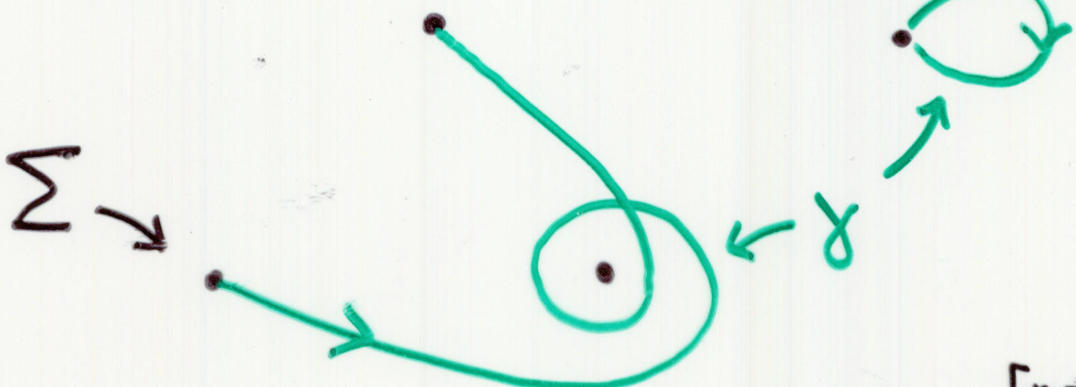
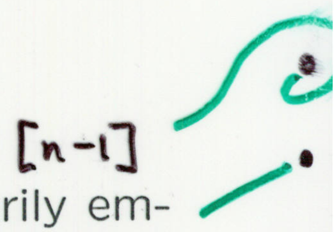
Let $\pi : M \to \mathbb{C}$ be a Lefschetz fibration with fibre dimension $n > 0$, and $\Sigma \subset \mathbb{C}$ its (finite) set of critical values. From the symplectic geometry of such fibrations one obtains a rich algebraic structure, which we will now try to characterize axiomatically. The description is made up of three different "animals".





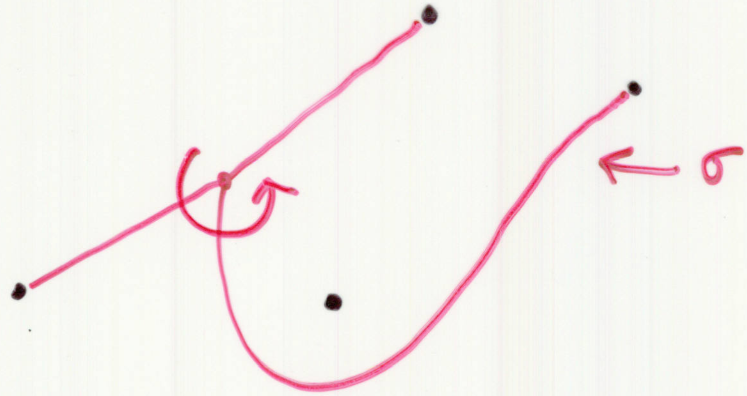
A <u>worm</u> is an oriented path γ in $\mathbb{C}$, not necessarily embedded, which intersects Σ precisely at its two endpoints. *To each worm γ is associated a $\mathbb{Z}$-graded chain complex $C_\gamma = (C_\gamma, \delta_\gamma)$*, with the following properties:

Homotopy invariance: With respect to homotopies in the given class of paths (endpoints fixed, and cannot pass over points of Σ).

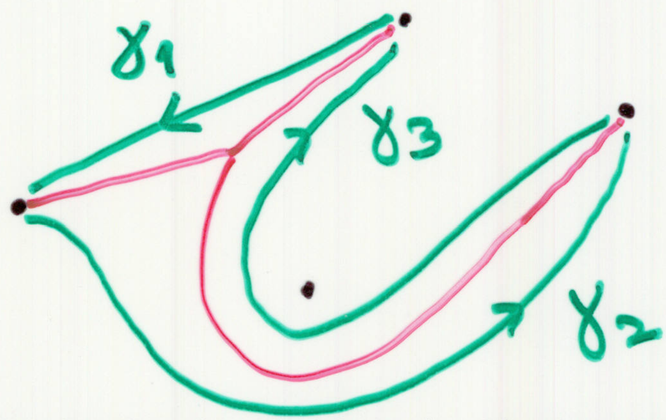
Symmetry: Reversing orientation dualizes the chain complex, more precisely $C_{-\gamma} = C_\gamma^\vee[-n]$. We denote the duality pairing by $\langle \cdot, \cdot \rangle$.

