *If X is a compact Kähler manifold, one has*

$$h^{p,q}(X) = h^{q,p}(X),$$

where $h^{p,q} := rk\, H^q(X, \Omega_X^p)$. Furthermore there exists a canonical $\mathbb{C}$-antilinear isomorphism

$$H^q(X, \Omega_X^p) \cong H^p(X, \Omega_X^q).$$

Indeed, the map $\eta^{p,q} \mapsto \overline{\eta^{p,q}}$ clearly induces an $\mathbb{C}$-antilinear isomorphism between the space $H^{p,q}(X)$ of de Rham cohomology classes of closed forms of type (p, q) and the space $H^{q,p}(X)$ of de Rham cohomology classes of closed forms of type (q, p). Thus the result follows from the canonical isomorphisms $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.

1.1.2 Hodge filtration and Hodge decomposition

Consider the Hodge filtration $F^pH^k(X, \mathbb{C})$ on the cohomology of a compact Kähler manifold. Recalling the spaces $H^{p,q}(X)$ introduced above, we have:

$$F^pH^k(X, \mathbb{C}) = \oplus_{r+s=k, r\geq p} H^{r,s}(X). \tag{1.4}$$

Indeed, the sum must be a direct sum, because if the class of $\sum_{r+s=k,r\geq p} \alpha^{r,s}$ is 0, with each $\alpha^{r,s}$ closed of type (r, s) then $\sum_{r,s} \alpha^{r,s} = d\phi$, and writing $\phi = \sum_{r'+s'=k-1} \phi^{r',s'}$, we get for type reasons :

$$\alpha^{r,s} = \overline{\partial}\phi^{r,s-1} + \partial\phi^{r-1,s},\ r + s = k.$$

The form $\partial\phi^{r-1,s}$ is then d-closed and ∂-exact, thus it is d-exact by Theorem 2 applied to its complex conjugate. Similarly the form $\overline{\partial}\phi^{r,s-1}$ is d-closed and $\overline{\partial}$-exact, thus it is d-exact by Theorem 2. Hence each $\alpha^{r,s}$ is exact.

Next the right hand side is certainly contained in $F^pH^k(X, \mathbb{C})$, because a closed form of type (r, s), $r \geq p$, $r + s = k$ defines a class in $\mathbb{H}^k(X, \Omega_X^{\geq p})$ using the total complex of the Dolbeault resolution. Finally, to see the reverse inclusion, we use Theorem 2. Let $\alpha \in F^pH^k(X, \mathbb{C})$; its image $\overline{\alpha}$ in

$$Gr_F^p H^k(X, \mathbb{C}) = E_\infty^{p,k-p} = E_1^{p,k-p} = H^{k-p}(X, \Omega_X^p)$$

is represented by a closed form γ of type $(p, k - p)$. Then the class $\overline{\gamma}$ of γ lies in $H^{p,k-p}(X)$ and $\alpha - \overline{\gamma} \in F^{p+1}H^k(X, \mathbb{C})$. Thus we conclude by (decreasing) induction on p. ∎

Recalling that $H^{p,q}(X)$ and $H^{q,p}(X)$ are complex conjugate inside $H^k(X, \mathbb{C})$, we also get

$$\overline{F^qH^k(X, \mathbb{C})} = \oplus_{r+s=k, s\geq q} H^{r,s}(X),$$

and we conclude that the subspaces $H^{p,q}(X) \subset H^{p+q}(X, \mathbb{C})$ are determined by the Hodge filtration, by the rule:

$$H^{p,q}(X) = F^pH^k(X, \mathbb{C}) \cap \overline{F^qH^k(X, \mathbb{C})},\ p + q = k.$$

Note that not any filtration is associated to a Hodge decomposition as in (1.4). In fact, given a decomposition of a complex vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ with real structure

$$V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q},\ V^{q,p} = \overline{V^{p,q}}, \tag{1.5}$$

the associated filtration

$$F^pV_{\mathbb{C}} := \oplus_{r+s=k,r\geq p}V^{r,s} \tag{1.6}$$

obviously satisfies the property

$$F^pV_{\mathbb{C}} \cap \overline{F^{q+1}V_{\mathbb{C}}} = \{0\},\ F^pV_{\mathbb{C}} \oplus \overline{F^{q+1}V_{\mathbb{C}}} = V_{\mathbb{C}},\ p+q=k. \tag{1.7}$$

Conversely, it is an easy exercise to show that a filtration satisfying condition (1.7) is associated to a decomposition (1.5), where the $V^{p,q}$ are defined as

$$V^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}},\ p+q=k.$$

Remark 1 We already mentioned that for X projective, we have an isomorphism

$$H^k(X,\mathbb{C}) = \mathbb{H}^k(X,\Omega_X^{\cdot}),$$

where we can take here hypercohomology in the Zariski topology of the algebraic de Rham complex. Thus, if X is defined over a field $K \subset \mathbb{C}$, the right hand side has a K-structure, which means that it is of the form

$$M \otimes_K \mathbb{C}$$

for a K-vector space M. This K-structure has nothing to do with the rational structure on the left hand side given by the change of coefficients:

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Q}) \otimes \mathbb{C}.$$

Even if X is defined over $\mathbb{R}$, the two $\mathbb{R}$-structures on $H^k(X,\mathbb{C})$ obtained by this argument do not coincide. Thus complex conjugation acting on $H^k(X,\mathbb{C})$ cannot be understood algebraically and the Hodge decomposition is not algebraic on $H^k(X,\mathbb{C})$, while the Hodge filtration is algebraic, as it is given by

$$F^p\mathbb{H}^k(X,\Omega_X^{\cdot}) = Im\,(\mathbb{H}^k(X,\Omega_X^{\geq p}) \to \mathbb{H}^k(X,\Omega_X^{\cdot})).$$

1.1.3 Hodge structures

The complex cohomology of a compact Kähler manifold carries the Hodge decomposition. On the other hand, it is not only a complex vector space, since it has a canonical integral structure, namely we have the change of coefficients theorem:

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C}.$$

In the sequel we will denote by $H^k(X,\mathbb{Z})$, the integral cohomology of X modulo torsion. Thus $H^k(X,\mathbb{Z})$ is a lattice, and Hodge theory provides us with an interesting continuous invariant attached to a Kählerian complex structure on X, namely the position of the complex spaces $H^{p,q}$ with respect to the lattice $H^k(X,\mathbb{Z})$. This leads to the notion of Hodge structure.

Definition 1 *A weight k (integral) Hodge structure is a lattice V, with a decomposition*

$$V_{\mathbb{C}} = \oplus_{p+q=k}V^{p,q},\ V^{q,p} = \overline{V^{p,q}},$$

where $V_{\mathbb{C}} := V \otimes \mathbb{C}$. The associated Hodge filtration on $V_{\mathbb{C}}$ is defined by (1.6).

The contents of the previous section can be summarized by saying that if X is a compact Kähler manifold, each cohomology group (modulo torsion) $H^k(X,\mathbb{Z})$ carries a canonical Hodge structure of weight k.

Given a weight k Hodge structure V, we can define a representation ρ of $\mathbb{C}^*$ on $V_\mathbb{R}$, defined by the condition that $z \in \mathbb{C}^*$ acts by multiplication by $z^p\overline{z}^q$ on $V^{p,q}$. Then the restriction of ρ to $\mathbb{R}^*$ is the map $\lambda \mapsto$ multiplication by λ^k. Conversely, given a representation of $\mathbb{C}^*$ on $V_\mathbb{R}$ satisfying the last condition, the associated character decomposition of $V_\mathbb{C}$ will provide a Hodge structure on V.

Remark 2 One should not ask that the sum is only over pairs of non negative integers. While the Hodge structures coming from geometry are "effective", in the sense that the Hodge decomposition is a sum only over pairs of non negative integers, we will see below that the Hodge structures obtained by natural operations on the set of Hodge structures (eg taking *Hom*'s) lead to non-effective Hodge structures. On the other hand, it is always possible to shift a non-effective Hodge structure to an effective one, by defining

$$V' = V,\ V'^{p,q} = V^{p-r,q-r},$$

where r is large. The new Hodge structure is then of weight $k+2r$. It is called a r-Tate twist of the Hodge structure $(V, F^pV_\mathbb{C})$.

Example 3 The simplest Hodge structures are trivial Hodge structures of even weight $2k$. Namely, one defines $V_\mathbb{C} = V^{k,k}$, $V^{p,q} = 0$, $(p,q) \neq (k,k)$.

Example 4 The next simplest Hodge structures are effective weight 1 Hodge structures. Such a Hodge structure is given by a lattice V (necessary of even rank $2n$), and a decomposition

$$V_\mathbb{C} = V^{1,0} \oplus V^{0,1},\ V^{0,1} = \overline{V^{1,0}}.$$

Given V, weight 1 Hodge decompositions as above on $V_\mathbb{C}$ are determined by the subspace $V^{1,0}$ of $V_\mathbb{C}$ which belongs to the dense open set of the Grassmannian $Grass(n,2n)$ of rank n complex subspaces W of $V_\mathbb{C}$ satisfying the property $W \cap \overline{W} = \{0\}$.

If V, $V_\mathbb{C} = V^{1,0} \oplus \overline{V^{1,0}}$ is such a Hodge structure, we have $V_\mathbb{R} \cap V^{1,0} = \{0\}$, and thus via the natural projection

$$V_\mathbb{C} \to V_\mathbb{C}/V^{1,0}$$

$V_\mathbb{R}$ projects isomorphically to the right hand side. As $V \subset V_\mathbb{R}$ is a lattice, the projection above sends V to a lattice in the complex vector space $V_\mathbb{C}/V^{1,0}$. It follows that the quotient

$$T = V_\mathbb{C}/(V^{1,0} \oplus V)$$

is a complex torus, the complex structure being given by the complex structure on $V_\mathbb{C}/V^{1,0}$.

Conversely, a n-dimensional complex torus T is a quotient of a complex vector space K of rank n by a lattice V of rank $2n$. The inclusion

$$V \subset K$$

extends by $\mathbb{C}$-linearity to a map of complex vector spaces

$$i : V_{\mathbb{C}} \to K,$$

which is surjective, as V generates K over $\mathbb{R}$. Thus, denoting $V^{1,0} = Ker\, i$, we find that $T = V_{\mathbb{C}}/(V^{1,0} \oplus V)$. As V is a lattice in the quotient $V_{\mathbb{C}}/V^{1,0} = K$, it follows that $V_{\mathbb{R}} \cap V^{1,0} = \{0\}$, or equivalently $V^{1,0} \cap \overline{V^{1,0}} = \{0\}$.

This way we have an equivalence of categories between effective Hodge structures of weight 1 and complex tori.

Remark 3 The weight 1 Hodge structure associated to T above is a particular example of a Hodge structure on a cohomology group. Namely, consider

$$H^{2n-1}(T, \mathbb{Z}) \cong H_1(V, \mathbb{Z}).$$

Then the weight $2n - 1$ Hodge structure on $H^{2n-1}(T, \mathbb{Z})$ associated to the complex structure on T is nothing but the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus \overline{V^{1,0}}$ above. More generally, any n-dimensional compact Kähler manifold X has an associated complex torus, which is called the Albanese torus of X, obtained as the torus associated to the Hodge structure on $H^{2n-1}(X, \mathbb{Z})$. This Hodge structure has in fact Hodge level 1, which means that it is a $n-1$-Tate twist of an effective Hodge structure of weight 1.

A number of operations can be done in the category of Hodge structures.

We can take the direct sum of two Hodge structures of weight k: the lattice is the direct sum of the two lattices, and the (p, q) components are the direct sum of the (p, q)-components of each term.

We can take the dual of a Hodge structure of weight k, which will have weight $-k$. Its underlying lattice is the dual of the original one, and its Hodge decomposition is dual to the original one, with the rule

$$V^{*p,q} = V^{-p,-q*}.$$

With this definition, we can verify that if X is a compact Kähler manifold of dimension n, whose cohomology has no torsion, the Hodge structures on $H^k(X, \mathbb{Z})$ and $H^{2n-k}(X, \mathbb{Z})$ are dual via Poincaré duality, up to a n-Tate twist.

The tensor product of two Hodge structures V, W of weight k, l is the Hodge structure of weight $k + l$ whose underlying lattice is $M = V \otimes W$ and which has

$$M^{p,q} = \oplus_{r+t=p, s+u=q} V^{r,s} \otimes W^{t,u}.$$

To conclude, let us introduce a very important notion: the Mumford-Tate group of a Hodge structure. Given a Hodge structure $(V, F^p V_{\mathbb{C}})$ of weight k, determined by a representation of $\mathbb{C}^*$ on $V_{\mathbb{R}}$, where $\lambda \in \mathbb{R}^*$ acts by multiplication by λ^k, the Mumford-Tate group of $(V, F^p V_{\mathbb{C}})$ is defined as the smallest algebraic subgroup of $Aut\, V_{\mathbb{R}}$ which is defined over $\mathbb{Q}$ and contains $\rho(S^1)$, where $S^1 \subset \mathbb{C}^*$ is the set of complex numbers of mudulus 1. We will see in next section nice applications of this notion.

