Historically, a typical application of Lefschetz fibrations was to compute topological invariants of the total space, such as Betti numbers, in terms of fibre and vanishing cycle data [Lefschetz]. What about the symplectic viewpoint and "new" (Gromov-Witten or related) invariants?

Symplectic cohomology: Let M be an affine variety, with Kähler form $\omega = -dd^c\psi$, ψ an exhausting function (one can obtain this from a projective compactification of M). Let ϕ^t be the Hamiltonian flow of ψ. Given two Lagrangian submanifolds $L_0, L_1 \subset M$, define

$$\widehat{HF}^*(L_0, L_1) = \varinjlim_t HF^*(\phi^t(L_0), L_1).$$

If at least one of the L_k is closed, then this is just the ordinary $HF^*(L_0, L_1)$. The interesting case is for non-compact L_k (which have to be suitably nice at infinity; for instance, if $\pi : M \to \mathbb{C}$ is a Lefschetz fibration, then the Lefschetz thimbles $\Delta \subset M$ qualify).

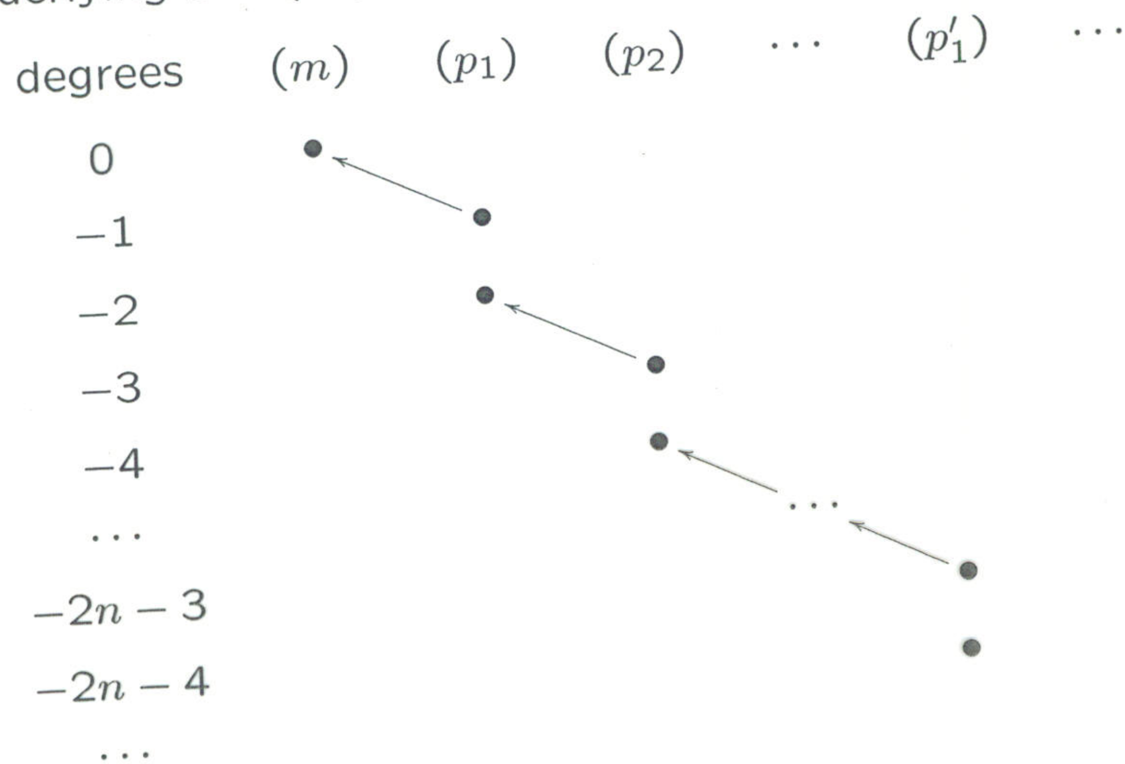
The "closed string" analogue is <u>symplectic cohomology</u> [Viterbo, Cieliebak-Hofer-Wysocki-Zehnder]

$$SH^*(M) = \varinjlim_t HF^*(\phi^t).$$

Here $HF^*(\phi^t)$ is the Hamiltonian or fixed-point Floer cohomology. It is actually better to think in terms of a sufficiently large sublevel set $M' = \{x : \psi(x) \leq C\}$. $\phi^t \,|\, \partial M'$ is called the <u>Reeb flow</u>. The chain complex defining $SH^*(M)$ has one generator for each cell of M' (taking ψ as Morse function), and a pair of generators for each periodic Reeb orbit (lying in adjacent degrees).

