Summary (Algebra)

A_∞-category $\mathcal{C}$

- set of objects, $Ob\,\mathcal{C}$
- morphisms $hom_{\mathcal{C}}(X_0, X_1)$, a graded vector space
- an ∞ hierarchy of operations

$$hom_{\mathcal{C}}(X_{d-1}, X_d) \otimes \cdots \otimes hom_{\mathcal{C}}(X_0, X_1) \xrightarrow{\mu^d_{\mathcal{C}}} hom_{\mathcal{C}}(X_0, X_d)[2-d] \qquad (d \geq 1)$$

μ^1 = differential, μ^2 = composition, $\mu^3, \mu^4 \ldots$ = Massey products

NEW <u>Ex.</u> Every dg-category is an A_∞-category. Conversely, every A_∞-cat is quasi-isomorphic to a dg-category.

Constructions

$\mathcal{C} \longrightarrow$ cohomological category $H(\mathcal{C})$ or $H^0(\mathcal{C})$: pass to μ^1-cohomology, forget $\mu^3, \mu^4 \ldots$

$\longrightarrow$ category of twisted complexes $\mathcal{C} \subset Tw\,\mathcal{C}$: allows mapping cone operation. In particular, given objects X, Y we have

$$T_X(Y) = \{ hom(X,Y) \otimes X \longrightarrow Y \}$$

$\longrightarrow$ derived category $D(\mathcal{C}) = H^0(Tw\,\mathcal{C})$, is triangulated.

$\longrightarrow$ Karoubi completion $D^\pi(\mathcal{C})$: contains $D(\mathcal{C})$ and has idempotent splittings,

NEW

$$X \circlearrowleft \pi,\ \pi^2 = \pi \Rightarrow X = X^\pi \oplus X^{1-\pi}$$

$\mathcal{C}$ and $\mathcal{X} = (X_1, \dots, X_m) \to$ directed A_∞-subcategory $\mathcal{A}$,

$$hom_{\mathcal{A}}(X_i, X_j) = \begin{cases} hom_{\mathcal{C}}(X_i, X_j) & i<j \\ \mathbb{K} \cdot e_{X_i} & i=j \\ 0 & i>j \end{cases}$$

with restrictions of μ^d maps

Summary (Symplectic)

$L_0, L_1 \subset M$ Lagrangian submanifolds $\longrightarrow$ chain complex $CF^*(L_0, L_1)$ generated by intersection points, differential = 

. The cohomology is called Floer cohomology

NEW $HF^*(L_0, L_1)$, and is isotopy invariant.

This is part of an A_∞-category structure, called Fukaya category $\mathcal{F}(M)$: objects L, hom $= CF$, μ^d counts holomorphic $(d+1)$-gons.

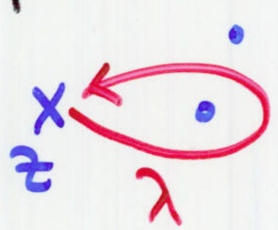
<u>Note</u>: $\phi : M \circlearrowleft$ symplectic autom.

$\longrightarrow \phi_* : \mathcal{F}(M) \circlearrowleft$. There is another Floer cohomology theory $HF^*(\phi)$, based on fixed points, and a map

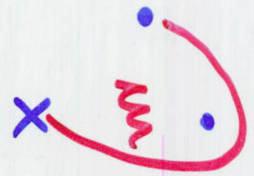
NEW $HF^*(\phi) \to \mathrm{Nat.\,transf.}(id \to \phi_*)$

<u>Summary (Picard-Lefschetz)</u>

$\pi : M \to \mathbb{C}$ Lefschetz fibration, Σ set of critical values. Fix a base point $z \notin \Sigma$.

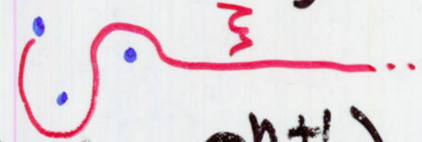
<u>Loop</u>  $\longrightarrow$ $\phi_\lambda : M_z \circlearrowleft$

monodromy

<u>Vanishing path</u> 

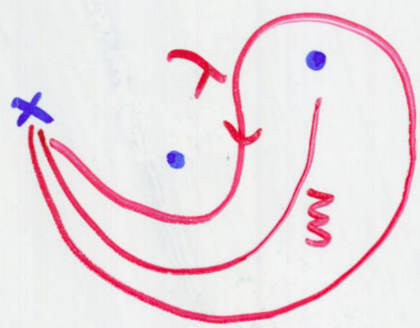
$\longrightarrow$

Lagrangian Lefschetz thimble $\Delta_\xi \subset M$, vanishing cycle
$V_\xi = \partial\Delta_\xi \subset M_z$ $(\Delta \cong D^{n+1}, V \cong S^n)$.