1.2 Morphisms of Hodge structures

1.2.1 Functoriality

Definition 2 *A morphism of Hodge structures* $(V, F^{\cdot}V_{\mathbb{C}})$, $(W, F^{\cdot}W_{\mathbb{C}})$ *of respective weights* k, $k+2r$ *is a morphism of lattices*

$$\phi : V \to W,$$

such that the $\mathbb{C}$*-linear extension* $\phi_{\mathbb{C}}$ *of* ϕ *sends* $V^{p,q}$ *to* $W^{p+r,q+r}$*. Such a morphism is said to be of bidegree* (r,r)*, as it shifts by* (r,r) *the bigraduation given by the Hodge decomposition.*

Natural examples of morphisms of Hodge structures are induced by holomorphic maps

$$f : X \to Y$$

between compact Kähler manifolds. The pull-back on cohomology

$$f^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is a morphism of Hodge structures of weight k, because the pull-back by f of a closed form of type (p,q) is again a closed form of type (p,q).

We also have the Gysin map

$$f_* : H^k(X, \mathbb{Z}) \to H^{k+2r}(Y, \mathbb{Z}),$$

where $r := dim\, Y - dim\, X$. It is defined on integral cohomology as the composition

$$PD_Y^{-1} \circ f_* \circ PD_X,$$

where PD_X is the Poincaré duality isomorphism

$$H^l(X, \mathbb{Z}) \cong H_{2n-l}(X, \mathbb{Z}),\ n = dim\, X$$

and similarly for PD_Y, and f_* at the middle is the natural push-forward map induced on homology by f. One can show that f_* is a morphism of Hodge structures of bidegree (r,r), observing the following fact: Under Poincaré duality, the Hodge structure on $H^l(X, \mathbb{Z})$ is dual to the Hodge structure on $H^{2n-l}(X, \mathbb{Z}$, up to a Tate twist of n. Using the definition of the dual Hodge structure, this amounts to say that $H^{p,q}(X)$, $p+q = l$ is orthogonal with respect to the intersection pairing to $\oplus_{p'+q'=2n-l,(p',q')\neq(n-p,n-q)} H^{p',q'}(X)$, which is obvious by degree reasons, and thus identifies to the dual of $H^{n-p,n-q}(X)$.

Up to now, we have been working with integral Hodge structures. It is sometimes more convenient to use rational Hodge structures, which are the same objects, except that the lattice is replaced by a rational vector space.

Morphisms of rational Hodge structures are defined in the same way as above, morphisms of lattices being replaced with morphisms of $\mathbb{Q}$-vector spaces. Given a morphism of Hodge structures

$$\phi : V_{\mathbb{Q}} \to W_{\mathbb{Q}}$$

there is an obvious induced Hodge structure on $Ker\, \phi$, due to the fact that, since ϕ preserves up to a shift the bigradation given by Hodge decomposition, its kernel is stable under Hodge decomposition. For the same reason there is an induced Hodge structure on $Coker\,\phi$. Thus we have kernels and cokernels in the category of rational Hodge structures.

The following lemma is also useful.

Lemma 1 *A morphism of Hodge structure is strict with respect to the Hodge filtrations.*

By this we mean the following: a morphism $\phi : V \to W$ of Hodge structures, of bidegree (r, r), shifts the Hodge filtrations by r:

$$\phi(F^p V_{\mathbb{C}}) \subset F^{p+r} W_{\mathbb{C}}.$$

Then ϕ being strict means that

$$F^{p+r} W_{\mathbb{C}} \cap Im\,\phi = \phi(F^p V_{\mathbb{C}}).$$

This is obvious using the Hodge decomposition.

1.2.2 Hodge classes

Definition 3 *Let $(V, F^{\cdot} V_{\mathbb{C}})$ be a Hodge structure of even weight $2k$. Then $Hdg(V)$, the set of Hodge classes of V, is defined as the set of classes $\alpha \in V \cap V^{k,k}$, where the intersection is inside $V_{\mathbb{C}}$.*

Examples of Hodge classes on Kähler compact manifolds are given by classes of closed analytic subsets (cf [44], I, 11.1). If $Z \subset X$ is a closed irreducible reduced analytic subset of codimension k, one can desingularize Z to get $j : \widetilde{Z} \to X$. Then the class $[Z]$ of Z is defined as $j_*(1_{\widetilde{Z}})$, which is a Hodge class because $1_Z \in H^0(\widetilde{Z}, \mathbb{Z})$ is a Hodge class. Other Hodge classes can be constructed as Chern classes of holomorphic vector bundles or coherent sheaves. In the case of compact Kähler manifolds, Chern classes of coherent sheaves cannot in general be expressed as rational combinations of classes of analytic subsets (think to the case of a complex torus admitting a holomorphic line bundle which is not topologically trivial, but not admitting any semi-positive not topologically trivial line bundle).

It is proven in [45] that on compact Kähler manifolds, Chern classes of coherent sheaves cannot be in general expressed as polynomials into Chern classes of holomorphic vector bundles, which implies (by the Whitney formula) the existence of coherent sheaves not admitting locally free resolutions. Thus on general Kähler manifolds, the construction of Hodge classes as rational combinations of Chern classes of coherent sheaves is strictly more general than the construction via classes of analytic subsets or Chern classes of holomorphic vector bundles.

On projective manifolds, the three constructions generate over $\mathbb{Q}$ the same space of rational Hodge classes [5], and rational Hodge classes are conjectured to be combinations with rational coefficients of classes of analytic subsets, or equivalently of Chern classes of coherent sheaves (the Hodge conjecture). In [42], it is shown that the Weil classes on Weil complex tori cannot be expressed as combinations with rational coefficients of Chern classes of coherent sheaves, thus showing the impossibility of extending the Hodge conjecture to compact Kähler manifolds.

One relation between Hodge classes and morphisms of Hodge structures is the following lemma, which follows readily from the definitions :

Lemma 2 *Let $(V, F^{\cdot}V_{\mathbb{C}})$, $(W, F^{\cdot}W_{\mathbb{C}})$ be Hodge structures of weights k, $k+2r$. Then the set of morphisms of Hodge structures from V to W identifies to the set of Hodge classes in the weight $2r$ Hodge structure*

$$Hom\,(V,W) = V^* \otimes W.$$

For example, consider the identity map $Id \in End(H^k(X,\mathbb{Q}))$, where X is a compact Kähler manifold. This is a Hodge class. On the other hand, via Poincaré duality, $End(H^k(X,\mathbb{Q}))$ identifies as a Hodge structure, up to a Tate twist of n to $H^{2n-k}(X,\mathbb{Q}) \otimes H^k(X,\mathbb{Q})$. The later is via Künneth decomposition a sub-Hodge structure of $H^{2n}(X \times X, \mathbb{Q})$. Hence we get this way natural Hodge classes on $X \times X$, called the Künneth components of the diagonal. It is not known whether they are algebraic when X is projective, except for certain cases (abelian varieties or more generally complex tori T for example, where this is obvious because one can express the Künneth components as adequate combinations of the classes of the graphs of the map $m : T \to T$ of multiplication by $m \in \mathbb{Z}$).

Let us conclude with the following important fact concerning the relation between the Mumford-Tate group of a rational Hodge structure and Hodge classes.

Theorem 4 *Let $(H, F^{\cdot}H)$ be a Hodge structure of weight k. Then the Mumford-Tate group of H is the algebraic subgroup of $Aut_{\mathbb{Q}}(H)$ consisting of elements acting as the identity on all the Hodge classes in the tensor products $(H^*)^{\otimes t} \otimes H^{\otimes s}$ for $k(s-t)$ even.*

It is clear by the definition of the Mumford-Tate group that it has to be contained in the algebraic group fixing Hodge classes in all these tensor products. Indeed, the S^1-action on H induces an S^1-action on each tensor power $(H^*)^{\otimes t} \otimes H^{\otimes s}$, which is the identity on the component of type (r,r), $2r = k(s-t)$ of $(H^*)^{\otimes t} \otimes H^{\otimes s}$; As Hodge classes are of type (r,r), S^1 acts trivially on them and thus it is contained in this group; On the other hand, as the Hodge classes are rational, this group is defined over $\mathbb{Q}$. Hence it must contain the whole Mumford-Tate group.

The reverse inclusion necessitates some notions of invariant theory.

An interesting application of this, due to Deligne [14], is the fact that if we have an algebraic family of projective complex manifolds $(X_t)_{t\in B}$ parameterized by a quasi-projective basis B, for general $t \in B$, the Mumford-Tate group of the Hodge structure on $H^l(X_t,\mathbb{Q})$ contains a subgroup of finite index of the monodromy group

$$Im\,\rho : \pi_1(B,t) \to Aut\,H^l(X_t,\mathbb{Q}).$$

This is due to the fact that the monodromy group preserves the space of Hodge classes in tensor products $H^l(X_t,\mathbb{Q})^{\otimes -s} \otimes H^l(X_t,\mathbb{Q})^{\otimes t}$ for general $t \in B$ (here, if "general" means "away from a countable union of Zariski closed algebraic subsets", we have to use the theorem of [10] for this result, but if we just want to know that the result holds on a countable intersection of dense open sets, it is quite easy). Furthermore the monodromy acts on these spaces of Hodge classes via a finite group, a result which is a consequence of the notion of polarization introduced later on.

1.2.3 Cohomology ring

Coming back to geometry, we have the cup-product between the cohomology groups $H^k(X,\mathbb{Z})$ of a manifold (or topological space):

$$\cup : H^k(X,\mathbb{Z}) \otimes H^l(X,\mathbb{Z}) \to H^{k+l}(X,\mathbb{Z}). \tag{1.8}$$

At the level of complex cohomology, where cohomology classes are represented via de Rham theory as classes of closed forms modulo exact ones, the cup-product is given by the exterior product, namely, if $\alpha \in H^k(X,\mathbb{C})$, $\beta \in H^l(X,\mathbb{C})$ are represented respectively by closed complex valued differential forms $\tilde{\alpha}$, $\tilde{\beta}$, then $\alpha \cup \beta$ is represented by the closed differential form $\tilde{\alpha} \wedge \tilde{\beta}$.

Now, if X is a complex manifold and $\tilde{\alpha}$, $\tilde{\beta}$ are respectively of type (r,s), $r+s = k$, (t,u), $t+u=l$, then $\tilde{\alpha} \wedge \tilde{\beta}$ is closed of type $(r+t, s+u)$.

Thus, if X is a compact complex manifold, the definition of the $H^{p,q}$ groups of X shows that

$$H^{r,s}(X) \cup H^{t,u}(X) \subset H^{r+s,t+u}(X).$$

Using the definition of the Hodge structure on the tensor product $H^k(X,\mathbb{Z}) \otimes H^l(X,\mathbb{Z})$, this amounts to say that the cup-product (1.8) is a morphism of Hodge structures of weights $k+l$.

Another application of this is the fact that Hodge classes $\alpha \in Hdg^{2r}(X,\mathbb{Z})$ on compact Kähler manifolds give rise to morphisms of Hodge structure

$$\cup\alpha : H^l(X,\mathbb{Z}) \to H^{l+2r}(X,\mathbb{Z}),$$

a fact which will be very much used in the sequel.

1.3 Polarizations

1.3.1 The hard Lefschetz theorem

A very deep application of Hodge theory is the hard Lefschetz theorem, which says the following: let X be a compact Kähler manifold of dimension n and $\omega \in H^2(X,\mathbb{R})$ the class of a Kähler form Ω on X. Cup-product with ω gives an operator usually denoted by

$$L : H^*(X,\mathbb{R}) \to H^{*+2}(X,\mathbb{R}).$$

Theorem 5 *For any $k \leq n$,*

$$L^{n-k} : H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R})$$

is an isomorphism.

The proof involves first a pointwise computation, saying that wedge product with Ω^{n-k} induces a pointwise isomorphism

$$\bigwedge^k \Omega_{X,\mathbb{C}} \to \bigwedge^{2n-k} \Omega_{X,\mathbb{C}}.$$

The second ingredient is the fact that wedge product with the Kähler form Ω preserves harmonic forms, which are the canonical de Rham representatives of cohomology classes on X, given the Kähler metric. Thus one has to check the result by looking at the wedge product with Ω^{n-k} on harmonic forms. And the last ingredient is the Poincaré duality which says that both spaces have the same dimension, so that bijectivity is equivalent to injectivity.

It is interesting to note that if X is projective, we can take for ω the first Chern class of a very ample line bundle (cf section 2.1), and then the hard Lefschetz theorem implies immediately the injectivity statement in the Lefschetz theorem on hyperplane sections, at least for smooth hyperplane sections and rational coefficients. The Lefschetz theorem on hyperplane sections says that if $j : H \hookrightarrow X$ is the inclusion of an ample divisor, the restriction map

$$j^* : H^*(X,\mathbb{Z}) \to H^*(H,\mathbb{Z})$$

is an isomorphism for $k < n-1$, injective for $k = n-1$. It is well-known that $j_* \circ j^* = L$ on $H^*(X,\mathbb{Z})$ (cf [44] I, p 287). Thus the injectivity of j^* on $H^k(X,\mathbb{Q})$ for $k < n$ follows from that of L, given by Theorem 5.