Remark: It would seem that all this depends strongly on the choice of Kähler potential ψ, but a suitable deformation invariance property holds. In fact, the natural framework is the theory of Stein manifolds up to deformation, or of noncompact symplectic manifolds with conical ends up to isomorphism [Eliashberg].

Example: $M = \mathbb{C}^{n+1}$, $\psi(x) = |x|^2$, so M' is a ball. The Reeb flow is rotation of the circle bundle $\partial M' = S^{2n+1} \to \mathbb{C}P^n$. After perturbing, one ends up with $n+1$ distinct periodic orbits and their iterates. The chain complex underlying $SH^*(M)$ is



where

(m) the 0-cell (minimum of ψ at $x = 0$)

(p_1) the first periodic Reeb orbit

(p_2) the second periodic Reeb orbit

...

(p_1') the first periodic Reeb orbit, iterated twice

...

This means that $SH^*(\mathbb{C}^{n+1}) = 0$ (as an aside, if we take $L = \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$, then also $\widehat{HF}^*(L, L) = 0$).

The same vanishing result for $SH^*(M)$ holds whenever M is subcritical, namely all critical points of ψ are nondegenerate of Morse index $< dim_{\mathbb{C}} M$ [Viterbo, Cieliebak]. There is also a relative version of this, which says that $SH^*(M)$ remains unchanged under adding handles of dimension $< dim_{\mathbb{C}} M$ [Cieliebak].

Example: If $M = T^*N$ is the cotangent bundle of a closed oriented n-manifold, then

$$SH^*(M) = H_{n-*}(\mathcal{L}N),$$

where $\mathcal{L}N$ is the free loop space [Viterbo, Abbondandolo-Schwarz, Weber] (similarly $\widehat{HF}^*(T_x^*L, T_y^*L) = H_{-*}(\Omega_{x,y}N)$, homology of the path space).

Example: $SH^*(M)$ successfully distinguishes different symplectic structures on the same differentiable manifold. For instance, take S to be Ramanujam's affine algebraic surface, which is contractible but not homeomorphic to $\mathbb{R}^4$. We know $SH^*(S) \neq 0$ [Seidel-Smith]. Now $M = S \times S$ is diffeomorphic to $\mathbb{R}^8$, but again

$$SH^*(M) = SH^*(S) \otimes SH^*(S) \neq 0$$

[Oancea]. Now take $\tilde{M} = M \#_\partial M$ (boundary connected sum, or joining the two components by a 1-handle). Then [Cieliebak]

$$SH^*(\tilde{M}) = SH^*(M) \oplus SH^*(M),$$

but since $SH^*(M)$ is potentially infinite-dimensional in each degree, this is not sufficient to distinguish M and $\tilde{M}$ (the ring structure might help, but it is again not known). There are in fact infinitely many examples of Ramanujam's type, and again we cannot distinguish between them at present.

Problem: If M is an affine algebraic variety, with projective closure $Y = M \cup Z$, one should be able to compute $SH^*(M)$ from $H^*(Y)$ and the relative Gromov-Witten invariants of (Y, Z). This can be seen quite clearly in the example of $\mathbb{C}^{n+1}$ above, where the differentials correspond to counting lines in $\mathbb{CP}^{n+1}$ and $\mathbb{CP}^n$.

Digression: In the general symplectic or Stein framework, $SH^*(M)$ is not algorithmically computable, not even for simply-connected M. To see that, let Γ be a finitely presented group with a fixed presentation, and construct from that a manifold N with $\pi_1(N) = \Gamma$. Take T^*N and attach two-handles (fattened discs) to the boundary, killing off the generators of π_1. The resulting M satisfies

$$\begin{aligned} &\text{rank}\, SH^n(M) = \text{rank}\, SH^n(T^*N) = \\ &= \text{rank}\, H_0(\mathcal{L}N) = \#\text{ of conjugacy classes in }\Gamma \end{aligned}$$

Hence, any algorithm to determine $SH^n(M)$ would solve the triviality problem for finitely presented groups, contradiction. The same argument implies no-go results for the algorithmic classification of simply-connected symplectic manifolds with conical ends (and, presumably, simply-connected closed contact manifolds). ↯

Localizing along a natural transformation: suppose that we have

A	category (linear over some field);
$F : A \longrightarrow A$	functor;
$N : F \longrightarrow Id_A$	natural transformation.