As a variation, 

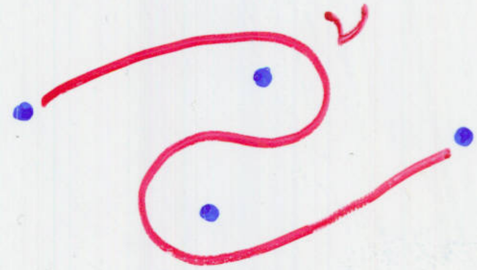
$\rightarrow \Delta_\xi$ noncompact $(\cong \mathbb{R}^{n+1})$.

Picard-Lefschetz:



<u>Dehn twist</u>

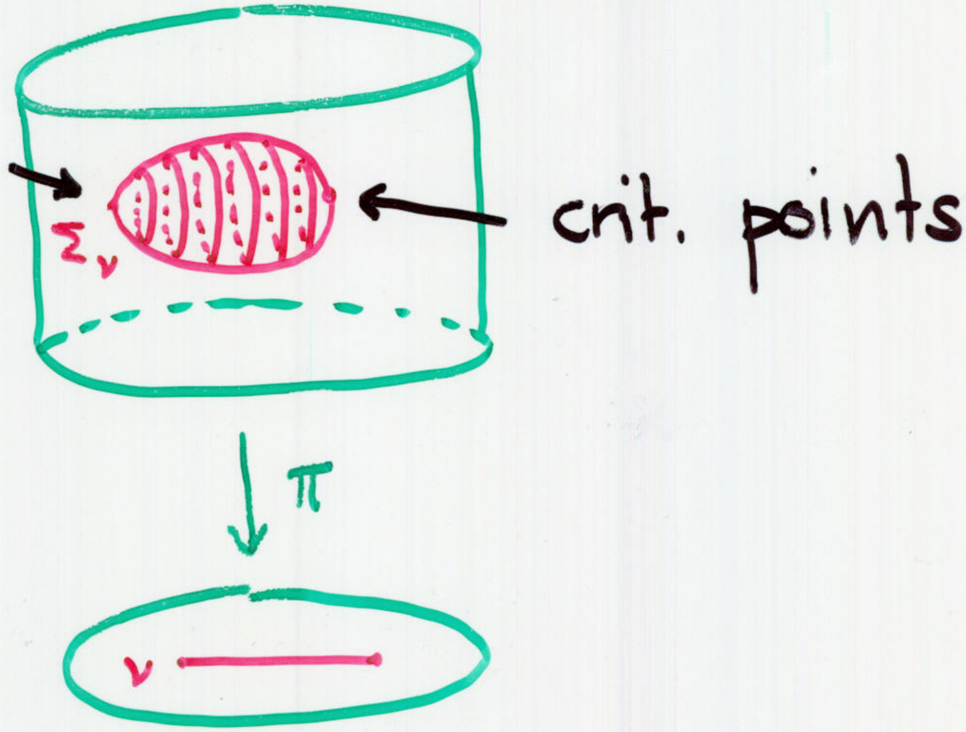
$\phi_\lambda = \tau_{V_\xi}$

<u>Matching path</u> 

NEW $\nu = \nu_+ \circ \nu_-$, if $V_{\nu_+} \cong V_{\nu_-}$ $\longrightarrow$
construct matching cycle, Lagrangian
$\Sigma_\nu \subset M$, $\Sigma_\nu \cong S^{n+1}$.

Naively, if $V_{\nu_+} = V_{\nu_-}$ then

$$\Sigma_\nu = \Delta_{\nu_-} \cup_{V_{\nu_\pm}} \Delta_{\nu_+}$$



Take Lefschetz fibrations
$\pi : M \to \mathbb{C}$, $q : M_2 \to \mathbb{C}$. Generically,
vanishing cycles of π
are matching cycles of q

("branch curve" or "braid monodromy" method).

Fix $\pi : M \longrightarrow \mathbb{C}$. We have

$\mathcal{F}(M)$ Fukaya cat. of total sp.