Small loops: If γ is a short path from a critical value to itself, then $C_\gamma = \mathbb{K}e \oplus \mathbb{K}t$, with $|e| = 0$, $|t| = n$, and vanishing differential. $\mathbb{K}$ is our coefficient field, $char(\mathbb{K}) = 0$.

A spider σ is a star-shaped graph with $d+1 \geq 3$ edges, mapped to $\mathbb{C}$ like this:



The spider's feet lie on Σ, and otherwise it is disjoint from Σ. The map to $\mathbb{C}$ must be an embedding near the central vertex, so that the legs are naturally cyclically ordered. Note that one can associate to σ a set of $d+1$ worms $\gamma_1, \dots, \gamma_{d+1}$ as follows:



The associated algebraic datum is that *every spider* σ *gives rise to a canonical linear map of degree* $2-d-n$,

$$c_\sigma : \bigotimes_{j=1}^{d+1} C_{\gamma_j} \longrightarrow \mathbb{K}$$

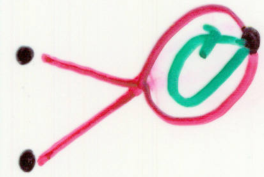
(one can use orientation-reversal to think of this in a variety of ways, for instance as a distinguished element of $\bigotimes C_{-\gamma_j}$).

Cyclic invariance: This is really implicit in our definition, since σ does not a priori come with an absolute ordering of the γ_j, only a cyclic one.

Homotopy invariance: as before.

Small spiders: If σ lies near a single critical value, then $c_\sigma = 0$ if $d > 2$. In the remaining case $d = 2$, we have $c_\sigma(e, e, t) = 1$, with all other combinations of generators vanishing for degree reasons.

Lopsided spiders: If two adjacent edges of σ are isotopic, so that γ_k is homotopic to a small loop, then the resulting map



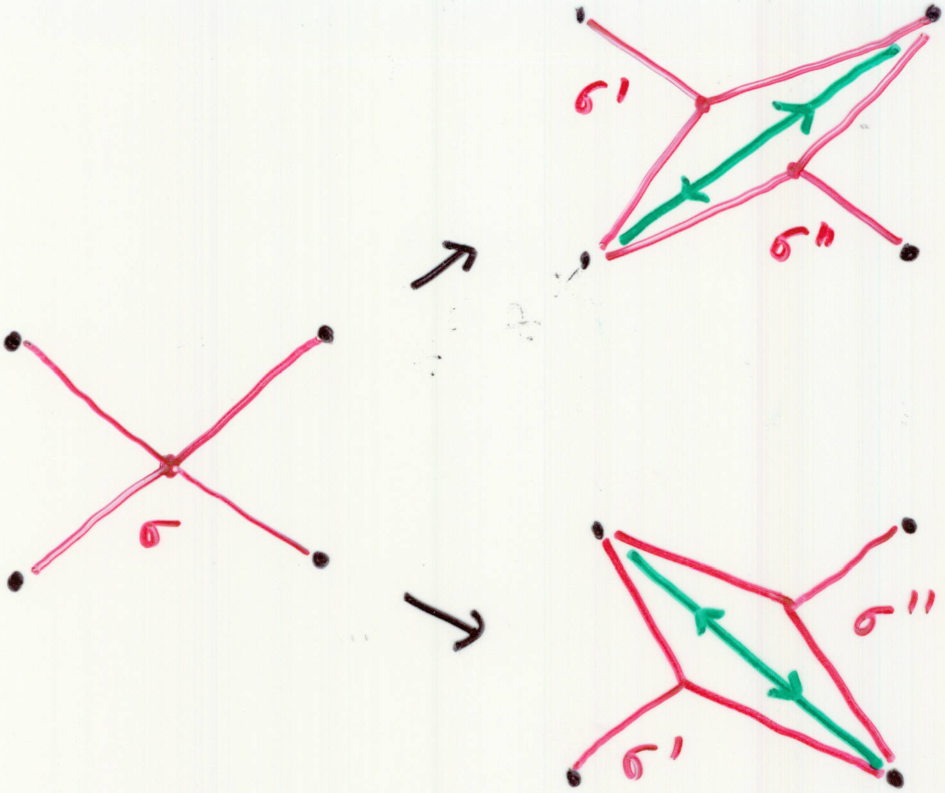
$$c_\sigma : (\mathbb{K}e \oplus \mathbb{K}t) \otimes \bigotimes_{j \neq k} C_{\gamma_j} \longrightarrow \mathbb{K}$$

satisfies $c_\sigma(e, \cdots) = 0$ if $d > 2$, and $c_\sigma(e, x, y) = \langle x, y \rangle$ for $d = 2$.