By composing with F on the left and right, we get two natural transformations

$$L_F N, R_F N : F^2 \longrightarrow F.$$

Assume that they are equal. Concretely, this means that $N_{FX} = F(N_X) \in Hom_A(F^2X, FX)$ for all X. We want to modify our category in such a way that "all the N_X become isomorphisms". Namely, define $\widehat{A}$ as follows:

$$Ob\,\widehat{A} = Ob\,A,$$
$$Hom_{\widehat{A}}(X_0, X_1) = \varinjlim Hom_A(F^k X_0, X_1),$$

where the map $Hom_A(F^k X_0, X_1) \to Hom_A(F^{k+1} X_0, X_1)$ is $x \mapsto x \cdot F^k(N_{X_0}) = x \cdot N_{F^k(X_0)}$; and composition of morphisms is the direct limit of

$$Hom_A(F^l X_1, X_2) \otimes Hom_A(F^k X_0, X_1) \longrightarrow$$
$$\longrightarrow Hom_A(F^{k+l} X_0, X_2),$$
$$(x_2, x_1) \longmapsto x_2 \cdot F^l(x_1).$$

Trivial example: A is an algebra (category with one object), $F = Id_A$, N a central element. Then $\widehat{A} = A[N^{-1}]$.

Motivating example: $Y = Y^{n+1}$ a Fano variety, s a nontrivial section of $\mathcal{K}_Y^{-1}$. Set $M = \{s \neq 0\}$. Take

$$A = D^b Coh(Y),$$
$$F = - \otimes \mathcal{K}_Y,$$
$$N = (\textit{multiplication with } s).$$

Then $\hat{A} = D^b Coh(M)$.

Set $Z = s^{-1}(0)$, with $j : Z \to Y$ the embedding. Take $B = D^b Coh(Z)$ (in the singular case, suitable full subcategory). Note that for any two objects X_0, X_1 of A, we have natural maps

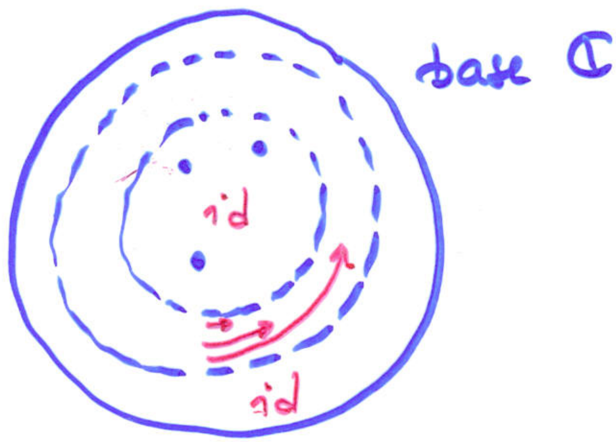
$$Hom_A(X_0, X_1) \longrightarrow Hom_B(j^* X_0, j^* X_1),$$
$$Hom_B(j^* X_0, j^* X_1[n]) \cong Hom_B(j^* X_1, j^* X_0)^\vee \to$$
$$\to Hom_A(X_1, X_0)^\vee = Hom_A(X_0, X_1 \otimes \mathcal{K}_Y[n+1])$$

and these form part of a long exact sequence, whose connecting homomorphism is composition with N_{X_1}.

To put this more abstractly, recall that for a general category A, the <u>Serre functor</u> $S_A : A \to A$ is characterized by $Hom_A(X_0, S_A X_1) = Hom_A(X_1, X_0)^\vee$. If $A = D^b Coh(Y)$, $S_A = - \otimes \mathcal{K}_Y[n+1]$ by Serre duality. Now Z is Calabi-Yau, so $S_B = [n]$. The maps above are actually natural transformations $Id_A \longrightarrow j_* j^*$, $j_* S_B j^* \longrightarrow S_A$, and these form an exact triangle of functors

$$Id_A \longrightarrow j_* j^* \longrightarrow S_A[-n] \xrightarrow{\ N\ } Id_A$$

Symplectic counterpart: Let $\pi : M \to \mathbb{C}$ be a Lefschetz fibration, and consider $\mathcal{A} = \mathcal{F}(M, M_\infty)$. The Serre A_∞-functor $\mathcal{S}_\mathcal{A}$ is induced by a canonical symplectic automorphism σ of (M, M_∞) [Barannikov-Kontsevich]:



In $Symp(M)$, σ is isotopic to the identity, and using that one defines canonical elements of $HF^*(\sigma(L), L)$ for any object L, which form a natural transformation $\mathcal{S}_\mathcal{A} \to Id_\mathcal{A}$. With that (and some related higher order algebraic data), we define the localization $\widehat{\mathcal{A}} = \widehat{\mathcal{F}}(M, M_\infty)$ on the level of A_∞-categories. This satisfies

$$H(hom_{\widehat{\mathcal{A}}}(L_0, L_1)) \stackrel{\text{def}}{=} \varinjlim HF^*(\sigma^k(L_0), L_1) = \\ = \widehat{HF}^*(L_0, L_1),$$

because σ is closely related to the Reeb flow, at least on the part of the boundary relevant to our L's. One expects the connection between "open" and "closed string" theory to go through Hochschild cohomology of A_∞-categories. Hence,

Conjecture: $SH^*(M) \cong HH^*(\widehat{\mathcal{A}}, \widehat{\mathcal{A}})$.

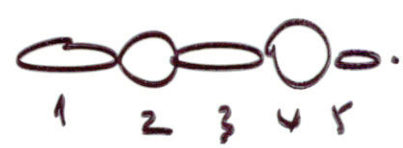
Trivial example: Projection $\pi : M = X \times \mathbb{C} \to \mathbb{C}$. Then $\mathcal{A}$ and hence $\widehat{\mathcal{A}}$ are empty, so $HH^*(\widehat{\mathcal{A}}, \widehat{\mathcal{A}}) = 0$. On the other hand, M is subcritical, so $SH^*(M) = 0$.

Trivial example: Given a Lefschetz fibration $\pi : M^{n+1} \to \mathbb{C}$, we can enlarge the fibre (on a symplectic or Stein manifold level) by attaching handles of dimension $\leq n$.

This does not affect the Lefschetz thimbles, so $\widehat{\mathcal{A}}$ and its Hochschild cohomology remain the same. On the other hand, it is a subcritical handle attachment from the point of view of the total space, hence $SH^*(M)$ is equally unchanged.

Example: Take $M = \mathbb{C}^2$ and $\pi(x, y) = x^2 + q(y)$, q a generic polynomial of degree $m + 1 \geq 3$. On the level of derived categories, the resulting $\mathcal{A} = \mathcal{F}(M, M_\infty)$ is equivalent to the category of modules over the path algebra P_m of the directed quiver

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \cdots \longrightarrow \overset{m-1}{\bullet} \longrightarrow \overset{m}{\bullet}$$



The Serre functor is periodic up to a shift [Bernstein-Gelfand-Ponomarev], $S^{m+1} \cong [m-1]$. Given $N : S \to Id$, we have $N^m = R_{S^{m-1}}(N) \circ R_{S^{m-2}}(N) \circ \cdots \circ N : S^{m+1} \to Id$, which is an element of $HH^{1-m}(P_m, P_m) = 0$. This implies that the localized category is zero, hence $HH^*(\hat{\mathcal{A}}, \hat{\mathcal{A}}) = 0$ in accordance with our computation of $SH^*(\mathbb{C}^2)$.

Example: Let Y be a toric Fano manifold, and $Z \subset Y$ an anticanonical divisor which is invariant with respect to the torus action, so $Z \setminus Y$ is the open torus $G = (\mathbb{C}^*)^{n+1}$. Localizing $A = D^bCoh(Y)$ using the resulting natural transformation yields $\hat{A} = D^bCoh(G)$, hence [Hochschild-Kostant-Rosenberg]

$$HH^*(\hat{A}, \hat{A}) = HH^*(G) = \Gamma(G, \Lambda^* TG) = \\ = \mathbb{C}[t_1^{\pm 1}, \ldots, t_{n+1}^{\pm 1}] \otimes \Lambda(\tau_1, \ldots, \tau_{n+1}).$$

Under mirror symmetry, Y corresponds to a Landau-Ginzburg theory, which is a Lefschetz fibration $\pi : G^\vee \to \mathbb{C}$ with total space $G^\vee \cong G \cong T^*(T^{n+1})$. A is mirror to the derived category of $\mathcal{A} = \mathcal{F}(M, M_\infty)$, and one expects $\hat{A}$ to correspond to the localization $\hat{\mathcal{A}}$. According to the conjecture,

$$HH^*(\hat{\mathcal{A}}, \hat{\mathcal{A}}) = SH^*(M) = H_{n+1-*}(\mathcal{L}T^{n+1}) = \\ = \mathbb{C}[\pi_1(T^{n+1})] \otimes H_*(T^{n+1}).$$