$\mathcal{F}(M_z)$ " " of fibre

(note: in lecture 1, $\mathcal{E} \subset \mathcal{F}(M_z)$ was the full subcategory of vanishing cycles)

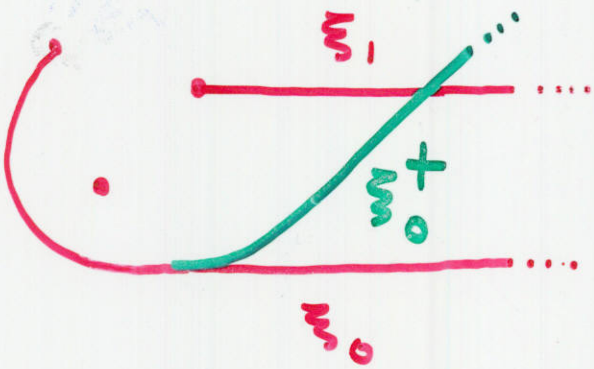
$\mathcal{F}(M, M_z)$ or better $\mathcal{F}(M, M_\infty)$, relative Fukaya category

Pick $\theta \in S^1$. Objects of $\mathcal{F}(M, M_\infty)$ are closed Lagrangian submanifolds of M, together with noncompact thimbles associated to paths $\xi : [0; \infty) \to \mathbb{C}$, $\xi^{-1}(\Sigma) = \{0\}$

$$\xi(t) = t\theta + \text{const.} \quad (t \gg 0)$$

If $L_0, L_1 \in \mathrm{Ob}\,\mathcal{F}(M, M_\infty)$ are thimbles, set

$$hom(L_0, L_1) = CF^*(L_0^+, L_1)$$



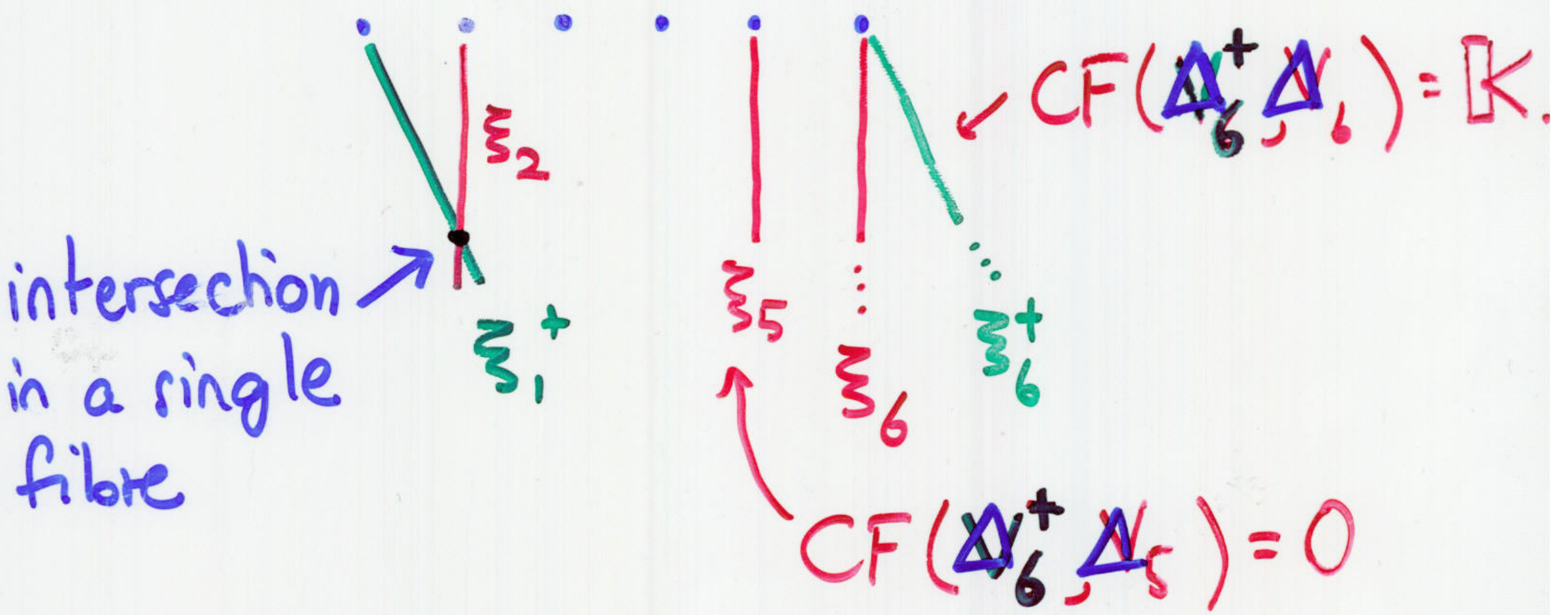
breaks duality: $\mathcal{F}(M, M_\infty)$ not cyclic