Coboundary:

$$\delta(c_\sigma) = \sum \langle c_{\sigma'}, c_{\sigma''} \rangle.$$

Here δ is the induced differential on $\bigotimes C_{\gamma_j}^\vee$, and the sum is over all splittings of σ into σ' and σ'', which share two adjacent edges ($d = d' + d'' - 1$).



Example: If $d = 2$, there are no splittings, so c_σ descends to a map on cohomology

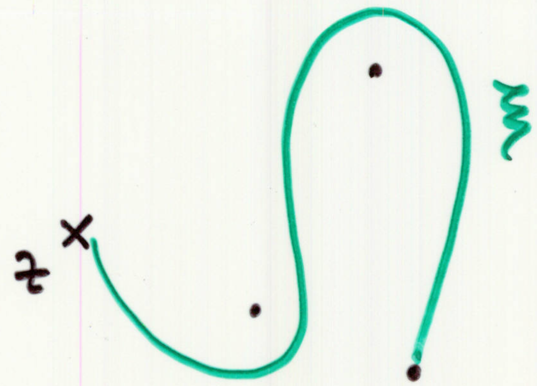
$$H(c_{\gamma_1}) \otimes H(c_{\gamma_2}) \to H(c_{-\gamma_3}). \qquad (1)$$

If $d = 3$, there are two splittings. The resulting formula shows that (1) is associative, even though the underlying cochain level map may not be.

Interpretation: fix a base point $z \in \mathbb{C} \setminus \Sigma$. Consider the set $Ob\,\mathcal{C} = \{X_\xi\}$ which has one element for each homotopy class of paths ξ going from z to a critical value of π (we call these *vanishing paths*). Given two vanishing paths, form $\gamma = \xi_1 \circ (-\xi_0)$ and define $hom_{\mathcal{C}}(X_{\xi_0}, X_{\xi_1}) = C_\gamma$. Write $\mu^1_{\mathcal{C}}$ for the differential δ_γ on this. Spiders centered at z give rise to further operations

$$\mu^d_{\mathcal{C}} : hom_{\mathcal{C}}(X_{\xi_{d-1}}, X_{\xi_d}) \otimes \cdots$$
$$\cdots \otimes hom_{\mathcal{C}}(X_{\xi_0}, X_{\xi_1}) \to hom_{\mathcal{C}}(X_{\xi_0}, X_{\xi_d})[2-d]$$

which have the following properties:



Homotopy associativity:

$$\sum \pm \mu_{\mathcal{C}}^{d-q+1}(a_d, \dots, \mu_{\mathcal{C}}^{q}(a_{p+q}, \dots, a_{p+1}), a_p, \dots, a_1) = 0, \tag{2}$$

where the sum is over all p and q with $q \geq 1$, $d-q+1 \geq 1$.

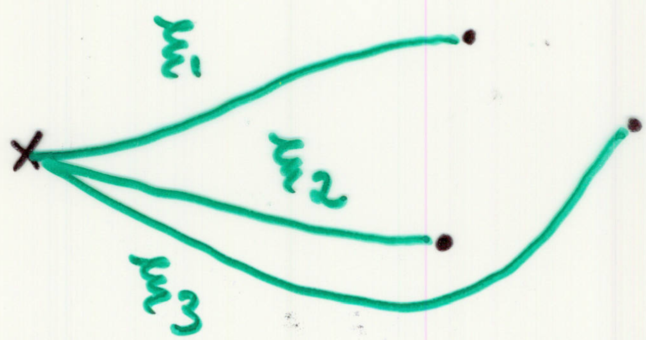
Unitality: The canonical elements $e = e_\xi \in hom_{\mathcal{C}}(X_\xi, X_\xi)$ satisfy $\mu_{\mathcal{C}}^2(e,a) = \mu_2^{\mathcal{C}}(a,e) = \pm a$, and $\mu_{\mathcal{C}}^d(a_d, \dots, a_1) = 0$ if $d \neq 2$ and at least one $a_k = e$.