∎

We will see later on that the surjectivity statement in Lefschetz restriction theorem can be recovered as a consequence of the second Hodge-Riemann bilinear relations. It is interesting to note that the Lefschetz theorem on hyperplane sections can be given a proof using vanishing theorems (cf [44], II, 1.3, and that the later ones can be proved algebraically (cf [16]). On the other hand, the hard Lefschetz theorem has no known algebraic proof.

A first formal consequence of the hard Lefschetz theorem is the so-called Lefschetz decomposition. With the same notations as before, define for $k \leq n$ the primitive cohomology of X by

$$H^k(X,\mathbb{R})_{prim} := Ker\,(L^{n-k+1} : H^k(X,\mathbb{R}) \to H^{2n+2-k}(X,\mathbb{R})).$$

For example, if $k = 1$, the whole cohomology is primitive, and if $k = 2$, primitive cohomology is the same as the orthogonal part, with respect to Poincaré duality, to $\omega^{n-1} \in H^{2n-2}(X,\mathbb{R})$.

The Lefschetz decomposition is the following (it can also be extended to $k > n$ using the hard Lefschetz isomorphism).

Theorem 6 *The cohomology groups $H^k(X,\mathbb{R})$ for $k \leq n$ decompose as*

$$H^k(X,\mathbb{R}) = \oplus_{2r\leq k} L^r H^{k-2r}(X,\mathbb{R})_{prim}.$$

We have to prove surjectivity and injectivity of the natural map

$$\oplus_{2r\leq k} H^{k-2r}(X,\mathbb{R})_{prim} \stackrel{\sum L^r}{\to} H^k(X,\mathbb{R}).$$

For the surjectivity, one uses induction on k and the hard Lefschetz theorem, which says that if $\beta \in H^k(X,\mathbb{R})$, then

$$L^{n-k+1}\beta = L^{n-k+2}\gamma$$

for some $\gamma \in H^{k-2}(X,\mathbb{R})$. Then $\beta - L\alpha$ is primitive, and we continue with α. The injectivity is also easy.

1.3.2 Hodge-Riemann bilinear relations

We consider a Kähler compact manifold X with Kähler class ω. We can define an intersection form q_ω on each $H^k(X,\mathbb{R})$ by the formula

$$q_\omega(\alpha,\beta) = \int_X \omega^{n-k} \cup \alpha \cup \beta.$$

By hard Lefschetz theorem and Poincaré duality, q_ω is a non-degenerate bilinear form. It is skew-symmetric if k is odd and symmetric if k is even. Furthermore, the extension of q_ω to $H^k(X,\mathbb{C})$ satisfies the property that

$$q_\omega(\alpha,\beta)=0,\ \alpha\in H^{p,q},\ \beta\in H^{p',q'},\ (p',q')\neq(q,p).$$

Another way to rephrase this is to say that the sesquilinear pairing h_ω on $H^k(X,\mathbb{C})$ defined by

$$h_\omega(\alpha,\beta)=i^k q_\omega(\alpha,\overline{\beta})$$

has the property that the Hodge decomposition is orthogonal with respect to h_ω.

This property is summarized under the name of first Hodge-Riemann bilinear relations.

Coming back to q_ω, note also that the Lefschetz decomposition is orthogonal with respect to q_ω. Indeed, if $\alpha=L^r\alpha'$, $\beta=L^s\beta'$, with $r<s$, and α', β' primitive, then

$$L^{n-k}\alpha\cup\beta=L^{n-k+r+s}\alpha'\cup\beta',$$

with $L^{n-k+r+s}\alpha'=0$ because $L^{n-k+2r+1}\alpha'=0$.

The second Hodge-Riemann bilinear relations play a crucial role, especially in the study of the period domains. Note first that, because the operator L shifts the Hodge decomposition by $(1,1)$, the primitive cohomology has an induced Hodge decomposition:

$$H^k(X,\mathbb{C})_{prim}=\oplus_{p+q=k}H^{p,q}(X)_{prim},$$

with $H^{p,q}(X)_{prim}:=H^{p,q}(X)\cap H^r(X,\mathbb{C})_{prim}$. We have now

Theorem 7 *The sesquilinear form h_ω is definite of sign $(-1)^{\frac{k(k-1)}{2}}i^{p-q-k}$ on the component $L^rH^{p,q}(X)_{prim}$, $2r+p+q=k$ of $H^k(X,\mathbb{C})$.*

The first application of this theorem is the well-known Hodge index theorem for the intersection form on H^2 of a compact Kähler surface X. As we are looking at the middle cohomology, the form q_ω is equal to the natural intersection pairing on $H^2(X)$. The primitive cohomology is in this case the orthogonal complement of the Kähler form and Theorem 7 says that q_ω is negative definite on the real part of $H^{1,1}(X)_{prim}$ and positive definite on the real part on the real part of $H^{2,0}(X)\oplus H^{0,2}(X)$. It is also obviously positive on the line $<\omega>$, which is perpendular to both of these spaces.

This shows that the Hodge numbers of compact Kähler surfaces are determined by their topology, which is not the case in higher dimension.

As a second application, let us prove the surjectivity part in Lefschetz hyperplane section theorem: If X is projective and $H\overset{j}{\hookrightarrow}X$ is the inclusion of a smooth ample divisor, then

$$j^*:H^k(X,\mathbb{Q})\to H^k(H,\mathbb{Q})$$

is surjective for $k<n-1=dim\,H$.

Indeed, choose for Kähler class ω the class of H, and consider the Lefschetz decomposition on $H^{k+2}(X,\mathbb{Q})$ with respect to ω:

$$H^{k+2}(X,\mathbb{Q})=H^{k+2}(X,\mathbb{Q})_{prim}\oplus LH^k(X,\mathbb{Q}).$$

Note that, as $L = j_* \circ j^*$, the first term is equal to

$$j_*(j^* H^k(X, \mathbb{Q})).$$

Thus, the surjectivity statement is equivalent to the fact that for $k+2 \leq n$, we have

$$j_* H^k(H, \mathbb{Q}) \cap H^{k+2}(X, \mathbb{Q})_{prim} = 0.$$

(Indeed, we use here the fact that j_* is injective on $H^k(H, \mathbb{Q})$ by hard Lefchetz theorem on H.)

Note that if $\alpha \in H^k(H, \mathbb{Q})$ satisfies

$$j_* \alpha \in H^{k+2}(X, \mathbb{Q})_{prim},$$

then $L^{n-k-2} j_* \alpha = 0$ and thus

$$j^* L^{n-k-2} j_* \alpha = 0 = L_H^{n-k-1} \alpha.$$

Thus $\alpha \in H^k(H, \mathbb{Q})_{prim}$.

Finally, note that the forms q_ω defined on $H^*(X)$, $H^*(H)$ are compatible with j_*:

$$q_{\omega,H}(\alpha, \beta) = q_\omega(j_* \alpha, j_* \beta),$$

for α, $\beta \in H^k(H, \mathbb{Q})$, $k \leq n-2$. Indeed,

$$q_\omega(j_* \alpha, j_* \beta) = \int_X L^{n-2-k} j_* \alpha \cup j_* \beta = \int_H L_H^{n-2-k} \alpha \cup j^*(j_* \beta) =$$

$$= \int_H L_H^{n-1-k} \alpha \cup \beta = q_{\omega,H}(\alpha, \beta).$$

Now let $K \subset H^k(H, \mathbb{Q})_{prim}$ be the set of classes α such that $j_* \alpha \in H^{k+2}(X, \mathbb{Q})_{prim}$. This is a sub-Hodge structure of $H^k(H, \mathbb{Q})_{prim}$. Thus it suffices to show that each (p, q)-component of K is 0. Now, if $0 \neq \alpha \in K^{p,q}$, we have $(-1)^{\frac{k(k-1)}{2}} i^{p-q} q_{\omega,H}(\alpha, \overline{\alpha}) > 0$, while $(-1)^{\frac{(k+2)(k+1)}{2}} i^{p-q} q_\omega(j_* \alpha, j_* \overline{\alpha}) > 0$, which is a contradiction. ∎

As another application of this which will be used later on, we can also conclude the following:

Lemma 3 *Let X be a compact Kähler manifold. Assume there is a rank 2 subspace $V \subset H^2(X, \mathbb{R})$ such that*

$$V \cup V = 0$$

in $H^4(V, \mathbb{R})$. Then the Hodge structure on $H^2(X, \mathbb{Q})$ is non trivial.

Indeed, if it were trivial, that is entirely of type $(1,1)$, then for ω a Kähler form on X, the non-degenerate intersection form q_ω on $H^2(X, \mathbb{R})$ would have one positive sign, and thus the dimension of a maximal isotropic subspace would be 1. But V is isotropic, which is a contradiction. ∎

Other first applications of the Hodge-Riemann bilinear relations concern the period domains for Kähler surfaces. Let X be a compact Kähler surface and let $V := H^2(X, \mathbb{Z})$, which is endowed with the intersection pairing $<,>$. To each Kähler complex structure X_t on X, one associates the subspace

$$H^{2,0}(X_t) \subset H^2(X, \mathbb{C}).$$

This is the so-called period map. The dimension $h^{2,0}$ of this subspace is determined by the topology of X. Thus this space is a point in the Grassmannian $Grass(h^{2,0}, H^2(X, \mathbb{C}))$. The Hodge-Riemann bilinear relations say the following:

1. This point belongs in fact to the isotropic Grassmannian of those subspaces on which the intersection form vanishes identically.

2. This point belongs to the open set of the isotropic Grassmannian consisting of those subspaces on which the Hermitian bilinear form $h(\alpha, \beta) = < \alpha, \overline{\beta} >$ is positive definite.

3. This point determines the whole Hodge decomposition on $H^2(X_t, \mathbb{Z})$ by the rules:

$$H^{0,2}(X_t) = \overline{H^{2,0}(X_t)},\ H^{1,1}(X_t) = (H^{2,0}(X_t) \oplus H^{0,2}(X_t))^{\perp}.$$

Note that if a subspace $V \subset H^2(X, \mathbb{C})$ of rank $h^{2,0}$ satisfies the properties 1 and 2, then certainly $V \cap \overline{V} = 0$ and one can define a Hodge decomposition on $H^2(X, \mathbb{C})$ by the two rules above, for which $V = H^{2,0}$ and satisfying the Hodge-Riemann bilinear relations. One thus may ask whether one can hope that any such Hodge structure comes from a deformation of the complex structure on X, at least in a neighbourhood of a given point. However, this is forbidden by the so-called transversality property, discovered by Griffiths, which imposes strong conditions on the differential of the period map, showing that it is not surjective on the tangent space of the period domain described by the conditions above. Only in the case where $h^{2,0} = 1$, the transversality condition is empty, and indeed, at least if the rank of $H^2(X)$ is not larger than 20, the period map for $K3$ surfaces allows to fill-in the period domain described by conditions 1 and 2 above.

1.3.3 Rational polarizations

The Lefschetz decomposition is particularly useful if the Kähler class can be chosen to be rational, or equivalently if the manifold X is projective (see section 2.1.1). Indeed, in this case, the Lefschetz decomposition is a decomposition into rational subspaces, and as each of these subspaces is stable under the Hodge decomposition, it is a decomposition into sub-Hodge structures. This is very important for using Hodge theory to study moduli spaces of projective complex manifolds. Indeed, the period map, which roughly speaking associates to a (Kähler or projective) complex structure the Hodge decomposition on the complex cohomology groups regarded as a varying decomposition on a fixed complex vector space, splits in the projective case into period map for each primitive component (considering deformations of the complex structure with fixed integral Kähler class). However the Hodge decomposition on primitive cohomology satisfies the second Hodge-Riemann bilinear relations (here, as

ω is fixed, q_ω is also fixed). The polarized period domain has much better properties, in particular curvature properties in the horizontal directions (those satisfying the transversality condition) and this leads by pull-back via the period map to many interesting curvature results on the moduli spaces themselves (see [23], [41]).

We shall focus on the notion which emerges from the Lefschetz decomposition on a compact Kähler manifold, with respect to a rational Kähler cohomology class.

Definition 4 *A rational polarized Hodge structure of weight k is a Hodge structure $(V, F^{\cdot}V)$ of weight k, together with a rational intersection form q on V, symmetric if k is even, skew-symmetric if k is odd, such that the associated Hermitian bilinear form h on $V_{\mathbb{C}}$, defined by $h(v,w) = i^k q(v,\overline{w})$ satisfies the Hodge-Riemann bilinear relations:*

1. *The Hodge decomposition is orthogonal with respect to h.*
2. *The restriction of h to each $V^{p,q}$ is definite of sign $(-1)^p$.*

This is (up to a sign) the structure we get on the primitive components of the cohomology of a compact Kähler manifold endowed with a rational Kähler cohomology class.

The category of polarized Hodge structures behaves much better than the one of general Hodge structures:

Lemma 4 *Let $(V, F^{\cdot}V_{\mathbb{C}})$ be a polarized Hodge structure, with intersection form q, and let $L \subset V$ be a sub-Hodge structure (that is $L_{\mathbb{C}}$ is stable under the Hodge decomposition). Then there is an orthogonal decomposition*

$$V = L \oplus L^{\perp},$$

where $\perp$ stands for the orthogonal complement with respect to q and $L^{\perp}$ is also a sub-Hodge structure of V.

