Rational Simple-Connectedness and Rational Points

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Joint work with A. J. de Jong

I. Three Problems

**Problem 1:** (Hilbert scheme/Chow variety problem)

Let $k$ be alg. closed. Give examples of smooth, projective $k$-schemes $X$ such that:

(i) $X$ is "close to being rational", and

(ii) the Hilbert schemes/Chow varieties of rational curves on $X$ are also "close to being rational."

**Problem 2:** (Existence of rational points)

Let $(K, X)$ be a pair of a field $K$—typically not alg. closed—and a smooth, projective, geometrically integral $K$-scheme. Give sufficient conditions on $(K, X)$ for $X$ to have a $K$-rational point.

**Problem 3:** (Period-Index Problem)

Let $(K, D)$ be a pair of a field $K$ and a division algebra $D$ whose center is $K$. Find a relationship b/w $\text{index}(D) := \sqrt{\text{dim}_K(D)}$ and $\text{period}(D) := \text{order of } [D] \text{ in the Brauer group of } K$. 
Observation. Problem 2 (existence of rational points) is related to the Brauer group of $K$:

There is a short-exact-sequence

$$\text{Pic}(X) \to \text{Pic}(X_{\overline{K}}) \to \text{Gal}(\overline{K}/K) \to \text{Br}(K)$$

If $X$ has a $K$-rational point, then $\delta_K$ is zero.

So $\delta_K$ is "a Brauer obstruction" (but almost certainly not "the only cohomological obstruction").

Explanation. "Close to being rational" equals "rationally connected": A smooth projective scheme $X$ over an alg. closed field $k$ is rationally connected if there exists an integral $k$-scheme $M$ and a $k$-morphism $u : M \times \mathbb{P}^1 \to X$ whose associated morphism $u^{(2)} : M \times \mathbb{P}^1 \times \mathbb{P}^1 \to X \times X$ is dominant:

$$(m, t_1, t_2) \mapsto (u(m, t_1), u(m, t_2)).$$

Equivalent Formulation in Char. 0: $X$ is $\text{RC}$ iff $X$ is connected and there exists:

$$f : \mathbb{P}^1 \to X$$

such that $f^*(T_X)$ is ample.

"very free rational curve"
Some Answers:

**Problem 1.** (B. Kim & R. Pandharipande). If $X$ is a projective homogeneous space, every connected component of the space of rational curves on $X$ is rational (builds on earlier work of Katsylo and Hirschowitz).

(J. Harris & -). If $d^2 \leq n$ and $X$ is a general hypersurface of degree $d$ in $\mathbb{P}^n$, then for every $e \in \mathbb{Z}$,

$$\text{ev}: \bar{M}_{0,2}(X, e) \rightarrow X \times X$$

$$\{(f, t_1, t_2) \mid f: \mathbb{P}^1 \rightarrow X \text{ degree } e, (f(t_1), t_2) \mapsto (f(t_1), f(t_2)) \}$$

is surjective and the geometric generic fiber is rationally connected (+ $e$, which I will come back to.)

(A.J. de Jong & -). If $d^2 \leq n+1$ (i.e., the extra cases when $d^2 = n+1$), the conclusion still holds. But this method does not give $+ e$.

(A.J. de Jong & -). A very simple argument gives the conclusion $+ e$ for Grassmannians, orthogonal Grassmannians & symplectic Grassmannians.

**Open Problem.** What about "Grassmannians = for algebraic groups, i.e., $G$/maximal parabolic. Most important case: $G = E_8"
PROBLEM 2. (C. Tsen) If \( K/k \) is the function field of a curve over an alg. closed field, and if \( X \subset \mathbb{P}^n_K \) is a hypersurface of degree \( \leq n \), then \( X \) has a \( K \)-rational point.

(C. Tsen, S. Lang). If \( K/k \) is the function field of an \( r \)-fold and \( X \subset \mathbb{P}^n_K \) is a hypersurface of degree \( d \) satisfying \( d^r \leq n \), then \( X \) has a \( K \)-rational point.