Cyclic symmetry: There is a nondegenerate pairing $\langle \cdot, \cdot \rangle$ between $hom_{\mathcal{C}}(X_{\xi_0}, X_{\xi_1})$ and $hom_{\mathcal{C}}(X_{\xi_1}, X_{\xi_0})$, such that the expressions $\langle a_{d+1}, \mu_{\mathcal{C}}^d(a_d, \dots, a_1)\rangle$ are cyclically symmetric.

These are merely reformulations of the axioms. The first two make $\mathcal{C}$ into a (strictly unital) A_∞-category, and the last one means that $\mathcal{C}$ is cyclic.

Variants: (1) The cohomological category $H(\mathcal{C})$ has the same objects as $\mathcal{C}$, morphisms $H(hom_{\mathcal{C}}(X_{\xi_0}, X_{\xi_1})) = H(C_\gamma)$, and composition (1). This is a genuine category (graded, linear over $\mathbb{K}$).

(2) Take a basis of vanishing paths $\mathcal{X} = \{\xi_1, \dots, \xi_m\}$, one for each critical value:



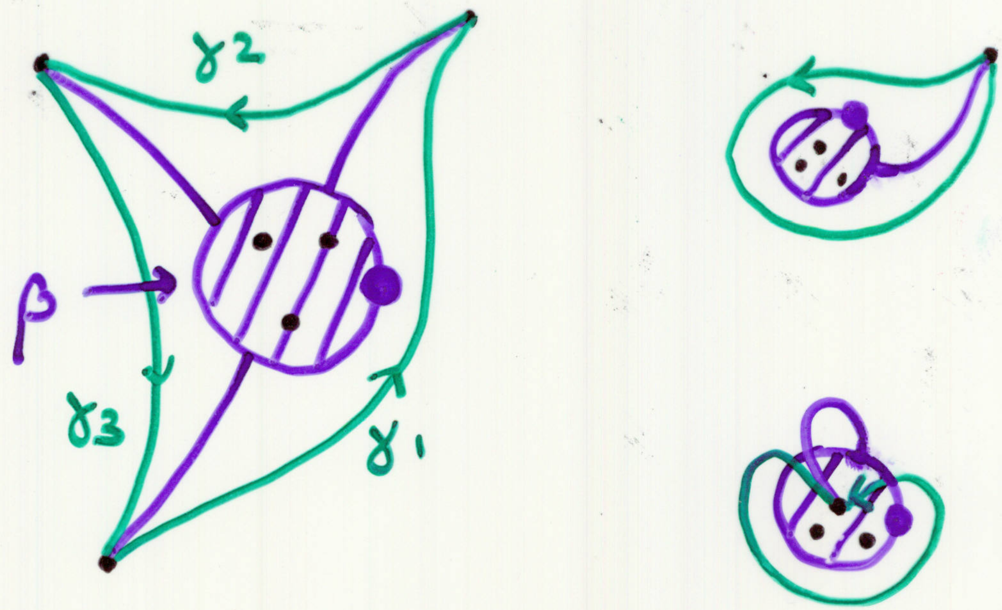
One can then consider the full subcategory $\mathcal{B} \subset \mathcal{C}$ formed using only the collection $(X_1, \dots, X_m)$, $X_k = X_{\xi_k}$.

(3) As before but we place our base point near infinity, and use a basis of vanishing paths which go inwards from there. This is then canonically ordered. The <u>directed A_∞-subcategory</u> $\mathcal{A} \subset \mathcal{B}$ has

$$hom_{\mathcal{A}}(X_j, X_k) = \begin{cases} hom_{\mathcal{B}}(X_j, X_k) & j < k, \\ \mathbb{K} \cdot e & j = k, \\ 0 & j > k. \end{cases}$$

By passing to the directed subcategory, we lose nothing on the level of morphism spaces and their cohomology (due to orientation-reversal duality). But since $\mu^d_{\mathcal{A}} = 0$ for $d > m-1$, we do lose a lot of information on composition maps.

For the final piece of data we consider ladybugs, which consist of a body (an embedded disc $D \subset \mathbb{C}$ with $\partial D \cap \Sigma = \emptyset$, containing at least one point of Σ in the interior), together with a head (marked boundary point of D), and $d+1 \geq 1$ spider-like legs joining ∂D (but not the head) to Σ. Again, one can associate to each ladybug a set of $d+1$ worms γ_j, which are now canonically ordered.