Embeddings

- $\mathcal{F}(M) \subset \mathcal{F}(M, M_\infty)$ full subcategory.
- $\mathcal{X} = (\xi_1, \dots, \xi_m)$ basis of vanishing paths going to ∞; $\Delta_1, \dots, \Delta_m$ Lefschetz thimbles, $V_1, \dots, V_m$ vanishing cycles in M_z, $z = t\Theta$ with $t \gg 0$.

$$\Delta_k \in \mathrm{Ob}\,\mathcal{F}(M, M_\infty)$$
$$V_k \in \mathrm{Ob}\,\mathcal{F}(M_z)$$

Then, full subcategory of $\mathcal{F}(M, M_\infty)$ with obj. Δ_k = directed subcategory of $\mathcal{F}(M_z)$ with objects V_k.



<u>Twisting</u> We now pass to twisted complexes, as a convenient formalism for explaining how different objects are built out of each other.

• $L_0 \subset M$ closed Lagrangian submf.
$L \subset M$ Lagrangian sphere; set

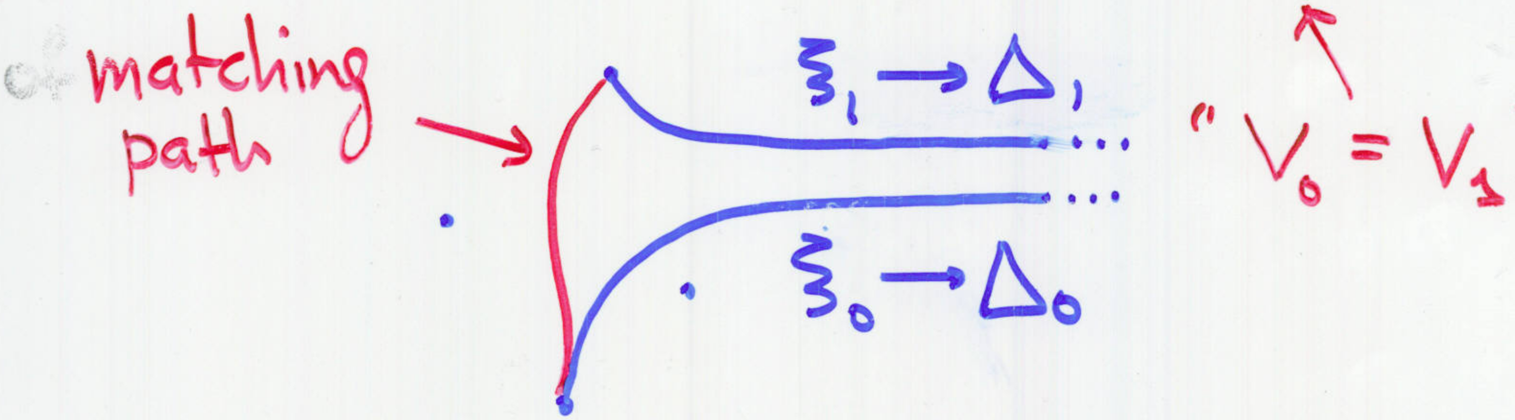
$$L_1 = \tau_L(L_0)$$

then in $H^0(Tw\,\mathcal{F}(M))$,

$$(1) \quad \boxed{L_1 \cong T_L(L_0)}$$

(same holds in $\mathcal{F}(M_z) \to$ lecture 1).