Indeed, it is clear by the first Hodge-Riemann bilinear relations that $L_{\mathbb{C}}^{\perp}$, which is also the orthogonal complement of $L_{\mathbb{C}}$ with respect to h, is also a sub-Hodge structure. Thus it suffices to show that $q_{|L}$ is non-degenerate. This is equivalent to say that $h_{|L_{\mathbb{C}}}$ is non-degenerate. But this follows from the fact that $L_{\mathbb{C}}$ is an orthogonal direct sum with respect to h:

$$L_{\mathbb{C}} = \oplus L_{\mathbb{C}}^{p,q}$$

and that each $h_{|L_{\mathbb{C}}^{p,q}}$ is non-degenerate, because h is definite on $V_{\mathbb{C}}^{p,q}$. ∎

Example 5 We have seen that a weight 1 integral Hodge structure is the same thing as a complex torus. Let us show that a weight 1 integral polarized Hodge structure is the same as a projective complex torus (an abelian variety) with a given integral Kähler cohomology class. Indeed, let $(V, V_{\mathbb{C}} = V^{1,0} \oplus \overline{V^{1,0}})$ be a weight 1 Hodge structure and

$$q : \bigwedge^2 V \to \mathbb{Z}$$

be a polarization. Then, recalling that the corresponding complex torus T is given by

$$T = V_{\mathbb{C}}/(V^{1,0} \oplus V),$$

we find that

$$\bigwedge^2 V^* \cong \bigwedge^2 H^1(T,\mathbb{Z}) = H^2(T,\mathbb{Z}),$$

and thus q can be seen as an integral cohomology class on T. Furthermore, q is represented in de Rham cohomology by the $\mathbb{R}$-linear extension $q_{\mathbb{R}}$ of q to $V_{\mathbb{R}}$, which is a 2-form on T, and $V_{\mathbb{R}}$ is isomorphic to $V^{0,1}$ by the projection, which gives the complex structure on the real tangent space $V_{\mathbb{R}}$ of T. Now one verifies ([44], I, 7.2.2) that the first Hodge-Riemann bilinear relation says that $q_{\mathbb{R}}$ is of type $(1,1)$ and the second Hodge-Riemann bilinear relation says that $q_{\mathbb{R}}$ is a positive real $(1,1)$ form, that is a Kähler form. Thus T is projective by the Kodaira criterion (see next section).

In this example, Lemma 4 says that if $A \subset B$ is an abelian subvariety of an abelian variety, then there is a spliting up to isogeny:

$$A \oplus C \xrightarrow{s} B,$$

where s is finite. This can be seen from an algebrogeometric as follows: let C be the quotient B/A and denote by $q : B \to C$ the quotient map. As B is projective, there is a subvariety

$$Y \subset B$$

which projects in a finite way to C via q. Then we can construct a rational map, which in fact must be holomorphic and necessarily a morphism of abelian varieties [31]:

$$\sigma : C \to B$$

by the rule:

$$\sigma(c) = \sum_{y\in Y, q(y)=c} y,$$

Clearly the composition $q \circ \sigma$ is multiplication by the degree of $q_{|Y}$ and thus we conclude that the map

$$A \oplus C \to B,\ (a,c) \mapsto a + \sigma(c)$$

is an isogeny. ∎

Example 6 There is a beautiful formal construction, due to Kuga and Satake [30], which associates to a polarized weight 2 Hodge structure H of $K3$ type (that is $h^{2,0} = 1$) an abelian variety A such that $H^2(S,\mathbb{Q})$ can be realized as a direct summand in $H^2(A,\mathbb{Q})$ as a Hodge structure. Hence, by lemma 2, assuming $H = H^2(S,\mathbb{Q})_{prim}$ for S an algebraic $K3$ surface, there is a degree 4 Hodge class on $S \times A$ which induces this inclusion

$$H^2(S,\mathbb{Q}) \hookrightarrow H^2(A,\mathbb{Q}), \tag{1.9}$$

and the Hodge conjecture thus predicts that there should be an algebraic cycle of codimension 2 in $S \times A$ inducing this inclusion at the cohomology level.

Such an inclusion does not exist for a general weight 2 Hodge structure coming from geometry, by a Mumford-Tate group argument (cf [14]).

The construction goes as follows. On starts with a Hodge structure H with $h^{2,0} = 1$, and polarization q. Let us introduce the Clifford algebra

$$C(H, q) = \bigotimes H / < v \otimes v = -q(v)1 > .$$

This is a rational vector space. A complex structure is defined on it using an orthonormal basis e_1, e_2 of $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$ and defining $e := e_2 e_1 \in C(H)$. (The orientation comes from the real isomorphism $H^{2,0} \cong (H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$, and one sees that e does not depend on the choice of the oriented orthonormal basis.) Then $e^2 = -1$ and left Clifford multiplication by e defines a complex structure on $C(H)_{\mathbb{R}}$. This defines a rational Hodge structure of weight 1, hence a complex torus up to isogeny. One can show that this complex torus is in fact an abelian variety.

Finally there is a morphism of Hodge structures

$$H \to Hom\,(C(H), C(H))$$

given by left Clifford multiplication. Thus for the abelian variety associated up to isogeny to the weight 1 rational Hodge structure on $C(H) \oplus C(H)^*$, there is an inclusion of Hodge structures as in (1.9).

To conclude this section, let us explain why the second Hodge-Riemann relations imply that for a projective family of varieties $(X_t)_{t \in B}$, (that is, there exists a line bundle on the total space of the family which restricts to an ample one on the fibers), the monodromy acts as a finite group on the set of Hodge classes in $H^{2k}(X_t, \mathbb{Z})$, for general t.

The point is that there is a relative Lefschetz decomposition on the cohomology of the fibers, relative to the (locally constant) integral Kähler class, given as the first Chern class of the relatively ample line bundle. The monodromy preserves this relative Lefschetz decomposition, which is a decomposition of a subgroup of finite index in $H^{2k}(X_t, \mathbb{Z})$. Furthermore, on each component of the Lefschetz decomposition, the monodromy preserves the integral structure and the intersection form q_ω. But as the Hodge classes are of type (k, k), the second Hodge-Riemann bilinear relations imply that on the set of Hodge classes in each component of the Lefschetz decomposition, the form q_ω is non degenerate and has a definite sign. Thus the monodromy acts on each of them via the orthogonal group of an integral positive definite form on a lattice, that is via a finite group. ∎

2 Kähler and complex projective manifolds

2.1 The Kodaira criterion

2.1.1 Polarizations on projective manifolds

The Kodaira criterion [27] characterizes projective complex manifolds inside the class of compact Kähler manifolds.

Theorem 8 *A compact complex manifold X is projective if and only if X admits a Kähler class which is rational, that is belongs to*

$$H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R}).$$

The only if comes from the fact that if X is projective, one gets a Kähler form on X by restricting the Fubini-Study Kähler form on some projective space $\mathbb{P}^N$ in which X is imbedded as a complex submanifold. But the Fubini-Study Kähler form has integral cohomology class, as its class is the first Chern class of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$.

Conversely, if the class β of a Kähler form Ω is rational, some multiple $\alpha = m\beta$ is integral, and as α is represented by a closed form of type $(1,1)$, its image in $H^2(X, \mathcal{O}_X)$ vanishes. Thus, via the long exact sequence induced by the exponential exact sequence:

$$Pic\, X = H^1(X, \mathcal{O}_X^*) \stackrel{c_1}{\to} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X),$$

one concludes that $\alpha = c_1(\mathcal{L})$ for some holomorphic line bundle $\mathcal{L}$. The conclusion then follows from the following two facts:

- $\mathcal{L}$ can be endowed with a Hermitian metric whose Chern form is equal to $m\Omega$, a non-trivial fact which involves the $\partial\overline{\partial}$-lemma, and uses the fact that X is Kähler.

- Kodaira's vanishing theorem for line bundles endowed with metrics of positive associated Chern forms, applied to the blow-up of X along points, which finally allow to conclude that $\mathcal{L}$ is ample.

Definition 5 *A polarization on a projective manifold X is the data of a rational Kähler cohomology class.*

As explained in the previous section, a polarization on X induces an operator L of cup-product with the given Kähler class and a Lefschetz decomposition on each cohomology group $H^k(X, \mathbb{Q})$ and a polarization on each component $L^r H^{k-2r}(X, \mathbb{Q})_{prim}$ of the Lefschetz decomposition, which is essential for most statements concerning the period map.

The polarizations play a crucial role in the construction of moduli spaces as quasi-projective varieties (see [40]). In particular, the notion of a family of polarized varieties is the following:

Definition 6 *A polarized family of complex manifolds, is a complex analytic space $\mathcal{X}$, together with a smooth proper map $f : \mathcal{X} \to B$, where B is an analytic space, and a rational cohomology class $\alpha \in H^2(\mathcal{X}, \mathbb{Q})$, such that each $\alpha_{|X_t}$ is a Kähler cohomology class.*

Thus by Kodaira's criterion, the fibers are smooth complex projective manifolds.

A standard example of a non polarized family of Kähler manifolds is the twistor family associated to a $K3$ surface S with given Kähler class α (see [8], [26], [33]). By Yau's theorem [47], the class α admits a representative Ω which defines a Kähler-Einstein metric. The $K3$ surface has trivial canonical bundle, and if $\eta \in H^0(S, K_S)$ is a generator, the metric is Kähler-Einstein if and only if η is Ricci-flat with respect to the metric. Note that Ω is also Ricci-flat. For each $s \in S$ we have the three complex 2-forms $\eta_s, \overline{\eta}_s, \Omega_s$ on the real tangent space $T_{S,s}$ at s, and by Ricci-flatness, we find that a combination

$$\mu_s = a\eta_s + b\overline{\eta}_s + c\Omega_s,\ a,\ b,\ c \in \mathbb{C}$$

satisfies $\mu_s^2 = 0$ in $\bigwedge^4 \Omega_{S,s,\mathbb{C}}$ if and only if it does at every point of S. The closed 2-form $\mu = a\eta + b\overline{\eta} + c\Omega$ then defines a new almost complex structure on $T_{S,s}$: indeed, as $\mu_s^2 = 0$ at every point s, μ_s determines a rank 2 subspace V_s of $T_{S,s,\mathbb{R}} \otimes \mathbb{C}$, which is transverse to its complex conjugate because $\mu \wedge \overline{\mu} > 0$ at any point.

This rank 2 subspace determines the almost complex structure at the considered point. As μ is closed, this almost complex structure is integrable. In conclusion, we have constructed a family of complex structures on S parameterized by the conic C in $\mathbb{P}^2(\mathbb{C})$ determined by equation

$$\mu^2 = 0.$$

With some more work, one can define a complex structure on $S \times C$, (note $C \cong \mathbb{P}^1$) such that the second projection is holomorphic and the induced complex structure on the fibers is precisely the one described above.

This family is not a polarized family of complex manifolds, although one can show that each fiber is Kähler (in fact the initial *metric* remains Kähler for each of these complex structures, although the Kähler form does not of course remain Kähler). Indeed, the total space $\mathcal{X}$ is diffeomorphic to $S \times \mathbb{P}^1$. Thus a rational cohomology class β on $\mathcal{X}$ can be written as $pr_1^* u + pr_2^* v$, $u \in H^2(S, \mathbb{Q})$, $v \in H^2(\mathbb{P}^1, \mathbb{Q})$, and saying that α polarizes the family means that u remains Kähler for any $\mu \in C$. However, note that by definition, for the complex structure defined by $\mu \in C$, $\mu = a\eta + b\overline{\eta} + c\Omega$ is holomorphic of type $(2,0)$ for this complex structure. As u is Kähler for this complex structure, we must have

$$< u, [\mu] >= 0,$$

where $[\mu]$ is the de Rham cohomology class of μ and $<,>$ is the intersection pairing on S. As this equality must be true for all $\mu \in C$, we conclude that in fact u has to be orthogonal to η, $\overline{\eta}$ and α. But then the Hodge index theorem, that is the second Hodge-Riemann bilinear relations for $H^2(S)$ says that $u^2 < 0$, contradicting the fact that u is Kähler.

LeBrun [33] uses this family to exhibit examples or real 6-dimensional manifolds admitting infinitely many complex structures with different Chern numbers. In contrast, the Chern numbers of compact almost complex surfaces depend only on topology, as a consequence of Hirzebruch's signature theorem 16.

One important property of polarized families of complex manifolds is the following theorem due to Deligne [12] and Blanchard:

Theorem 9 *Let $\pi : \mathcal{X} \to B$ be a polarized family of compact complex manifolds. Then the Leray spectral sequence of π with rational coefficients degenerates at E_2. In particular any cohomology class $\alpha \in H^0(B, R^k\pi_*\mathbb{Q})$ comes from a global cohomology class $\tilde{\alpha} \in H^k(\mathcal{X}, \mathbb{Q})$.*

The proof plays on the relative Lefschetz decomposition associated with the polarization of the family, and the fact that it is compatible with the differentials d_i of the Leray spectral sequence, leading to the conclusion that $d_i = 0$, $i \geq 2$.

Remark 4 For this theorem, a real polarization suffices. Hence its range of applications does not restrict to families of projective manifolds.

2.1.2 Applications of the Kodaira criterion

The simplest application of Kodaira characterization of projective complex manifolds is the following.

