Rational Simple-Connectedness and Rational Points ①

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Joint work with A. J. de Jong

I. Three Problems

PROBLEM 1: (Hilbert scheme/Chow variety problem)
Let k be alg. closed. Give examples of smooth, projective k-schemes X such that:
(i) X is "close to being rational", and
(ii) the Hilbert schemes/Chow varieties of rational curves on X are also "close to being rational."

PROBLEM 2: (Existence of rational points)
Let (K, X) be a pair of a field K — typically not alg. closed — and a smooth, projective, geometrically integral K-scheme. Give sufficient conditions on (K, X) for X to have a K-rational point.

PROBLEM 3: (Period-Index Problem)
Let (K, D) be a pair of a field K and a division algebra D whose center is K. Find a relationship b/w $\text{index}(D) := \sqrt{\dim_K(D)}$ and $\text{period}(D) :=$ order of $[D]$ in the Brauer group of K.

Observation. Problem 2 (existence of rational points) is related to the Brauer group of K: There is a short-exact-sequence

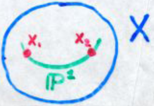
$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{K}})^{\mathrm{Gal}(\overline{K}/K)} \xrightarrow{\delta_K} \mathrm{Br}(K)$$

If X has a K-rational point, then δ_K is zero. So δ_K is "a Brauer obstruction" (but almost certainly not "the only cohomological obstruction").

Explanation. "Close to being rational" equals "rationally connected". A smooth projective scheme X over an alg. closed field k is rationally connected if there exists an integral k-scheme M and a k-morphism $u: M \times \mathbb{P}^1 \to X$ whose associated morphism $u^{(2)}: M \times \mathbb{P}^1 \times \mathbb{P}^1 \to X \times X$ is dominant.

$(m, t_1, t_2) \mapsto (u(m,t_1), u(m,t_2))$

Picture.



Equivalent Formulation in Char. 0. X is RC iff X is connected and there exists:

$$\underbrace{f: \mathbb{P}^1 \to X \text{ s.t. } f^*(T_X) \text{ is ample}}_{\text{"very free rational curve"}}$$

Some Answers:

PROBLEM 1. (B. Kim & R. Pandharipande). If X is a projective homogeneous space, every connected component of the space of rational curves on X is rational (builds on earlier work of Katsylo and Hirschowitz).

(J. Harris & —). If $d^2 \leq n$ and X is a general hypersurface of degree d in $\mathbb{P}^n$, then for every $e \geq 2$,

$$\mathrm{ev}: \overline{M}_{0,2}(X, e) \longrightarrow X \times X$$

$$\{(f, t_1, t_2) \mid f: \mathbb{P}^1 \to X \text{ degree } e, \ t_1, t_2 \in \mathbb{P}^1\}, \quad (f, t_1, t_2) \mapsto (f(t_1), f(t_2))$$

is surjective and the geometric generic fiber is rationally connected (+ ε, which I will come back to.)

(A.J. de Jong & —). If $d^2 \leq n+1$ (i.e., the extra cases when $d^2 = n+1$), the conclusion still holds. But this method does not give "+ ε".

(A.J. de Jong & —). A very simple argument gives the conclusion "+ ε" for Grassmanians, orthogonal Grassmannians & symplectic Grassmannians

OPEN PROBLEM. What about "Grassmannians" for algebraic groups, i.e., G/maximal parabolic. MOST IMPORTANT CASE: $G = E_8$

PROBLEM 2. (C. Tsen) If K/k is the function field of a curve over an alg. closed field, and if $X \subset \mathbb{P}^n_K$ is a hypersurface of degree $\leq n$, X has a K-rational point.

(C. Tsen, S. Lang). If K/k is the function field of an r-fold and $X \subset \mathbb{P}^n_K$ is a hypersurface of degree d satisfying.

$$d^r \leq n$$

This result is sharp. For $d^r \geq n+1$, there are counter-examples

then X has a K-rational point.

(T. Graber, J. Harris & — in char. 0; A. J. de Jong & — in char. p).
If K/k is the function field of a curve and $X_{\bar{K}}$ is (separably) rationally connected, then X has a K-rational point.

PROBLEM 3. (C. Tsen) If K/k is the function field of a curve, the Brauer group is trivial, i.e.,
index (D) = period $(D)^0$.