(T. Graber, J. Harris & -- in char. 0; A.J. de Jong & -- in char. \( p \)). If \( K/k \) is the function field of a curve and \( X \subset \overline{K} \) is (separably) rationally connected, then \( X \) has a \( K \)-rational point.

Problem 3. (C. Tsen) If \( K/k \) is the function field of a curve, the Brauer group is trivial, i.e., \( \text{index}(D) = \text{period}(D)^0 \).

Challenge. Deduce this from previous Tsen's theorem!

(A.J. de Jong) If \( K/k \) is the function field of a surface and \( \text{char}(k) \) does not divide \( \text{period}(D) \), then \( \text{index}(D) = \text{period}(D)^1 \).
EXPLANATION OF "+ ε": Existence of a very twisting family of pointed lines, i.e.,

where \( i \) is a closed immersion of a flat family of lines in \( X \), \( \sigma \) is a section of \( \pi := \text{pr}_X \circ i \) and

(i) the normal bundle of \( i \), \( N \), is \( \pi \)-relatively globally generated, i.e., \( N\xi_\sigma \) is globally generated for \( \xi_\sigma \);
(ii) \( \pi_X(N(-\sigma(P'))) \) is ample, and,
(iii) \( (\sigma(P'), \sigma(P'))_\xi \geq 0 \).

This condition is close to a very free rational curve in \( \overline{M}_{0,1}(X,1) \) (but it is stronger).

III. Connection by Problems 2+3: A division algebra \( D \) of index \( n \) determines a (fppt) \( PGL_n \)-torsor \( T \) over \( K \). For every integer \( 1 \leq l \leq n \), there is a \( K \)-scheme \( T_{\text{Grass}}(l,n) \) representing the functor of right ideals in \( D \) of rank \( ln \). \( \beta_k \) is zero iff \( l \cdot [D] \) is zero in \( \text{Br}(K) \).
To prove \( \text{index}(D) = \text{period}(D) \), it suffices to prove that for \( X = \text{a twisted form of Grass}(\mathbb{C}) \), if \( \mathcal{E}_X \) is zero, then \( X \) has a \( K \)-rational point; this then implies \( D \) has a right ideal of rank \( \ln \Rightarrow 2 \) equality, i.e. period \( \text{(D)} \) = index \( \text{(D)} \).

**Question:** What additional hypotheses are necessary so that \( \mathcal{E}_X = 0 \) implies \( X \) has a \( K \)-rational point? Do these hypotheses hold for \( X = \text{twisted form of Grass}(\mathbb{C}) \)?

### IV.

**One Answer**

(possibly not the "final answer")

1. **Hypothesis 1.** \( \mathcal{E}_X \) is zero.
2. **Hypothesis 2.** \( \text{Pic}(X_K) \cong \mathbb{Z} \).
3. **Hypothesis 3.**
   1. \( \forall e > 0 \), ev: \( \overline{M}_{0,2}(X_K, e) \to X_K \) is surj. & the geom. generic fiber is RC.
   2. There exists a very twisting family of pointed lines on \( X_K \).

\((i) + (ii) \) Together \( \approx \) Rationally simply-connected
The last hypothesis involves a "fibered projective model":

\[ X \xrightarrow{f} B \xrightarrow{\pi} S \]

f and \( \pi \) surjective, projective morphisms, \( k(B) \) equals \( K \) and \( X_{k(B)} \) equals \( X \).

**Hypothesis 4.** There exists a fibered projective model \( X/B/S \) satisfying,

(i) For a dense open \( S^0 \subset S \), \( B^0 := S^0 \times_B B \to S^0 \)
and \( X^0 := S^0 \times_S X \to B^0 \) are both smooth, and

(ii) The evaluation morphism,

\[ \text{ev} : \overline{M}_{0,1}(X^0/B^0, 1) \to X^0 \]

where

\[ \{(b, L, x) \mid b \in B^0, L \subset f^{-1}(b), x \in L \text{ a point} \} \]

is smooth and the geometric fibers are \( R.C. \).