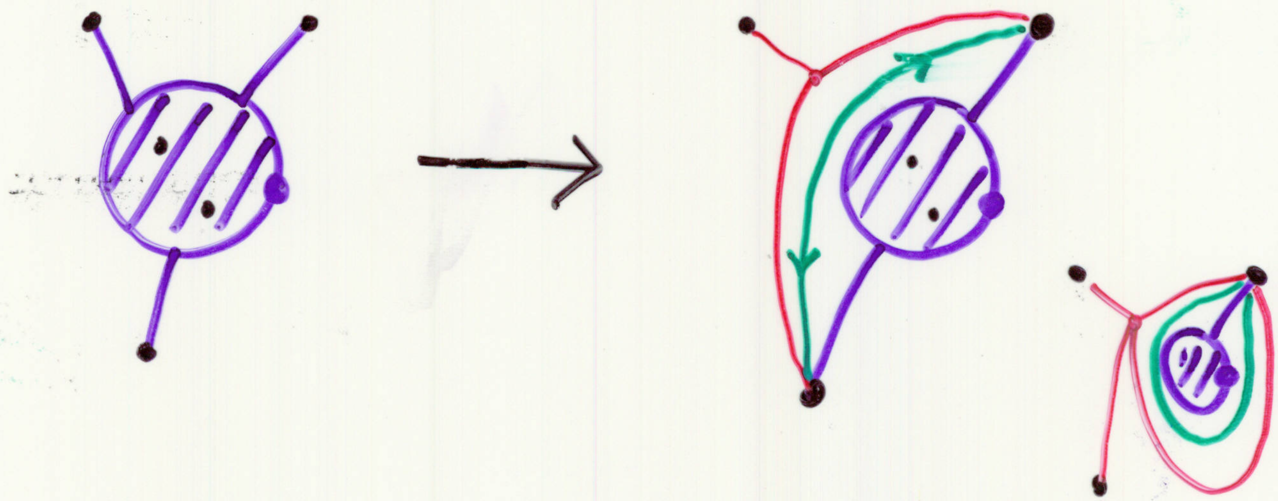


Our final piece of structure is this: *every ladybug gives rise to a linear map of degree $n-d$,*

$$c_\beta : \bigotimes_{j=1}^{d+1} C_{\gamma_j} \longrightarrow \mathbb{K}.$$

Homotopy invariance: As always.

Coboundary: $\delta(c_\beta)$ is obtained by pairing other spiders and ladybugs (with the same body D) as follows.



In particular, if $d = 0$ then c_β is a degree 0 cocycle in C_γ, $\gamma = -\gamma_1$. We have not finished the ladybugs axioms yet, but here is an update of the

Interpretation: Recall that the category $\mathcal{C}$ defined above involved a choice of base point z. Up to equivalence, z is irrelevant, but a loop $[\lambda] \in \pi_1(\mathbb{C} \setminus \Sigma, z)$ yields a nontrivial monodromy automorphism

$$\mathcal{G}_\lambda : \mathcal{C} \longrightarrow \mathcal{C}.$$

defined by dragging vanishing paths around the loop. If λ is the boundary of an embedded disc D, and we take z to be the head, then all possible ways of attaching legs and the resulting c_β define a natural transformation, in the sense of A_∞-functors,

$$N_D : \mathcal{G}_\lambda \longrightarrow Id_{\mathcal{C}}.$$

The next axiom can be conveniently formulated in this language:

Composition: Splitting of a disc D into two pieces D', D'' ~~corresponds~~ yields two natural transformations which satisfy $N_D = N_{D'} \circ R_{\mathcal{G}_{\lambda'}}(N_{D''})$.

We now return to the more elementary viewpoint of directly looking at the operations c_β.

Small ladybugs: Suppose that D is a small disc containing a single critical value, and that the legs of the ladybug remain nearby, so one gets

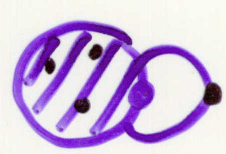
$$c_\beta : (\mathbb{K}e \oplus \mathbb{K}t)^{\otimes d+1} \longrightarrow \mathbb{K}.$$

This is zero for all $d \neq 1$; and for $d = 1$, $c_\beta(t \otimes t) = 1$, with other coefficients vanishing for degree reasons.