Theorem 10 *Let X be a compact Kähler manifold such that $H^2(X, \mathcal{O}_X) = 0$. Then X is projective.*

Indeed, by the Hodge decomposition theorem, the assumption implies that $H^2(X, \mathbb{R})$ can be represented by real closed forms of type $(1,1)$. Working a little more, one can even choose a representative to vary continuously with the class. Now start from a Kähler class α represented by a Kähler form $\tilde{\alpha}$. Then as $H^2(X, \mathbb{Q})$ is dense inside $H^2(X, \mathbb{R})$, α can approximated by rational cohomology classes α_n. Choosing representative $\tilde{\alpha}_n$ of α_n, which are real closed forms of type $(1,1)$ converging to α, we conclude that $\tilde{\alpha}_n$ must be positive non-degenerate for n large enough, as it is an open property of real $(1,1)$-forms on compact complex manifolds. Thus α_n is Kähler for large n, and X is projective.

Another simple application is due to Campana [7]. We described above a varying family of Kähler $K3$ surfaces but showed that the total space could not be Kähler. On the other hand, if we allow singular fibers, it is easy to construct non isotrivial families of projective $K3$ surfaces whose total space is projective. For example, if f, g are two generic homogeneous polynomials of degree 4 on $\mathbb{P}^3$, then denote $f_t := t_0 f + t_1 g$, $t = (t_0, t_1) \in \mathbb{P}^1$, and let

$$\mathcal{X} = \{(x,t) \in \mathbb{P}^3 \times \mathbb{P}^1,\ f_t(x) = 0\},$$

with application $\pi = pr_2 : \mathcal{X} \to \mathbb{P}^1$.

$\mathcal{X}$ is smooth, being the blow-up of $\mathbb{P}^3$ along the base-locus of the pencil, and the generic fiber is the $K3$ surface defined by the equation $f_t = 0$ in $\mathbb{P}^3$.

Campana shows the following (we refer to [7] for a more general statement):

Theorem 11 *Let $\mathcal{X}$ be Kähler, and $\pi : \mathcal{X} \to B$ be holomorphic with generic fiber a $K3$ surface. Then either the generic fiber is projective, or the family is generically isotrivial (which means that the complex structure on the fiber X_t is constant on the open set where the fiber remains a smooth $K3$ surface).*

Indeed, one has the Torelli theorem for $K3$ surfaces (cf [37]), which says that the complex structure on a $K3$ surface S is determined by corresponding the Hodge structure on $H^2(S, \mathbb{Z})$. One then looks at the restriction map:

$$H^2(\mathcal{X}, \mathbb{Z}) \to H^2(S_t, \mathbb{Z}) \tag{2.10}$$

whose image is a constant sub-Hodge structure. If the induced map between the $(0,2)$-pieces

$$H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^2(S_t, \mathcal{O}_{S_t}) \tag{2.11}$$

is non-zero, then it is surjective because the right hand side has rank 1. Then the cokernel of (2.10) has a trivial Hodge structure, and it follows that the Hodge structure on $H^2(S_t, \mathbb{Z})$ is constant, thus implying isotriviality. But if the map (2.11) is 0, then the image of (2.10) consists of classes of type $(1,1)$. Consider the restriction map:

$$rest : H^2(\mathcal{X}, \mathbb{R}) \to H^2(S_t, \mathbb{R})$$

On one hand, the image is defined over $\mathbb{Q}$, so that rational cohomology classes are dense in it. On the other hand, it consists of classes of type $(1,1)$, hence by Lemma 1, it is equal to the image of the map

$$rest^{1,1} : H^{1,1}(\mathcal{X})_{\mathbb{R}} \to H^2(S_t, \mathbb{R}).$$

We then apply the same reasoning as in the proof of the previous lemma, approximating the restriction of a Kähler class by rational classes in $Im\,rest = Im\,rest^{1,1}$.
$\blacksquare$

2.2 Kodaira's theorem on surfaces

2.2.1 Some deformation theory

Kodaira's characterization theorem can also be used to show that certain compact Kähler manifolds X become projective after a small deformations of their complex structure. The point is that the Kähler classes belong to $H^{1,1}(X)_{\mathbb{R}}$, the set of degree 2 cohomology classes which can be represented by a real closed $(1,1)$-form. They even form an open cone, the Kähler cone, in this real vector subspace of $H^2(X,\mathbb{R})$. This space deforms differentiably with the complex structure, and by Kodaira's criterion we are reduced to see whether one can arrange that after a small deformation, the Kähler cone contains a rational cohomology class.

Let us recall a few facts about deformations of complex structure and variations of Hodge structure. Let B be a ball in $\mathbb{C}^n$, and let $\pi : \mathcal{X} \to B$ be a family of complex manifolds parameterized by B. Thus π is smooth and proper. If $X = X_0$ is the central fiber, $\mathcal{X}$ is called a family of deformations of X. The exact sequence

$$0 \to T_X \to T_{\mathcal{X}|X} \to T_{B,0} \otimes \mathcal{O}_X \to 0$$

of vector bundles on X induces the Kodaira-Spencer map

$$\rho : T_{B,0} = H^0(X, T_{B,0} \otimes \mathcal{O}_X) \to H^1(X, T_X).$$

Concretely, to get a representative of $\rho(u)$, one chooses a $\mathcal{C}^\infty$ trivialization of $\mathcal{X}$ over B

$$T : \mathcal{X} \cong X \times B,$$

which has the property that the $T^{-1}(x \times B)$, $x \in X$ are complex submanifolds of $\mathcal{X}$, and then $T_*^{-1}u$ gives a $\mathcal{C}^\infty$ section χ of $T_{\mathcal{X}}$ which lifts u: $\pi_*\chi = u$. Then $\overline{\partial\chi_{|X}}$ gives a $\overline{\partial}$-closed section of $\mathcal{A}^{0,1}(T_X)$, which represents $\rho(u)$. Next, we want to see how the Hodge filtration on $H^k(X_t,\mathbb{C}) \cong H^k(X,\mathbb{C})$ varies with the point $t \in B$.

Recall that $F^pH^k(X_t,\mathbb{C})$ is the set of classes α representable by a closed form $\tilde{\alpha}$ which is of type $(k,0)+\ldots+(p,k-p)$. This subspace varies differentiably with $t \in B$ because it has constant rank. (In fact each $h^{p,q}$ must be constant because their sum $\sum_{p+q=k} h^{p,q}$ is constant equal to b_k, while each of them is upper-semicontinuous.)

Furthermore, it is a consequence of Hodge theory that when t varies and $\alpha \in F^pH^k(X_t,\mathbb{C})$ varies, $\tilde{\alpha}$ can be chosen to vary differentiably with α. Choose a differentiably varying $\alpha_t \in F^pH^k(X_t,\mathbb{C})$. Using the lift $\tilde{\alpha}_t$, and the trivialization t, we can construct a k-form β on $\mathcal{X}$ which has the property that $\beta_{|X_t} = \tilde{\alpha}_t$ and that for any vector field v tangent to the $t^{-1}(x \times B)$, $x \in X$, $int(v)(\beta) = 0$.

Because the $T^{-1}(x \times B)$, $x \in X$, are complex submanifolds, the form β is of type $(k,0)+\ldots+(p,k-p)$ on $\mathcal{X}$. It is not closed, and in fact the Cartan formula for the Lie derivative says that for $u \in T_{B,0}$, one has

$$d_u(\alpha_t) = [int(\chi)(d\beta)_{|X}]. \tag{2.12}$$

Here we see α_t as a differentiable from B to $H^k(X,\mathbb{C})$. The $[\,]$ on the right means that we take the cohomology class of the considered form, which is closed. A more intrinsic formula would involve the Gauss-Manin connection.

As a corollary we get the famous transversality theorem of Griffiths:

Theorem 12 *For $\alpha_t \in F^pH^k(X_t,\mathbb{C})$, and $u \in T_{B,0}$, we have*

$$d_u\alpha_t \in F^{p-1}H^k(X,\mathbb{C}).$$

Indeed, as β is of type $(k,0)+\ldots+(p,k-p)$, $d\beta$ is of type $(k+1,0)+\ldots+(p,k-p+1)$, and as χ is of type $(1,0)$, we get that the closed form $int(\chi)(d\beta)_{|X}$ is of type $(k,0)+\ldots+(p-1,k-p+1)$. ∎

The formula (2.12) also gives us an explicit computation of the map

$$\overline{d}_u : F^pH^k(X,\mathbb{C}) \to H^{p-1,k-p+1}(X) = F^{p-1}H^k(X,\mathbb{C})/F^pH^k(X,\mathbb{C}),\ u \in T_{B,0},$$

which computes the infinitesimal deformation of $F^pH^k(X,\mathbb{C}) \subset H^k(X,\mathbb{C})$ with the complex structure in the direction u. (Here we use implicitly the fact that the tangent space of the Grassmannian $Grass(l,W)$ at a point $V \subset W$ is canonically $Hom(V,W/V)$.) We have

Theorem 13 *(Griffiths) $\overline{d}_u(\alpha)$ vanishes for $\alpha \in F^{p+1}H^k(X,\mathbb{C})$ so that*

$$\overline{d}_u \in Hom(H^{p,k-p}(X), H^{p-1,k-p+1}(X)),$$

and we have

$$\overline{d}_u(\alpha) = int(\rho(u))(\alpha),\ \forall \alpha \in H^{p,q}(X). \tag{2.13}$$

Here, $\rho(u) \in H^1(X,T_X)$, and we use the obvious product-contraction

$$H^1(X,T_X) \otimes H^{k-p}(X,\Omega_X^p) \to H^{k-p+1}(X,\Omega_X^{p-1})$$

to define $int(\rho(u))(\alpha)$.

The first statement is obvious by the transversality property applied to $F^{p+1}H^k(X_t)$. As for the second it is an immediate consequence of formula (2.12), observing that the $(p-1,k-p+1)$-component of $int(\chi)(d\beta)_{|X}$ is equal to $int(\chi)(\overline{\partial}\beta)_{|X}$ and that this is equal to $int(\overline{\partial}\chi)(\beta)_{|X}$ modulo a $\overline{\partial}$-exact form. ∎

2.2.2 Density of projective complex manifolds

Given a compact Kähler manifold X, there is the following criterion for the existence of a projective small deformation of the complex structure on X.

Proposition 2 *Assume deformations of X are unobstructed, and for some Kähler class $\alpha \in H^{1,1}(X)_{\mathbb{R}}$ the product-contraction map:*

$$.\alpha : H^1(X,T_X) \to H^2(X,\mathcal{O}_X),\ u \mapsto int(u)(\alpha)$$

is surjective. Then there exist arbitrarily small deformations of the complex structure of X which are projective.

Recall that “unobstructed” means that there exists a family of deformations of X

$$\pi : \mathcal{X} \to B,$$

such that the Kodaira-Spencer map $\rho : T_{B,0} \to H^1(X, T_X)$ is an isomorphism.

Buchdahl stated this criterion in [6] without the assumption that X is unobstructed, but the proof (which is much more analytic) does not seem to be complete.

Let us sketch the proof of the criterion. The proof is the same as in [44], II, 5.3.4, where it is used to prove the density of the Noether-Lefschetz locus. We claim that the condition $.\alpha$ be surjective is equivalent to the fact that the natural map

$$\phi : \mathcal{H}^{1,1}_{\mathbb{R}} \to H^2(X, \mathbb{R})$$

is submersive at α, where $\mathcal{H}^{1,1}_{\mathbb{R}}$ is the vector bundle over B with fiber $H^{1,1}(X_t)_{\mathbb{R}}$ at $t \in B$ and the map sends $\alpha_t \in H^{1,1}(X_t)_{\mathbb{R}}$ to $\alpha_t \subset H^2(X_t, \mathbb{R}) = H^2(X, \mathbb{R})$.

The claim implies the theorem, because if ϕ is submersive at α, it is open, hence $\phi(V)$ contains an open set U in $H^2(X, \mathbb{R})$, for V a small open neighbourhood of α that we may assume to be contained in $\mathcal{K}$, where $\mathcal{K} \subset \mathcal{H}^{1,1}_{\mathbb{R}}$ is the union of the Kähler cones:

$$\mathcal{K} = \cup_{t \in B} K_t \subset \cup_{t \in B} H^{1,1}(X_t)_{\mathbb{R}} = \mathcal{H}^{1,1}_{\mathbb{R}}.$$

But the rational classes are dense in U, and thus there is a point t close to 0, together with a Kähler class α_t close to α, with α_t rational.

To see the claim, we note that the submersivity of ϕ is implied by the submersivity of the map

$$\Phi : F^1\mathcal{H}^2 \to H^2(X, \mathbb{C}),$$

where $F^1\mathcal{H}^2$ is the vector bundle with fiber $F^1H^2(X_t)$ over $t \in B$, and the map Φ is defined on $F^1H^2(X_t)$ as the inclusion $F^1H^2(X_t) \subset H^2(X_t, \mathbb{C})$ followed by the identification $H^2(X_t, \mathbb{C}) = H^2(X, \mathbb{C})$. Indeed, one notices that

$$H^{1,1}(X_t)_{\mathbb{R}} = F^1H^2(X_t) \cap H^2(X_t, \mathbb{R}).$$

Hence ϕ is nothing but the restriction of Φ to $\Phi^{-1}(H^2(X, \mathbb{R}))$.