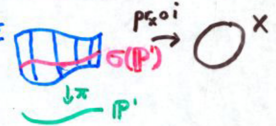
Challenge. Deduce this from previous Tsen's theorem!

(A. J. de Jong) If K/k is the function field of a surface and char(k) does not divide period(D), index(D) = period$(D)^1$.

EXPLANATION OF "$+\varepsilon$": Existence of a very twisting family of pointed lines, i.e., a diagram

$$\begin{tikzcd} \Sigma \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & \mathbb{P}^1 \times X \arrow[d, "pr_{\mathbb{P}^1}"] \\ & \mathbb{P}^1 \arrow[ul, bend left, "\sigma"] \end{tikzcd}$$

,



where i is a closed immersion of a flat family of lines in X, σ is a section of $\pi := pr_{\mathbb{P}^1} \circ \sigma$ and

(i) the normal bundle of i, N, is π-relatively globally generated, i.e., $N|_{\Sigma_t}$ is globally gen'd $\forall t \in \mathbb{P}^1$,

(ii) $\pi_*(N(-\sigma(\mathbb{P}^1)))$ is ample, and,

(iii) $(\sigma(\mathbb{P}^1).\sigma(\mathbb{P}^1))_\Sigma \geq 0$.

This condition is close to the condition to be a very free rational curve in $\overline{M}_{0,1}(X,1)$ (but it is stronger).

III.

Connection b/w Problems 2+3: A division algebra D of index n determines a (fppf) PGL_n-torsor T over K. For every integer $1 \leq \ell \leq n$, there is a K-scheme $T \times^{PGL_n} Grass(\ell, n)$ representing the functor of right ideals in D of rank ℓn. δ_K is zero iff $\ell \cdot [D]$ is zero in $Br(K)$.

To prove index (D) = period(D), it suffices to prove that for X = a twisted form of Grass(ℓ, n), if δ_K is zero, then X has a K-rational point: this then implies D has a right ideal of rank $\ell n \Rightarrow \ell$ equals n, i.e. period (D) = index (D). ⑥

<u>Question</u>: What additional hypotheses are necessary so that $\delta_K = 0$ implies X has a K-rational point? Do these hypoth. hold for X = twisted form of Grass(ℓ, n)?

<u>IV.</u> One Answer
(certainly <u>not</u> the "final answer")

<u>Hypothesis 1</u>. δ_K is zero. } RECALL: This is a necessary hypothesis.

<u>Hypothesis 2</u>. $\mathrm{Pic}(X_{\overline{K}}) \cong \mathbb{Z}$.

<u>Hypothesis 3</u>. (i) $\forall e \gg 0$, ev: $\overline{M}_{0,2}(X_{\overline{K}}, e) \to X_{\overline{K}} \times X_{\overline{K}}$ is surj. & the geom. generic fiber is RC.

(ii) "$+\varepsilon$" There exists a very twisting family of pointed lines on $X_{\overline{K}}$.

$\underbrace{\text{(i) + (ii) Together}}$ $\approx$ Rationally simply-connected

The last hypothesis involves a "fibered projective model":

$$\mathcal{X} \xrightarrow{f} \underset{\text{surface}}{B} \xrightarrow{\pi} \underset{\text{curve}}{S}$$

f and π surjective, projective morphisms, $k(B)$ equals K and $\mathcal{X}_{k(B)}$ equals X.

Hypothesis 4. There exists a fibered projective model $\mathcal{X}/B/S$ satisfying.

(i) For a dense open $S^o \subset S$, $B^o := S^o \times_S B \to S^o$ and $X^o := S^o \times_S X \to B^o$ are both smooth, and

(ii) The evaluation morphism,

$$ev : \overline{M}_{0,1}(\mathcal{X}^o/B^o, 1) \longrightarrow \mathcal{X}^o$$

$$\wr\wr$$

$$\left\{ (b, L, x) \,\middle|\, b \in B^o,\ \underset{x \in L \text{ a point}}{L \subset f^{-1}(b) \text{ a line},} \right\} \quad (b,L,x) \mapsto x$$

is smooth and the geometric fibers are R.C.

THEOREM (de Jong & –). Let K/k be the function field of a surface over an alg. closed field of char. 0. Let X be a smooth, proj K-scheme. Under Hypotheses 1 – 4, X has a K-rational point.