The next step is to give a more concrete construction of (part of) $\hat{\mathcal{A}}$. We summarize the analogies established by homological mirror symmetry:

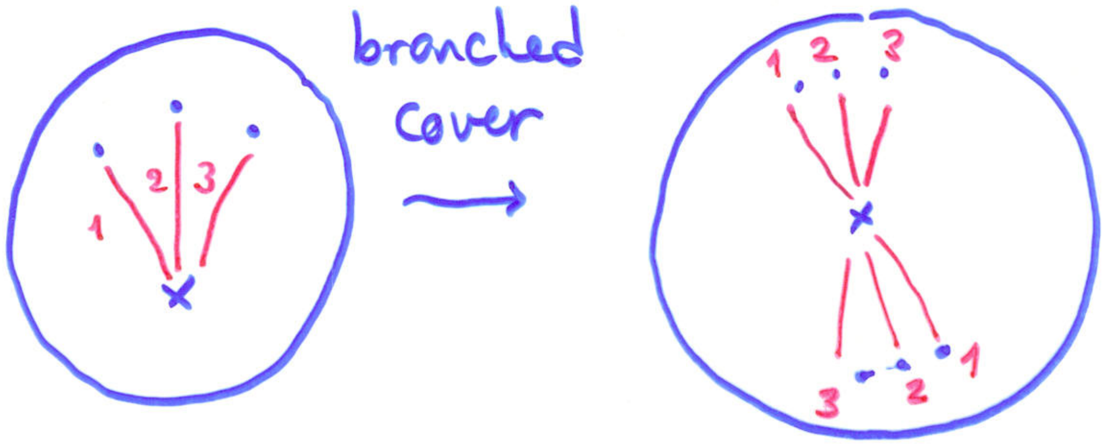
sheaves	Lagrangian submanifolds
Y, $s \in \Gamma(\mathcal{K}_Y^{-1})$, $Z = s^{-1}(0)$	$\pi : M \to \mathbb{C}$
$D^bCoh(Y)$	$\mathcal{F}(M, M_\infty)$
$D^bCoh(Z)$	$\mathcal{F}(M_z)$, generic z
$D^bCoh(Y \setminus Z)$	$\hat{\mathcal{F}}(M, M_\infty)$

For both $D^b(Z)$ and $\mathcal{F}(M_z)$, the Serre functor is $[n]$, in the latter case because it is a cyclic A_∞-category. In the sheaf case, we explained how the localized category $D^bCoh(Y \setminus Z)$ arose from an exact triangle of functors based on the restriction $D^bCoh(Y) \to D^bCoh(Z)$. There is a corresponding functor $\mathcal{F}(M, M_\infty) \to \mathcal{F}(M_z)$, and the idea is that $\hat{\mathcal{F}}(M, M_\infty)$ should be constructed entirely from that.

Guessing the answer: Let $\pi : M \to \mathbb{C}$ be a Lefschetz fibration. Take a basis $\mathcal{V} = (V_1, \dots, V_m)$ of vanishing cycles, $V_j \subset M_z$. We can also form the repeated bases,

$$\begin{aligned}\mathcal{V}^{(k)} = (V_1^{(k)} = V_1, \dots, V_m^{(k)} = V_m,\\ V_{m+1}^{(k)} = V_1, \dots, V_{2m}^{(k)} = V_m,\\ \dots\\ V_{(k-1)m+1}^{(k)} = V_1, \dots, V_{km}^{(k)} = V_m);\end{aligned}$$

geometrically, these form a basis of vanishing cycles for the k-fold cover $M^{(k)}$ of M branched along M_z.