Finally the fact that submersivity of ϕ at α is equivalent to the surjectivity of $.\alpha$ follows from Griffiths’ formula (2.13).

■

Example 7 $K3$ surfaces and more generally hyper-Kähler manifolds satisfy the criterion. A hyper-Kähler manifold is a Kähler compact manifold which admits one holomorphic 2-form $\eta \in H^0(X, \Omega^2_X)$ which is everywhere non-degenerate (see [1], [26]). By Hodge symmetry, such a manifold has $H^2(X, \mathcal{O}_X)$ of rank 1. It is a well-known theorem due to Tian, Bogomolov, Todorov (cf [39]) that these manifolds are unobstructed. If α is a Kähler form on X, then $\alpha \in H^1(X, \Omega_X)$. As η is non degenerate, the map

$$.\eta : H^1(X, T_X) \mapsto H^1(X, \Omega_X)$$

is an isomorphism, hence

$$\alpha = int(u)(\eta)$$

for some $u \in H^1(X, T_X)$. Then $int(u)(\alpha)$ is equal up to a coefficient to $q(\alpha)\overline{\eta}$, where q is the Beauville quadratic form (cf [1]). Hence, as q is positive on the Kähler cone, the map $\cdot\alpha$ is surjective.

Example 8 Complex tori satisfy the criterion. Again they are unobstructed, this is immediate because they are locally parameterized by an open set in the Grassmannian $Grass(n, 2n)$ of n-dimensional subspaces of a $2n$-dimensional complex space (cf Example 4) and it is easy to verify that the Kodaira-Spencer map for this family is an isomorphism. The infinitesimal criterion is easy to check using $H^1(X, T_X) = T_0 \otimes H^1(X, \mathcal{O}_X)$, where T_0 is the tangent space at 0 of the complex torus X, $H^1(X, \Omega_X) = \Omega_0 \otimes H^1(X, \mathcal{O}_X)$ and $H^2(X, \mathcal{O}_X) \cong \bigwedge^2 H^1(X, \mathcal{O}_X)$.

To conclude this section, let us state the beautiful theorem of Kodaira which was at the origin of the work [42].

Theorem 14 *Let S be a compact Kähler surface. Then there is an arbitrarily small deformation of S which is projective.*

Kodaira proved this theorem using his classification of surfaces. We will sketch in the next section an argument due to Buchdahl, proving this theorem without classification, in the case of unobstructed surfaces.

2.2.3 Buchdahl's approach

Buchdahl [6] starts with a compact Kähler surface which he assumes to be minimal, without loss of generality. He uses Proposition 2 in order to prove Kodaira's theorem 14 for unobstructed surfaces. His main result can be stated as follows:

Theorem 15 *Let S be a compact Kähler surface, and $\tilde{\alpha}$ be a Kähler form on S. If the map*

$$.\alpha : H^1(S, T_S) \to H^2(S, \mathcal{O}_S),\ u \mapsto int(u)(\alpha)$$

is not surjective, then S is projective.

The idea of his proof is as follows: one notes that by Serre duality, the condition is equivalent to the following:

There exists $0 \neq \eta \in H^0(S, K_S)$, such that

$$\eta\alpha = 0 \text{ in } H^1(S, \Omega_S(K_S)).$$

Indeed $H^1(S, \Omega_S(K_S))$ is Serre dual to $H^1(S, T_S)$ and one has up to sign:

$$< \eta\alpha, u >=< \eta, int(u)(\alpha) >, \forall u \in H^1(S, T_S).$$

Thus it suffices to take for η an element vanishing on $Im\,(\cdot\alpha)$.

Note that this means concretely that

$$\eta\tilde{\alpha} = \overline{\partial}\omega$$

for some section ω of $\Omega_S(D)$ where D is the divisor of η. It will be convenient to consider η as a section of K_S identifying K_S to $\mathcal{O}_S(D)$ and write s for the canonical section of $\mathcal{O}_S(D)$. Then equation above can be better written as

$$\tilde{\alpha} = \overline{\partial}(\frac{\omega'}{s}),$$

for some $\mathcal{C}^\infty$ section ω' of $\Omega_S(D)$ on S.

Next the goal is to show that if S is not projective, this implies that α is supported on D. Then, since $\alpha^2 > 0$, it will follow that S carries a divisor of positive self-intersection, which gives a contradiction (see [2], p 126).

Observe that as S is minimal we must have $D^2 \geq 0$, and as S is assumed not algebraic, $D^2 \leq 0$. Thus $D^2 = 0$.

The only divisors on S are the components of D; in particular any line bundle L on S satisfies $h^0(S, L) \leq 1$. Indeed, let D' be an irreducible divisor not supported on D. Then D' does not intersect D since otherwise $(D + D')^2 > 0$ and S is algebraic. But then, as $\tilde{\alpha} = \overline{\partial}(\frac{\omega'}{s})$ is $\overline{\partial}$-exact away from D, we find that

$$\int_{D'} \alpha = 0$$

which is absurd.

Let us come back to the equation

$$\tilde{\alpha} = \overline{\partial}(\frac{\omega'}{s}).$$

It can be also written as

$$\tilde{\alpha} = d(\frac{\omega'}{s}) - \partial(\frac{\omega'}{s}). \tag{2.14}$$

Note that $\partial(\frac{\omega'}{s})$ is a holomorphic 2-form on $S \setminus D$, with at most order 2 poles along D, because

$$\overline{\partial}\partial(\frac{\omega'}{s}) = -\partial\tilde{\alpha} = 0.$$

Thus, as $h^0(S, K_S(2D)) \leq 1$, we must have a relation

$$\partial(\frac{\omega'}{s}) = \mu\eta,$$

for some coefficient $\mu \in \mathbb{C}$. One deduces from this and equation (2.14) that $\tilde{\alpha} + \mu\eta$ is exact on $S \setminus D$.

However the kernel of the restriction map

$$H^2(S, \mathbb{C}) \to H^2(S \setminus D, \mathbb{C})$$

is generated by the classes of the components of D (see [44], I, 11.1.2). Thus it follows that in fact $\mu = 0$ and the cohomology class α of $\tilde{\alpha}$ is supported on D. But as S is not algebraic, the intersection form on the subspace generated by the classes of the components of D is non positive, contradicting the fact that $\alpha^2 > 0$. ■

2.3 The Kodaira problem

2.3.1 The Kodaira problem

Kodaira's theorem 14 leads immediately to ask a number of questions in higher dimensions:

Question 1: *(The Kodaira problem) Does any compact Kähler manifold admit an arbitrarily small deformation which is projective?*

In order to disprove this, it suffices to find rigid Kähler manifolds which are not projective. However, the paper [17] shows that it is not so easy: if a complex torus T carries three holomorphic line bundles L_1, L_2, L_3 such that the deformations of T preserving the L_i are trivial, then T is projective. The relation with the previous problem is the fact that from (T, L_1, L_2, L_3), one can construct a compact Kähler manifolds whose deformations identify to the deformations of the 4-uple (T, L_1, L_2, L_3).

A weaker question concerns global deformations.

Question 2: *(The global Kodaira problem) Does any compact Kähler manifold X admit a deformation which is projective?*

Here we consider any deformation parameterized by a connected analytic space B, that is any smooth proper map $\pi : \mathcal{X} \to B$ between connected analytic spaces, with $X_0 = X$ for some $0 \in B$. Then any fiber X_t will be said to be a deformation of X_0. In that case, even the existence of rigid Kähler manifolds which are not projective would not suffice to provide a negative answer, as there exist varieties which are locally rigid but not globally (consider for example the case of $\mathbb{P}^1 \times \mathbb{P}^1$ which deforms to a different Hirzebruch surface). This means that we may have a family of smooth compact complex manifolds $\pi : \mathcal{X} \to B$ whose all fibers X_t for $t \neq 0$ are isomorphic but are not isomorphic to the central fiber X_0.

Note that if X is a deformation of Y, then X and Y are diffeomorphic, because the base B is path connected, and the family of deformations can be trivialized in the $\mathcal{C}^\infty$-category over any path in B.

In particular, X and Y should be homeomorphic, hence have the same homotopy type, hence also the same cohomology ring. Thus Question 2 can be weakened as follows :

Question 3: *(The topological Kodaira problem) Is any compact Kähler manifold X diffeomorphic or homeomorphic to a projective complex manifold?*

Does any compact Kähler manifold X have the homotopy type of a projective complex manifold?

Our results will provide a negative answer to all these questions, answering negatively the last one.

It should be mentioned that the examples built in [42] had the property that they are bimeromorphically equivalent to complex tori or Kummer manifolds, which, as explained in the previous section have small projective deformations. Thus, a natural question was the following, asked by Buchdahl, Campana, Yau:

Question 4: *(The birational Kodaira problem) Is any compact Kähler manifold X bimeromorphic to a smooth compact complex manifold which deforms to a projective complex manifold?*

This is also disproved by topological methods in [43]. However, the compact Kähler manifolds constructed there do not have nonnegative Kodaira dimension, as they are bimeromorphic to $\mathbb{P}^1 \times \mathbb{P}^1$-bundles on a product of Kummer manifolds. Thus the following remains open:

Conjecture 1 *(Campana) Is any compact Kähler manifold X of nonnegative Kodaira dimension bimeromorphic to a smooth compact complex manifold which deforms to a projective complex manifold?*

2.3.2 Topological restrictions

In this section, we want to give a perspective to our results by recalling a number of other results providing topological obstructions for differentiable manifolds to be either symplectic, almost complex, complex or Kähler.

Starting from the notion of compact differentiable manifold M, we have a number of more or less restrictive conditions that we can impose to M. The weakest one is the existence of an almost complex structure, namely a complex structure on the tangent bundle. It is an observation due to Gromov that the existence of an almost complex structure is equivalent to the existence of a non degenerate 2-form on M. The easiest direction is from almost structures to non-degenerate 2-forms, as one can obtain these as the imaginary part of a Hermitian metric on the tangent bundle. In the other direction, this follows from the fact that compatible complex structures I on a real vector space endowed with a symplectic form ω form a contractible set. Here the compatibility means that ω is positive of type $(1,1)$ with respect to I. Not every even dimensional compact manifold admits an almost complex structure [36]. For example, if a 4-dimensional manifold M admits an almost complex structure, one has the Hirzebruch signature theorem [25]:

Theorem 16 *The signature $b_2^+ - b_2^-$ of a compact 4 dimensional manifold M admitting an almost complex structure (compatible with the orientation) is given by*

$$b_2^+ - b_2^- = \frac{1}{3}(c_1^2 - 2c_2),$$

where the c_i's are the Chern classes of T_M considered as a complex (rank 2) vector bundle.

Thus S^4 does not admit an almost complex structure, because one would have $c_1^2 = 0 = b_2^+ = b_2^-$, because $H^2(S^4, \mathbb{Z}) = 0$, while $c_2(S^4) = e(T_{S^4}) = \chi_{top}(S^4) = 2$.

When we introduce integrability conditions, we are led to two distinct notions, that of complex structure, where one asks that the almost complex structure comes from a true complex structure :this is characterized by the the Newlander-Nirenberg theorem :

Theorem 17 *An almost complex structure comes from a complex structure if and only if the bracket of two vector fields of type $(1,0)$ is again of type $(1,0)$.*

In the symplectic direction, we might want that the non-degenerate 2-form ω be closed, which leads to the notion of symplectic manifold. There are obvious obstructions for a $2n$-dimensional manifold to be symplectic: namely, if M is symplectic, the closed 2-form ω has a cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$, and ω^n being a volume form, we see that $[\omega]^n$ does not vanish in $H^{2n}(M, \mathbb{R})$. Thus in particular $[\omega] \neq 0$. For example, the sphere S^6, which admits a non-degenerate 2-form (cf [36], p 118), is not a symplectic manifold. It is not known however if S^6 carries a complex structure, while it carries an almost complex structure.

Finally, one might want to impose both integrability conditions, namely that I be integrable and that ω be closed, and this is exactly the Kähler condition. As we have seen before, this imposes a lot of supplementary topological restrictions. The easiest one is the fact that the odd Betti numbers should be even, because of the Hodge decomposition

$$H^{2k+1}(X) = \oplus_{p+q=2k+1} H^{p,q}(X)$$

together with the Hodge symmetry $h^{p,q}(X) = h^{q,p}(X)$ (Theorem 3). It is well known that for complex surfaces, being Kähler is equivalent to the condition that b_1 (hence also b_3 by Poincaré duality) is even.

More subtle restrictions are the following:

- The Hodge decomposition on the cohomology of a compact Kähler manifold is compatible with the cup-product, or in other words, the complex cohomology ring is bigraded, with a bigradation which satisfies the Hodge symmetry. It is easy to see that not any cohomology ring admits such a bigradation.

- The hard Lefschetz theorem implies that the operator L is injective on $H^k(X, \mathbb{R})$, for $k < dim\, X$. Thus we get that the even Betti numbers $b_{2i}(X)$ are increasing with i in the range $2i \leq n$, and similarly for the odd Betti numbers.