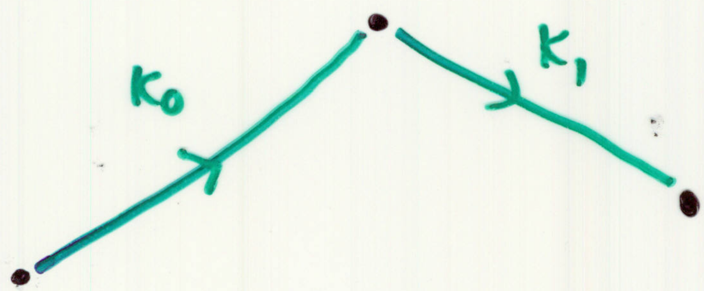
Lopsided ladybugs: Suppose that two adjacent legs of the ladybug are homotopic, so that

$$c_\beta : (\mathbb{K}e \oplus \mathbb{K}t) \otimes \bigotimes_{j \neq k} C_{\gamma_j} \longrightarrow \mathbb{K}.$$

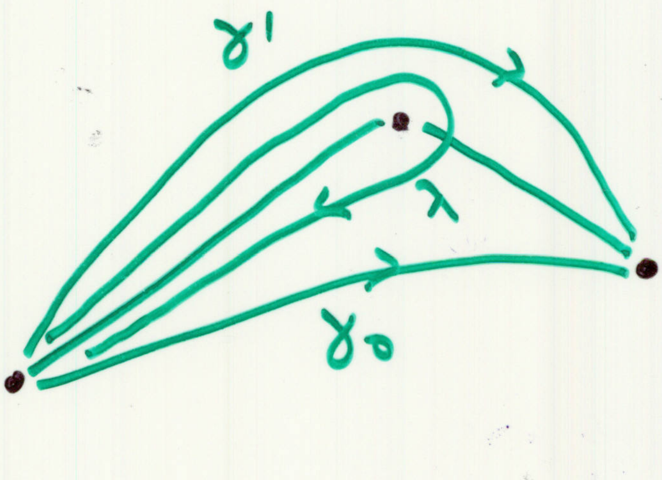
If $d > 0$, then $c_\beta(e, \cdots) = 0$.



The remaining axiom is the most crucial, but also the most complicated one. Suppose that we have three (not necessarily distinct) critical values, and worms κ_0, κ_1 joining them:

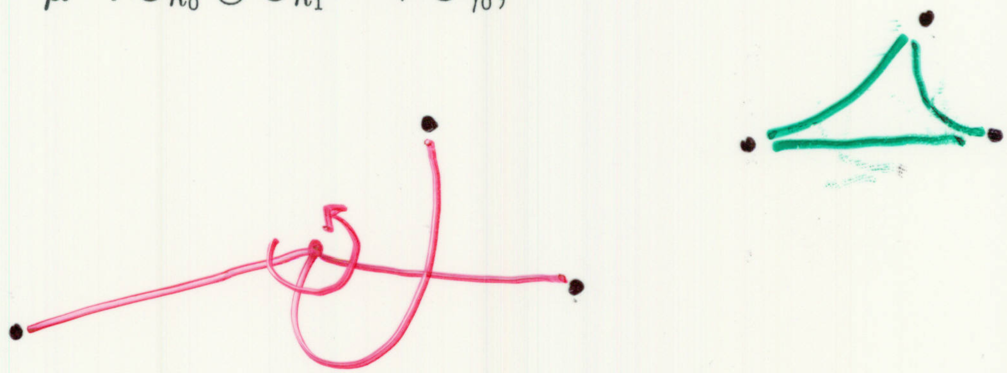


We then draw additional worms $\gamma_0, \gamma_1, \lambda$, as follows:



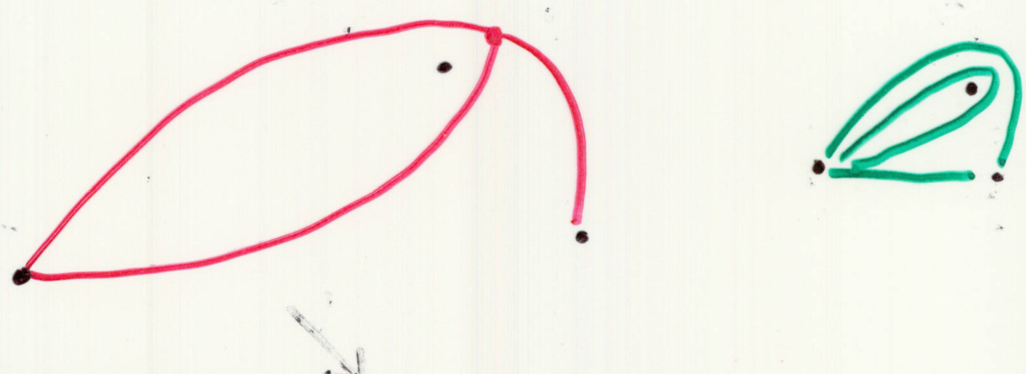
The three basic operations in this context are