**THEOREM (de Jong & -).** Let \( K/k \) be the function field of a surface over an alg. closed field of char. 0. Let \( X \) be a smooth, proj. \( K \)-scheme.
Under Hypotheses 1 - 4, \( X \) has a \( K \)-rational point.
Using a slight trick, every twisted form of \( \text{Gross}(E,n) \) can be reduced to \( X \) satisfying Hypotheses 1-4. So this gives another proof of the Period-Index Theorem in char. 0.

(In fact, de Jong and I can reduce the positive characteristic case — with no condition that \( \text{char}(k) \nmid \text{period}(D) \) — to the char. 0 case; but that’s another story.)
V. Very Rough Sketch of Proof

- As Joe Harris explained in his Lecture 3, there is a scheme \( \text{Sec}(X^0/B^0/\mathcal{S}^0) = \{(s, s') | s \circ s' \}

- \( s : B_s \to X_s \) a section of \( f_s : X_s \to B_s \).

- To prove \( X^0 \to B^0 \) has a rational section, it is equivalent to prove \( \text{Sec}(X^0/B^0/\mathcal{S}^0) \to \mathcal{S}^0 \) has a rational section.

- Since \( \mathcal{S}^0 \) is a curve, by the "RC fibration theorem", it suffices to find a subvariety \( Y \subset \text{Sec}(X^0/B^0/\mathcal{S}^0) \) s.t. \( Y \to \mathcal{S}^0 \) is (essentially) a rationally connected fibration.

- Consider the irreducible components of \( \text{Sec}(X^0/B^0/\mathcal{S}^0) \) (of which there are countably many). The idea is to prove that as the components increase (roughly, as the degrees of the section curves increase) the fibers of \( \text{I}_s \to \mathcal{S}^0 \) become "more rationally connected", i.e., the dimn of the MRC fibration drops.

- This (essentially) no longer concerns \( \mathcal{S}^0 \); it only concerns the geometric generic fiber of \( \text{Sec} \to \mathcal{S}^0 \).
Hypothesis 3(ii) (essentially) implies the MRC quotients of $I_\omega$ are non-increasing:

\[ I_{\omega, \mu} = \{ \text{line} \} \xrightarrow{\text{all deformations of } \begin{array}{c}
\hline
\end{array}} I_\omega. \]

Simplifying Hypothesis:
\[ g(B) = 0 \]

The very twisting family guarantees we can find $P^3$'s in $I_\omega$, intersecting $I_{\omega, \omega}$, $\Rightarrow$ MRC qtt of $I_\omega$ dominates MRC eff of $I_\omega$.

- Hypothesis 3(i) implies the MRC quotients of $I_\omega$ actually decrease: Given $\sigma_1, \sigma_2 \in I_\omega$, form

\[ \overline{M}_{0,2}(X,\xi) \times_{\text{ev}, X \times X, \sigma_1 \times \sigma_2} B = \{ (b, g, t_1, t_2) \mid g : P^1 \rightarrow f \}
\]

By Hyp 3(i), this is an RC fibration over $B$. By "RC fibration thm", there is a section.
A section is the same as a diagram

\[ \Sigma = \text{rational surface whose general fiber over } B \text{ is } \leq 1 \text{P}, \sigma_1(B), \sigma_2(B) \text{ contained in } \Sigma. \]

If the images of \( \sigma_1(B), \sigma_2(B) \) under the "Abel map" are equal, then

\( [\sigma_1(B)] - [\sigma_2(B)] \in \text{CH}_1(\Sigma) \) is linearly equivalent to a linear combination of irreducible parts of fibers of \( \Sigma \to B \), i.e.,

\[ \text{linearly equivalent} \]

Each comb is a point in some \( I_{x,y} \).
So these points in \( I_{x,y} \) are rationally connected.
\( \Rightarrow I_{x,y} \) is more rationally connected than \( I_x \).

- Eventually, "MRC \( q^+ (I_x) = \text{Pic}_e(B) \), which then implies \( I_x \to \text{Pic}_e(B) \) is a BC fibration.
- Since \( \text{Pic}_e(B) \) certainly has a point over \( k(C) \), the theorem follows from the BC fibration theorem.