- Further restrictions are given by the second Hodge-Riemann bilinear relations, for example, in the surface case, they say that the intersection form is positive definite on the space $(H^{2,0}(S) \oplus H^{0,2}(S)) \cap H^2(S, \mathbb{R}) \oplus \mathbb{R}[\omega]$ and negative on the supplementary space $H^{1,1}(S)_{\mathbb{R}} \cap [\omega]^{\perp}$. As the first space has odd dimension, one concludes that b_2^+ is odd for a compact Kähler surface.

These restrictions are purely topological and do not depend on the symplectic structure. One might wonder if some of them are already satisfied in the symplectic case, or ask whether those properties which involve only the Kähler class are satisfied in the symplectic case. But the answer to this is negative. There are known examples of symplectic non Kähler manifolds, starting from real dimension 4 (cf [36] p 89 for a non simply connected example, quotient of $\mathbb{R}^4$, and [35] for simply connected examples). The hard Lefschetz theorem imposes a non trivial condition for a symplectic compact manifold to be Kähler, as it asks that the L operator, which depends only on the symplectic structure, satisfies the Lefschetz theorem 5. There are examples of symplectic manifolds not satisfying the Lefschetz property. Namely, the Lefschetz property implies that odd Betti numbers are even, because it implies by Poincaré duality that the skew-symmetric form

$$q_\omega(\alpha, \beta) = \int_X \omega^{n-k} \cup \alpha \cup \beta),\ \alpha,\ \beta \in H^k(X, \mathbb{R}),\ k\ odd,$$

is non-degenerate.

Thus the (real) 4-dimensional example of [36], p 89, which has $b_1 = 3$, does not satisfy the Lefschetz property. It is also possible to construct examples of compact symplectic manifolds with no odd cohomology, and which do not satisfy the Lefschetz property.

Note that if M is symplectic, with symplectic form Ω, a small perturbation of Ω will have rational cohomology class. This is the starting point in Donaldson's construction of symplectic submanifolds of codimension 2, namely one can choose the symplectic form to be integral and hence the first Chern class of a complex line bundle.

The contents of the results we will present in the next section is that in the Kähler context, imposing that the Kähler form be of rational class, that is choosing the Kähler complex structure to be projective, may be impossible for topological reasons.

2.3.3 Statement of the results

We want to state here the existence of compact Kähler manifolds for which there are topological obstructions to the existence of projective complex structures on them. As mentioned in section 2.3.1, this disproves any higher dimensional generalization of Kodaira's theorem 14, except maybe Conjecture 1. Our first result is the following (cf [42]).

Theorem 18 *There exist, in any dimension ≥ 4, examples of compact Kähler manifolds which do not have the integral cohomology ring of a projective complex manifold.*

The first example we constructed was non simply connected and in fact bimeromorphic to a complex torus. Deligne provided then us with lemma 6, which allowed him to prove the following ([11], [42]):

Theorem 19 *There exist, in any dimension ≥ 4, examples of compact Kähler manifolds, which do not have the rational or even complex cohomology ring of a projective complex manifold.*

We then realized that Deligne's lemma 6, combined with Hodge index theorem, could be used to produce examples of simply connected compact Kähler manifolds satisfying the conclusion of Theorem 18, at least in dimension ≥ 6 (cf [42], section 3).

All these examples were built by blowing-up in an adequate way compact Kähler manifolds which had the property of deforming to projective ones, namely self-products of complex tori, or self-products of Kummer varieties. This led open the possibility that under bimeromorphic transformations, the topological obstructions we obtained above for a Kähler manifold to admit a projective complex structure would disappear. However we proved in [43] the following result.

Theorem 20 *In dimensions ≥ 10, there exist compact Kähler manifolds, no smooth bimeromorphic model of them has the rational cohomology ring of a projective complex manifold.*

We will give the examples and the detail of the argument in the next sections. Let us say that the topological obstruction that we exhibit comes from the Hodge-Riemann bilinear relations. The point is that the Hodge decomposition on the cohomology groups of a compact Kähler manifold is compatible with the ring structure (cf section 1.2.3). If the ring structure is rich enough, this may force the Hodge structures to admit endomorphisms of Hodge structures. But certain endomorphisms of Hodge structures are forbidden by the existence of a polarization.

3 Hodge theory and homotopy types

3.1 Construction of examples

3.1.1 The torus example

The simplest example of a compact Kähler manifold which cannot admit a projective complex structure for topological reasons is based on the existence of endomorphisms of complex tori which prevent the complex tori in question to be algebraic. Let Γ be a rank $2n$ lattice, and let ϕ be an endomorphism of Γ.

Assume that the eigenvalues of ϕ are all distinct and none is real.

Choosing n of these eigenvalues $\lambda_1, \ldots, \lambda_n$, so that no two of them are complex conjugate, then one can define

$$\Gamma^{1,0} := \textit{eigenspace associated to the } \lambda_i' s \subset \Gamma_{\mathbb{C}},$$

and

$$T = \Gamma_{\mathbb{C}}/(\Gamma^{1,0} \oplus \Gamma).$$

Clearly, the extended endomorphism $\phi_{\mathbb{C}}$ of $\Gamma_{\mathbb{C}}$ preserves both $\Gamma^{1,0}$ and Γ, and thus descends to an endomorphism ϕ_T of T.

We have then the following [42]:

Proposition 3 *If $n \geq 2$ and the Galois group of the field $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, \overline{\lambda}_1, \ldots, \overline{\lambda}_n)$, (that is the splitting field of $\mathbb{Q}(\phi)$,) acts as the full symmetric group $\mathfrak{S}_{2n}$ on the eigenvalues of ϕ, then T has $Hdg^2(T, \mathbb{Q}) = 0$ and thus T is not projective.*

Remark 5 In fact it would suffice here to know that the Galois group acts bitransitively on the eigenvalues. However, for the purpose of [43], which needs also the absence of Hodge classes of higher degree on $T \times \hat{T}$, except for the obvious ones, this stronger condition on the Galois group is needed.

Indeed, one looks at the action ϕ_T^* of ϕ_T on $H^2(T, \mathbb{Q}) = \bigwedge^2 \Gamma_{\mathbb{Q}}^*$. ϕ_T^* identifies to $\wedge^{2t}\phi$. The assumption on the Galois group then shows that this action is irreducible.

On the other hand, this action preserves the subspace $Hdg^2(T, \mathbb{Q})$, which must then be either 0 or the whole of $H^2(T, \mathbb{Q})$. As $n \geq 2$, we have $H^{1,1}(T) \neq H^2(T, \mathbb{C})$ and thus $Hdg^2(T, \mathbb{Q}) = 0$. ■

Our first example was the following. Let (T, ϕ_T) be as before, satisfying the assumptions of Proposition 3. Inside $T \times T$ we have the four subtori

$$T_1 = T \times 0,\ T_2 = 0 \times T,\ T_3 = Diag,\ T_4 = Graph(\phi_T),$$

which are all isomorphic to T.

These tori meet pairwise transversally in finitely many points $x_1, \ldots, x_N$. Blowing-up these points, the proper transforms $\widetilde{T_i}$ are smooth and do not meet anymore. We can thus blow-up them all to get a compact Kähler manifold X. This is our example.

Theorem 21 *If Y is a compact Kähler manifold such that there exists an isomorphism*

$$\gamma : H^*(Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$$

of graded rings, then Y is not projective.

In other words, X does not have the cohomology ring of a projective complex manifold. We shall explain later on a simple proof of that. With the help of Lemma 6, to be explained later on, Deligne then proved:

Theorem 22 *If Y is a compact Kähler manifold such that there exists an isomorphism*

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q})$$

of graded algebras, then Y is not projective.

Deligne also modified our example X in such a way that in the statement above, rational cohomology can be replaced with complex cohomology (cf [42], section 3.1).

3.1.2 Simply connected examples

This example is very similar, but in our first geometric argument to prove Theorem 21, we used heavily the Albanese map and this made the argument unlikely to work for simply connected varieties. However a combination of Deligne's lemma and Hodge index theorem allows finally to adapt the argument to the simply connected case, replacing the study of the Hodge structure on H^1 by a study of the Hodge structure on H^2.

We start with the same torus T as before, but we ask now that $n \geq 3$. Let us introduce the Kummer variety K of T, which is the desingularization of the quotient $T/\{\pm Id\}$ obtained by blowing-up the 2-torsion points (which are the fixed points of the involution $-Id$). $K \times K$ is a compact Kähler manifold, which contains two submanifolds birationally isomorphic to K, namely the diagonal Δ and the graph G of the rational map $\phi_K : K \dashrightarrow K$ deduced from ϕ_T. (ϕ_K is not necessarily holomorphic because ϕ_T may send a non 2-torsion point to a 2-torsion point.) We can then blow-up Δ, and then blow-up the proper transform of G. This gives our second example X'. X' is simply connected as K is simply connected.

This last fact can be seen as follows: K is an étale quotient of T away from its exceptional divisors over the 2-torsion points. Denoting

$$U := K \setminus \{exceptional\ divisors\ over\ 2 - torsion\ points\},$$

we thus have a surjection $\pi_1(U, u) \to \pi_1(K, u)$, where $\pi_1(U)$ identifies to the subgroup of $End\,(\Gamma_{\mathbb{C}})$ generated by translations by Γ and $-Id$. This surjection contains in its kernel the classes of the loops which start from u, turn around one of the exceptional divisors and then come back to u by the same path in reversed way. These are well defined under conjugation by the considered exceptional divisor. But the class γ of any of these loops sends to $-Id$, via the exact sequence

$$0 \to \Gamma \to \pi_1(U) \to \pm Id \to 0.$$

As Γ is commutative and the brackets $[-Id, \Gamma]$ generate 2Γ, one concludes that 2Γ goes to 0 in $\pi_1(K)$, and thus that $\pi_1(K)$ is a quotient of $\Gamma/2\Gamma \oplus \{\pm Id\}$. But as one can see, for an exceptional divisor D_η associated to a 2-torsion point $\eta \in \Gamma/2\Gamma$, the class of the loop γ_η described above is equal modulo 2Γ to $\eta + (-Id)$. Hence we have the equalities $\eta + (-Id) = 0$ in $\pi_1(K)$ for all $\eta \in \Gamma/2\Gamma$, which shows that $\pi_1(K) = 0$.

∎

We shall sketch in the next section the proof of the following

Theorem 23 *If Y is a compact Kähler manifold such that there exists an isomorphism*

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(X', \mathbb{Q})$$

of graded algebras, then Y is not projective.

3.1.3 Bimeromorphic examples

The previous examples are all bimeromorphic to the simplest possible compact Kähler compact manifolds, namely self-products of complex tori, or self-products of Kummer manifolds. These manifolds admit arbitrarily small deformations which are projective (Example 8). We sketch now the construction of [43].

We start again from a pair (T, ϕ_T) satisfying the assumptions of Proposition 3, but ask that $n \geq 4$. We introduce the dual complex torus $\widehat{T} = Pic^0(T)$ which parameterizes topologically trivial holomorphic line bundles on T. Recalling that T corresponds to a weight 1 Hodge structure (Example 4), $\widehat{T}$ is the complex torus corresponding to the dual weight 1 Hodge structure.

On $T \times \widehat{T}$ we have the Poincaré line bundle $\mathcal{L}$ which is determined by the conditions:

1. The restrictions $\mathcal{L}_{|T \times 0}$ and $\mathcal{L}_{|0 \times \widehat{T}}$ are trivial.
2. For each $t \in \widehat{T}$ parameterizing a line bundle L_t on T, the line bundle $\mathcal{L}_{|T \times t}$ is isomorphic to L_t.

We also have the line bundle $\mathcal{L}_\phi := (\phi_T, Id)^*\mathcal{L}$.

Consider the two vector bundles on $T \times \widehat{T}$

$$\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{-1},\ \mathcal{E}_\phi = \mathcal{L}_\phi \oplus \mathcal{L}_\phi^{-1}.$$

They have associated $\mathbb{P}^1$-bundles

$$\mathbb{P}(\mathcal{E}),\ \mathbb{P}(\mathcal{E}_\phi),$$

and we can take the fibered product

$$\mathbb{P}(\mathcal{E}) \times_{T \times \widehat{T}} \mathbb{P}(\mathcal{E}_\phi).$$

Recall now that K is birationally the quotient of T by the involution $\iota := -Id_T$ and introduce similarly $\widehat{K}$ which is the quotient of $\widehat{T}$ by $\hat{\iota} := -Id_{\widehat{T}}$. Thus $K \times \widehat{K}$ is birationally the quotient of $T \times \widehat{T}$ by the group generated by (ι, Id) and $(Id, \hat{\iota})$. One can show that this group lifts to a group $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{P}(\mathcal{E}) \times_{T \times \widehat{T}} \mathbb{P}(\mathcal{E}_\phi)$ (The point is that $(\iota, Id)^*\mathcal{L} \cong \mathcal{L}^{-1}$ so that we can lift (ι, Id) to an action on $\mathcal{L} \oplus \mathcal{L}^{-1}$ permuting the two summands, and similarly for $\mathcal{L}_\phi$.)

Our manifold X'' will be any desingularization of the quotient of $\mathbb{P}(\mathcal{E}) \times_{T \times \widehat{T}} \mathbb{P}(\mathcal{E}_\phi)$ by G.