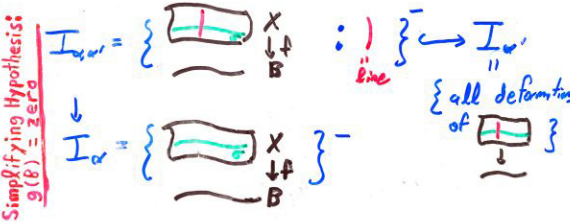
Using a slight trick, every twisted form of $\mathrm{Grass}(\ell, n)$ can be reduced to X satisfying Hypotheses 1–4. So this gives another proof of the Period-Index Theorem in char. 0.

(In fact, de Jong and I can reduce the positive characteristic case – with no condition that $\mathrm{char}(k) \nmid \mathrm{period}(D)$ – to the char. 0 case; but that's another story.)

V. Very Rough Sketch of Proof

- As Joe Harris explained in his Lecture 3, there is a scheme $\mathrm{Sec}(\mathcal{X}^\circ/B^\circ/S^\circ) = \{(s,\sigma) \mid s \in S^\circ, \sigma: B_s \to \mathcal{X}_s \text{ a section of } f_s: \mathcal{X}_s \to B_s\}$.
- To prove $\mathcal{X}^\circ \to B^\circ$ has a rational section, it is equivalent to prove $\mathrm{Sec}(\mathcal{X}^\circ/B^\circ/S^\circ) \longrightarrow S^\circ$ has a rational section.
- Since S° is a curve, by the "RC fibration theorem", it suffices to find a subvariety $Y \subset \mathrm{Sec}(\mathcal{X}^\circ/B^\circ/S^\circ)$ s.t. $Y \to S^\circ$ is (essentially) a rationally connected fibration. $\{I_\alpha\}$
- Consider the irreducible components of $\mathrm{Sec}(\mathcal{X}^\circ/B^\circ/S^\circ)$ (of which there are countably many). The idea is to prove that as the components increase (roughly, as the degrees of the section curves increase) the fibers of $I_\alpha \to S_0$ become "more rationally connected", i.e., the dim'n of the MRC fibration drops.
- This (essentially) no longer concerns S°; it only concerns the geometric generic fiber of $\mathrm{Sec} \to S^\circ$.

· Hypothesis 3(ii) (essentially) implies the MRC quotients of I_α are non-increasing:

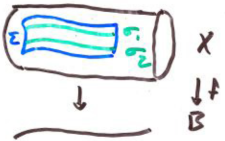


The very twisting family guarantees we can find $\mathbb{P}^1$'s in $I_{\alpha'}$ intersecting $I_{\alpha,\alpha'}$ $\Rightarrow$ MRC q'tt of I_α dominates MRC q'tt of $I_{\alpha'}$.

· Hypothesis 3(i) implies the MRC quotients of I_α actually decrease: Given $\sigma_1, \sigma_2 \in I_\alpha$, form

$$\overline{M}_{0,2}(X,e) \times_{ev, X \times X, \sigma_1 \times \sigma_2} B = \{(b, g, t_1, t_2) \mid g : \mathbb{P}^1 \to f^{-1}(\ldots, g(t_1) = \sigma_1(\ldots, g(t_2) = \sigma_2(\ldots$$

By Hyp. 3(i), this is an RC fibration over B. By "RC fibration thm", there is a section, A section is the same as a diagram:



Σ = rational surface whose general fiber over B is $\cong \mathbb{P}^1$; $\sigma_1(B), \sigma_2(B)$ contained in Σ.

If the images of $\sigma_1(B)$, $\sigma_2(B)$ under the "Abel map" are equal, then $[\sigma_1(B)] - [\sigma_2(B)] \in CH_1(\Sigma)$ is linearly equivalent to a linear combination of irred. cpts. of fibers of $\Sigma \to B$, i.e.,

Each comb is a point in some $I_{\alpha,\alpha'}$.
So these points in $I_{\alpha,\alpha'}$ are rationally conn'd.
$\Rightarrow I_{\alpha,\alpha'}$ is <u>more</u> rat. conn'd than I_α.

- Eventually, "MRC quot $(I_\alpha) = \mathrm{Pic}^e(B)$", which then implies $I_\alpha \to \mathrm{Pic}^e(B)$ is an RC fibration. Since $\mathrm{Pic}^e(B)$ certainly has a point over $k(S)$, the thm. follows from the RC fibration thm.