• $\Sigma \subset M$ matching cycle, split it into a pair of thimbles Δ_0, Δ_1

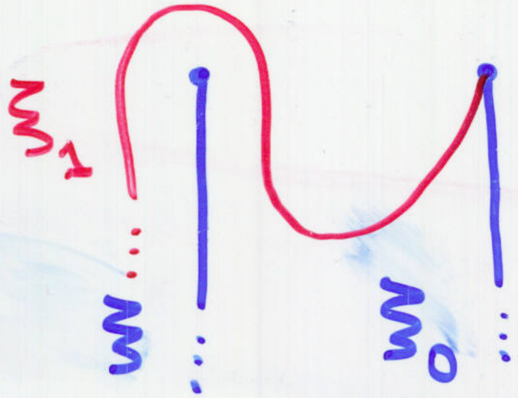


$$hom(\Delta_0, \Delta_1) = CF(\Delta_0^+, \Delta_1)$$
$$= CF_{M_z}(V_0, V_1) \ni e \text{ (preferred)}$$

In $H^0(Tw\,\mathcal{F}(M, M_\infty))$,

(2) $\boxed{\Sigma \cong \mathrm{Cone}(\Delta_0 \longrightarrow \Delta_1)}$

- Consider 3 Lefschetz thimbles as follows:



Then in $H^0(Tw\,\mathcal{F}(M, M_\infty))$,

(3) $\boxed{\Delta_1 \cong T_\Delta(\Delta_0)}$

So, if we know the <u>directed</u> subcategory of $\mathcal{F}(M_z)$ formed by one distinguished basis of vanishing cycles $(V_1, \dots, V_m)$

$\xrightarrow{\text{emb.}}$ <u>full</u> subcategory of $\mathcal{F}(M, M_\infty)$ formed by $(\Delta_1, \dots, \Delta_m)$

$\xrightarrow{③}$ full subcat. of $\mathcal{F}(M, M_\infty)$ formed by (any finite number of) thimbles

$\xrightarrow{②}$ full subcat. of $\mathcal{F}(M)$ formed by (any finite number of) matching cycles

! $\xrightarrow{\text{forget}}$ directed subcat. of $\mathcal{F}(M)$ formed by any ordered collection of matching cycles

... dimensional induction "machine" for computing $HF^*(L_0, L_1)$

spheres.

Recall that in our definition of Lefschetz fibration $M = X \setminus X_\infty$, $\pi = s_0/s_\infty$ with $s_0, s_\infty \in H^0(\mathcal{L})$, $X_\infty = s_\infty^{-1}(0)$, $K_X^{-1} \cong \mathcal{L}^r$ for some $r \in \mathbb{Z}$. Assume now that X_∞ is smooth and $r \neq 2$. Pick a basis of vanishing cycles $V_1, \dots, V_m \subset M_z$. Then in

$\widetilde{Symp}(M_z)$

$$\tau_{V_1} \cdots \tau_{V_m} \simeq \text{monodromy at } \infty = [4-2r]$$

shift ($\neq 0$)

<u>Corollary</u> $(V_1, \dots, V_m)$ split-generate the idempotent-closed derived cat. $D^\pi \mathcal{F}(M_z)$.

[<u>Corollary</u> $\mathcal{B} \subset \mathcal{F}(M_z)$ full subcat. with objects $(V_1, \dots, V_m)$, then
$D^{\pi}\mathcal{B} \cong D^{\pi}\mathcal{F}(M_z)$.

<u>Corollary</u> $D^{\pi}\mathcal{F}(M_z)$ can be fully described by a finite amount of computable data.]

<u>Proof</u> Exact triangles

$$
\begin{tikzcd}
L \arrow[r] & \tau_{V_k}(L) \arrow[dl] \\
\bigoplus V_k \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & L[4-2r] \arrow[d, phantom, "\|"] \\
L \arrow[rr, "0"] & & (\tau_{V_1} \cdots \tau_{V_m})^k(L) \arrow[dl] \\
 & \text{built from } V_1, \dots, V_m \arrow[ul] &
\end{tikzcd}
$$

$\xrightarrow{\text{octahedral axiom}}$

$\Rightarrow$ $L \oplus L[\text{shifted}]$ built from $\{V_k\}$

Mirror symmetry

no multiplicities

Y toric Fano

Y_∞ toric (singular) anticanonical div.

Y_z smoothing of Y_∞ (Calabi-Yau)

$\pi (= W) : M (= \mathbb{C}^* \times \cdots \times \mathbb{C}^*) \to \mathbb{C}$

the mirror LG theory

$D^{(\pi)}\mathcal{F}(M) \longleftrightarrow$ subcategory of $D^b Coh(Y)$ generated by points in the open orbit

$D^{(\pi)}\mathcal{F}(M, M_\infty) \longleftrightarrow D^b Coh(Y)$

$D^\pi \mathcal{F}(M_z) \longleftrightarrow$ (subcat. of) $D^b Coh(Y_\infty)$

$D^\pi \mathcal{F}(\overline{M}_z) \longleftrightarrow D^b Coh(Y_z)$