$$\mu^2 : C_{\kappa_0} \otimes C_{\kappa_1} \longrightarrow C_{\gamma_0},$$



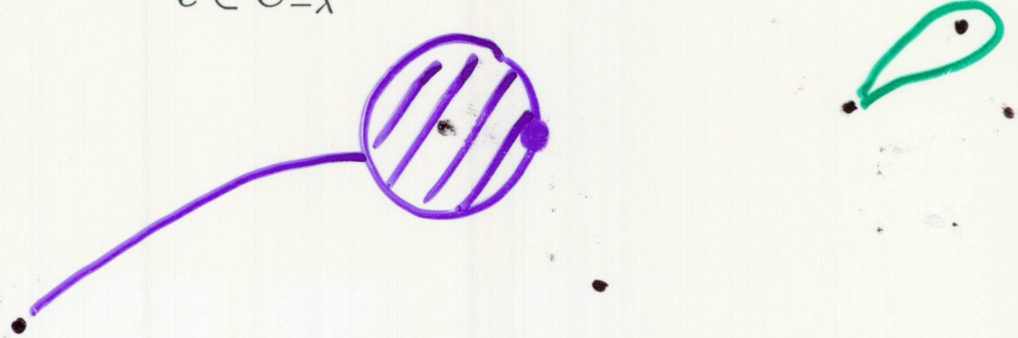
(note the specific choice of cyclic ordering of the edges)

$$\mu^2 : C_{-\lambda} \otimes C_{\gamma_0} \longrightarrow C_{\gamma_1}$$



and a distinguished cocycle

$$c \in C_{-\lambda}$$



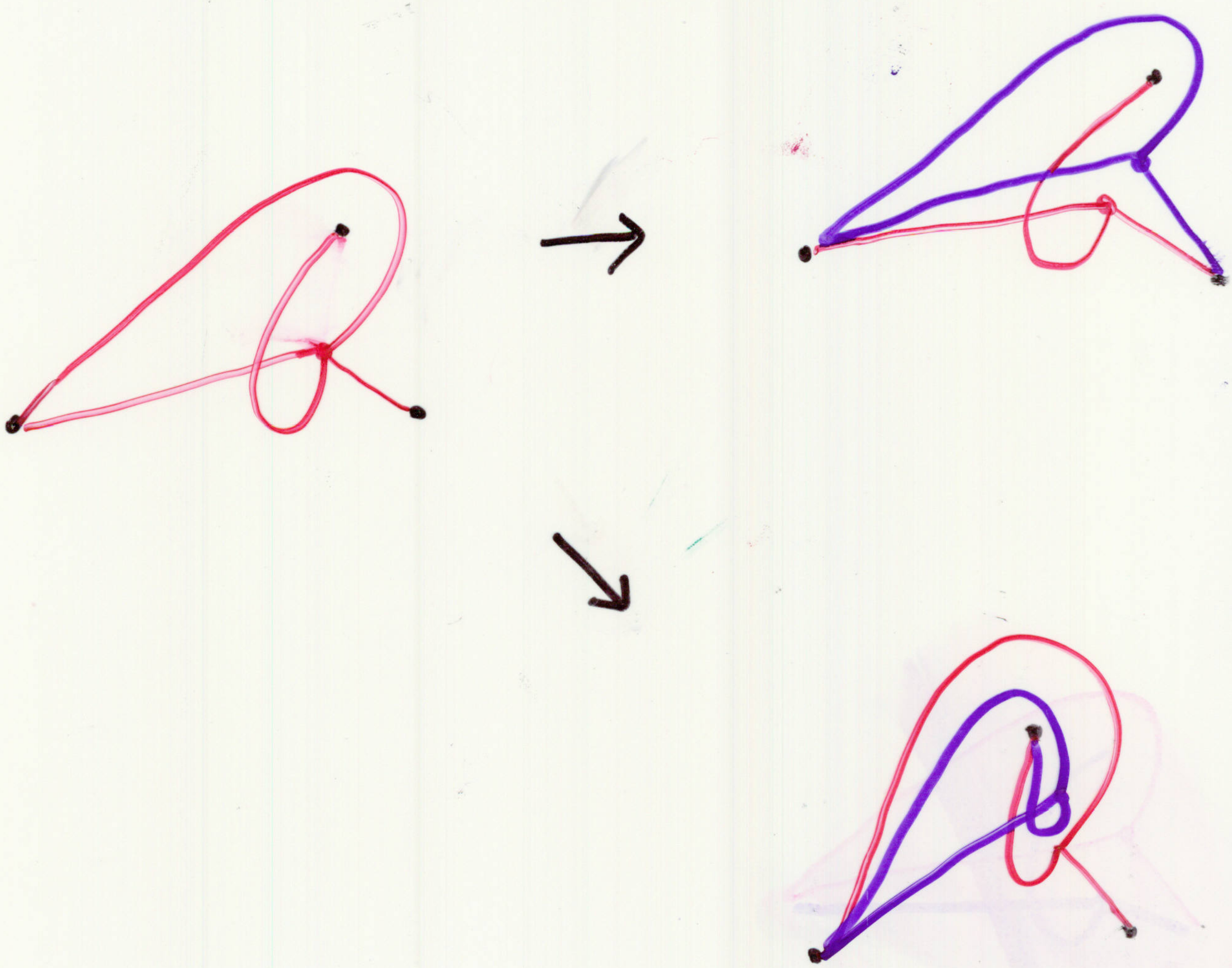
Claim: From the previous axioms we know that the composition of chain maps,