Set

$\mathcal{B}$ = full A_∞-subcategory of $\mathcal{F}(M_z)$ with objects V_j,

$\mathcal{A}$ = directed A_∞-subcategory with the same objects,

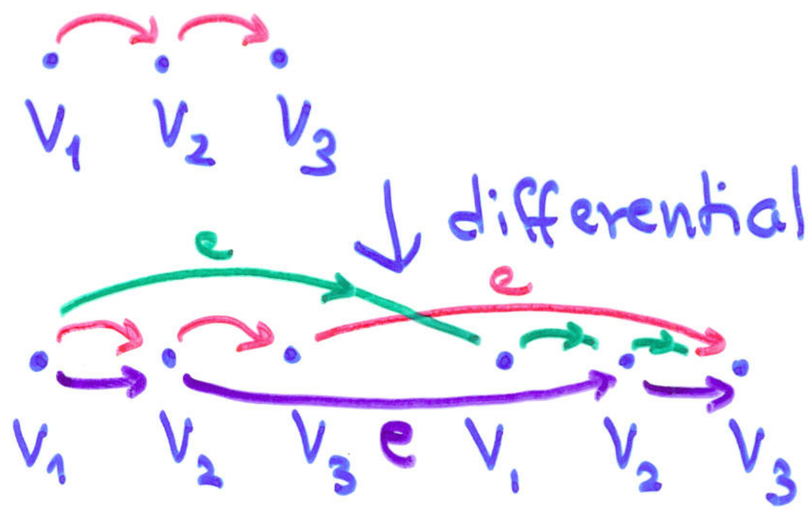
$\mathcal{A}^{(k)}$ = the same as $\mathcal{A}$, but now with $\mathcal{V}^{(k)}$

$\mathcal{B}$ has finitely many objects, but an a priori unbounded number of higher order A_∞-operations μ^d. In contrast, each $\mathcal{A}^{(k)}$ has only up to μ^{km-1} (and can be computed from $\mathcal{B}$). Note that on the derived level, $\mathcal{A}$ is equivalent to $\mathcal{F}(M, M_\infty)$ [Lecture 2]; the same applies to $\mathcal{A}^{(k)}$ and $M^{(k)}$.

Define a new A_∞-category (in fact dg-category) $\hat{\mathcal{A}}$. Its objects are $\Delta_1, \dots, \Delta_m$. To define the morphism spaces, consider

$$C^{(k)}(\Delta_j, \Delta_l) = \bigoplus hom_{\mathcal{A}^{(k)}}(V_{i_0}^{(k)}, V_{i_1}^{(k)})[1] \otimes \cdots$$
$$\cdots \otimes hom_{\mathcal{A}^{(k)}}(V_{i_{r-1}}^{(k)}, V_{i_r}^{(k)})[1]$$

where the sum goes over all r and $j = i_0 < i_1 < \cdots < i_r = (k-1)m + l$; in the special case $j = l$ and $k = 1$, there is also an extra one-dimensional summand for $r = 0$. Each $C^{(k)}$ carries a differential inherited from the A_∞-structure of $\mathcal{A}^{(k)}$ (bar complex). Moreover, there are natural degree -1 chain maps $C^{(k)} \to C^{(k+1)}$, which insert the identity morphism $V_i^{(k)} \to V_{i+m}^{(k)}$ in all possible positions:



Define

$$hom_{\widehat{\mathcal{A}}}(\Delta_j, \Delta_l) = \bigoplus_{k \geq 1} C^{(k)}(\Delta_j, \Delta_l)^\vee[2k-2],$$

with the combined (dualized) differential. Product structure is induced by the obvious tensor coproduct

$$C^{(k)} \longrightarrow \bigoplus_{i+j=k+1} C^{(i)} \otimes C^{(j)}$$

Conjecture: $HH^*(\widehat{\mathcal{A}}, \widehat{\mathcal{A}}) \cong SH^*(M)$.

Remarks: (1) The left hand side can be "written down" (as opposed to "algorithmically computed") in essentially any given case, starting with a knowledge of $\mathcal{B} \subset \mathcal{F}(M_z)$ [Lecture 2]. Some examples, such as the mirror of $\mathbb{C}\mathbf{P}^2$, have been checked in this way.

(2) The right hand side depends only on M, whereas a priori, the left hand side depends on π and on the choice of vanishing cycles. Here is one nontrivial check (for simplicity, in $dim_{\mathbb{C}} M = 2$): attach a 1-handle to the fibre boundary, and a new critical point to the fibration, whose vanishing cycle V_{m+1} is disjoint from the previous ones and goes through our 1-handle. This gives a new fibration $\tilde{\pi} : \tilde{M} \to \mathbb{C}$, however the total space $\tilde{M} \cong M$ remains the same (symplectically, which is the level on which the construction works). The category $\mathcal{B}$ gets enlarged by a new (orthogonal) summand corresponding to V_{m+1}, but the resulting new object of $\hat{\mathcal{A}}$ is isomorphic to zero, hence $HH^*(\hat{\mathcal{A}}, \hat{\mathcal{A}})$ is unchanged.