Theorem 24 *For any compact Kähler manifold Y, and any smooth bimeromorphic model $\widetilde{X}''$ of X'', if there exists an isomorphism*

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(\widetilde{X}'', \mathbb{Q})$$

of graded algebras, then Y is not projective.

The proof uses a sophisticated elaboration of the arguments used for the proof of Theorem 23.

3.2 Proofs of the theorems

3.2.1 The torus case

We want to sketch here the proof of Theorem 21. Thus let X be constructed as in section 3.1.1, and $Y, \gamma : H^*(Y,\mathbb{Z}) \cong H^*(X,\mathbb{Z})$ be as in the Theorem. Our goal is to show that the Hodge structure on $H^1(Y,\mathbb{Z})$ cannot be polarized, thus proving that Y is not projective.

The cohomology group $H^2(X,\mathbb{Z})$ contains the classes e_i of the exceptional divisors E_i over the $\widetilde{T}_i$. We claim the following:

Lemma 5 *The classes $a_i := \gamma^{-1}(e_i)$ are Hodge classes on Y.*

Assuming this, it follows that the morphisms of Hodge structures

$$\cup a_i : H^1(Y,\mathbb{Z}) \to H^3(Y,\mathbb{Z})$$

have for kernels sub-Hodge structures L_i of $H^1(Y,\mathbb{Z})$. Of course $L_i = \gamma^{-1}(Ker \cup e_i)$.

Recall now that X is obtained from $T \times T$ by blow-ups. Thus $H^1(X,\mathbb{Z}) = H^1(T,\mathbb{Z}) \oplus H^1(T,\mathbb{Z})$. Furthermore, an easy computation involving the cohomology ring of a blow-up (cf [44], I, 7.3.3) shows that $Ker \cup e_i$, $i = 1,\ldots,4$, are equal respectively to

$$pr_2^* H^1(T,\mathbb{Z}),\ \ pr_1^* H^1(T,\mathbb{Z}),\ \ \Delta^-,\ \ Graph(\phi_T^*)^-,$$

where

$$\Delta^- := \{(\alpha, -\alpha),\, \alpha \in H^1(T,\mathbb{Z})\},$$
$$Graph(\phi_T^*)^- = \{(\phi_T^*\alpha, -\alpha),\, \alpha \in H^1(T,\mathbb{Z})\}.$$

But it follows that the 4 sub-Hodge structures L_i of $H^1(Y,\mathbb{Z})$ satisfy

$$L_1 \oplus L_2 = H^1(Y,\mathbb{Z})$$

as Hodge structures, and furthermore

$$L_3 \subset L_1 \oplus L_2,\, L_4 \subset L_1 \oplus L_2$$

can be seen as the graphs of two isomorphisms of Hodge structure $L_1 \cong L_2$. Thus we can set $L_1 = L_2 =: L$, and the second isomorphism gives an automorphism ψ of L. It is immediate to see that ψ identifies to ϕ_T^*.

Thus we proved that the Hodge structure on $H^1(Y,\mathbb{Z})$ is a direct sum $L \oplus L$, and that L carries an automorphism wich is conjugate to ϕ_T^*. By proposition 3, L is not polarizable, and neither the Hodge structure on $H^1(Y,\mathbb{Z})$. ∎

It remains to prove the Lemma. We will be brief here, as the use of Deligne's lemma 6 gives a much better approach to this statement, working for the rational cohomology ring as well. The point is that looking at the isomorphism γ, we can conclude that the Albanese map a_Y of Y must be birational to its image, as it is the case for X. Indeed this property can be seen on the cohomology ring of a m-dimensional Kähler compact manifold, because it is equivalent to the fact that the natural map given by cup-product:

$$\bigwedge^{2m} H^1(Y,\mathbb{Z}) \to H^{2m}(Y,\mathbb{Z})$$

is an isomorphism.

Having this, one checks that the $\gamma^{-1}(e_i)$ must be in the kernel of the Gysin map

$$(a_Y)_* : H^2(Y,\mathbb{Z}) \to H^2(Alb(Y),\mathbb{Z}),$$

and because $a_Y$is birational, this kernel consists of Hodge classes. ∎

3.2.2 Deligne's lemma and applications

As we have seen in the previous proof, the key point under the assumptions of Theorem 21 was to show that certain classes in $H^2(Y,\mathbb{Z})$ must be Hodge classes, and then use them to show the existence of automorphisms of Hodge structures which prevent the existence of a polarization on the Hodge structure of $H^1(Y,\mathbb{Z})$.

The following provides an alternative proof for Lemma 5, namely the fact that the classes $\gamma^{-1}(e_i)$ must be Hodge classes, even if γ is only an isomorphism of rational cohomology rings, which leads to the proof of Theorem 22.

Let $A^* = \oplus A^k$ be a graded $\mathbb{Q}$-algebra, and assume that each A^k carries a weight k Hodge structure, compatible with the product. (Recall that this means that the product map

$$A^k \otimes A^l \to A^{k+l} \tag{3.15}$$

is a morphism of weight $k+l$ Hodge structures.)

Lemma 6 *Let $Z \subset A^k_{\mathbb{C}}$ be a closed algebraic subset defined by homogeneous equations expressed only in terms of the product map on A^*, and let $Z' \subset Z$ be an irreducible component of Z. Assume the vector space $< Z' >$ generated by Z' is defined over $\mathbb{Q}$, that is*

$$< Z' >= B \otimes \mathbb{C},$$

for some $\mathbb{Q}$-vector subspace B of A^k.

Then B is a rational sub-Hodge structure of A^k.

Here, by "defined by homogeneous equations expressed only in terms of the product map on A^*", we mean eg the following kind of algebraic subsets:

1. $Z = \{\alpha \in A^k_{\mathbb{C}},\, \alpha^l = 0 \text{ in } A^{kl}\}$, where l is a fixed integer.
2. $Z = \{\alpha \in A^k_{\mathbb{C}},\, rk\,(\alpha\cdot : A^l \to A^{k+l}) \leq m\}$ where l, m are fixed integers.

The proof is immediate. Indeed, as B is rational, to say that it is a sub-Hodge structure of A^k is equivalent to say that $B_{\mathbb{C}}$ is stable under the Hodge decomposition, or equivalently, that $B_{\mathbb{C}} =< Z' >$ is stable under the $\mathbb{C}^*$-action (see section 1.1.3) defining the Hodge decomposition.

But this is immediate because the compatibility of the Hodge structures with the product is equivalent to the fact that the product map (3.15) is equivariant with respect to the $\mathbb{C}^*$-actions on both sides. Hence Z, and also Z', are stable under the $\mathbb{C}^*$-action, and thus, so is $< Z' >$. ∎

A first application of this is the following proof of Lemma 5. (In fact we will prove a slightly weaker statement, but which is enough for our purpose.) First of all, consider the subset $P \subset H^2(X,\mathbb{Q})$ generated by classes of exceptional divisors over $T \times T$. This set P is characterized intrinsically by the cohomology ring of X, as being the subspace annihilating (under cup-product) the image of $\bigwedge^{4n-2} H^1(X,\mathbb{Q})$ in $H^{4n-2}(X,\mathbb{Q})$.

Inside P, there is the subspace P' generated by the classes of the total transforms of exceptional divisors over points. This P' has the property that for any $a \in P'$, the cup-product map $a\cup : H^1(X,\mathbb{Q}) \to H^3(X,\mathbb{Q})$ vanishes, and in fact P' is the subspace characterized by this property. Thus by Deligne's lemma, we find that both $\gamma^{-1}(P')$ and $\gamma^{-1}(P)$ are sub-Hodge structures of $H^2(Y,\mathbb{Q})$.

Finally, we look at the natural map induced by cup-product on Y:

$$\mu : \gamma^{-1}(P)/\gamma^{-1}(P') \to Hom\,(H^1(Y,\mathbb{Q}), H^3(Y,\mathbb{Q})).$$

Looking at the structure of the cohomology ring of X, we find that the set of elements p of $(\gamma^{-1}(P)/\gamma^{-1}(P')) \otimes \mathbb{C}$ for which $\mu(p) : H^1(Y,\mathbb{Q}) \to H^3(Y,\mathbb{Q})$ is not injective is the union of the four lines generated by $a_i = \gamma^{-1}(e_i)$ (or more precisely their projections modulo P'). Hence Deligne's lemma shows that the projection of each a_i in $\gamma^{-1}(P)/\gamma^{-1}(P')$ is a Hodge class. The rest of the proof then goes as before, because we conclude that the $\mu(a_i)$ are morphisms of Hodge structures, which is the only thing we need. ■

Let us give a few other applications of Deligne's lemma, towards the proof of Theorem 23.

Consider the manifold X' constructed in section 3.1.2. Recall that X' admits a holomorphic map ψ to $T/\pm Id \times T/\pm Id$. There are thus two $\mathbb{Q}$-subalgebras

$$A_1^* := (pr_1 \circ \psi)^* H^*(T/\pm Id,\mathbb{Q}), \quad A_2^* := (pr_1 \circ \psi)^* H^*(T/\pm Id,\mathbb{Q})$$

of $H^*(X,\mathbb{Q})$.

One can show the following:

Lemma 7 *Irreducible components of*

$$Z = \{\alpha \in H^2(X,\mathbb{Q}),\, \alpha^2 = 0\}$$

are given as

$$Z_i = \{(pr_i \circ \psi)^*\alpha,\, \alpha \in H^2(T/\pm Id,\mathbb{Q}),\, \alpha^2 = 0\}.$$

Furthermore we have $< Z_i >= A_i^2$.

Let Y, γ be as in Theorem 23. Deligne's Lemma combined with Lemma 7 gives us:

Lemma 8 *The two subspaces* $\gamma^{-1}(A_i^2)$ *are rational sub-Hodge structures of* $H^2(Y,\mathbb{Q})$.

The conclusion of the proof of Theorem 23 uses now two ingredients.

First of all, Lemma 3 guarantees us that the Hodge structures on $\gamma^{-1}(A_i^2) \subset H^2(Y,\mathbb{Q})$ are non trivial.

Next, we consider $P \subset H^2(Y,\mathbb{Q})$ defined as the annihilator of the image of $\bigotimes^{2n-1}(\gamma^{-1}(A_1^2) \oplus \gamma^{-1}(A_2^2))$ in $H^{4n-2}(Y,\mathbb{Q})$. This space is generated by the $\gamma^{-1}(e)$, where e are the classes of the exceptional divisors of the map ψ. This P is a sub-Hodge structure of $H^2(Y,\mathbb{Q})$ by Deligne's lemma. Looking at the cup-product map

$$\cup p : \gamma^{-1}(A_1^2) \oplus \gamma^{-1}(A_2^2) \to H^4(Y,\mathbb{Q}),\, p \in P$$

and applying Deligne's Lemma again, one concludes that P has a lot of Hodge classes, and finally, as in the end of the previous section, that the two rational Hodge structures $\gamma^{-1}(A_1^2)$ and $\gamma^{-1}(A_2^2)$ are isomorphic, say to L, and that L carries an automorphism of rational Hodge structures conjugated to $\wedge^2\phi_T^*$.

But the action of $\wedge^2\phi_T^*$ on $H^2(T/\pm Id,\mathbb{Q}) = H^2(T,\mathbb{Q})$ is irreducible, and as the Hodge structure on $\gamma^{-1}(A_i^2)$ is non trivial, we conclude that $\gamma^{-1}(A_i^2)$ contains no non zero Hodge classes. Thus all the Hodge classes of $H^2(Y,\mathbb{Q})$ lie in P, and it is then easy to conclude that there is no rational Kähler class on Y, as classes in P cannot polarize the Hodge structure on $H^2(Y,\mathbb{Q})$. ∎

3.3 Concluding remarks

In conclusion, the results in [42], [43], [45] show that there is an important gap between Kähler compact and complex projective geometry. Results of [45] point out a gap from the point of view of analytic geometry, while results of [42] show that there is even a gap from the point of view of topology.

It would be interesting to investigate other important conjectures from the point of view of Kähler geometry. A number of results are not known to hold in the general Kähler case. For example, the main results of Mori et al on the minimal model programm are not known in Kähler geometry. It is conjectured (and proved up to dimensions 3) that projective manifolds of negative Kodaira dimension are uniruled. Is this conjecture reasonable in the general Kähler case?

We would like to point out also a number of interesting questions asked by Catanese, concerning his Q.E.D. equivalence. Q.E.D. equivalence is the equivalence relation between complex projective (or singular compact Kähler) varieties with canonical singularities generated by:

1. Birational equivalence.
2. Deformation equivalence (allowing only singular fibers with canonical singularities).
3. Quasi-étale maps.

Here quasi-étale means étale in codimension 1. One question asked in [9] is the following.

Question. *Is any Kähler compact manifold QED equivalent to a projective one?*

This is not a priori disproved by our example of section 3.1.3, which up to birational equivalence is a quotient of a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle on a product $T \times \widehat{T}$. The fixed points of the group action are in codimension ≥ 2, so that the quotient map is quasi-étale. One can show that the $\mathbb{P}^1 \times \mathbb{P}^1$-bundle on the product $T \times \widehat{T}$ does not deform to a projective manifold; however it is unclear if this property remains true for any of its smooth bimeromorphic models.

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