$$C_{\kappa_0} \otimes C_{\kappa_1} \xrightarrow{\mu^2} C_{\gamma_0} \xrightarrow{\mu^2(c,\cdot)} C_{\gamma_1} \tag{3}$$

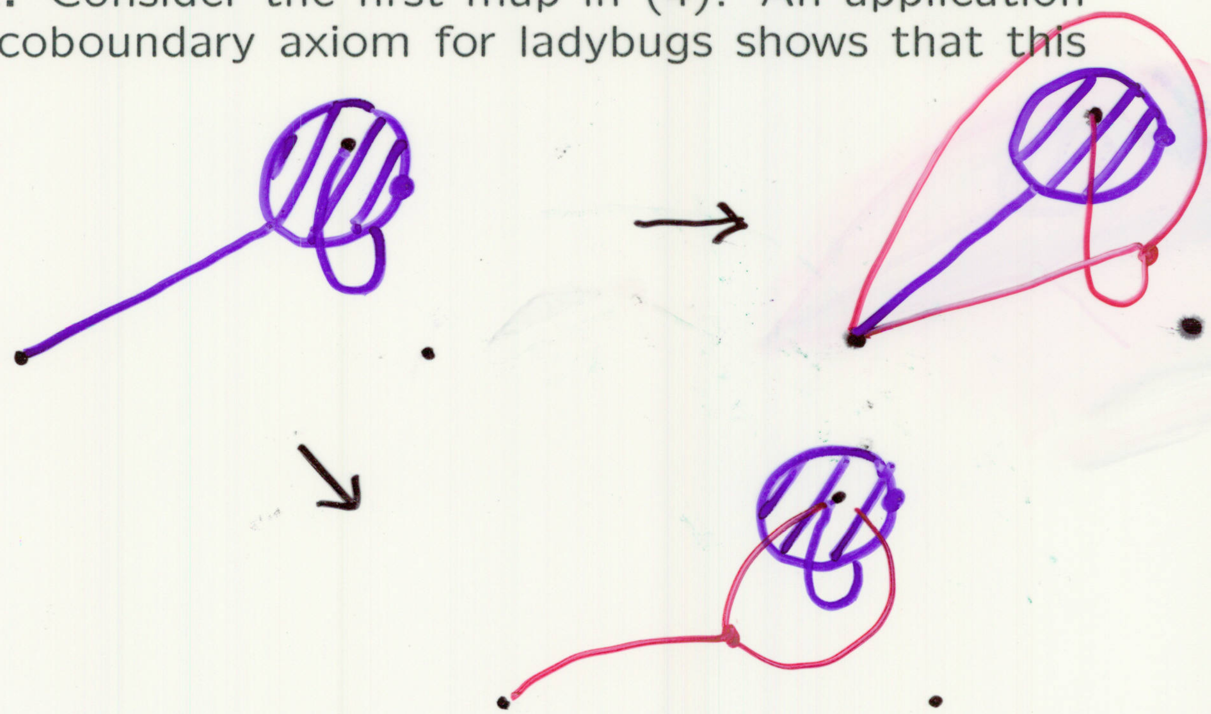
is canonically nullhomotopic. **Step 1**: applying the homotopy associativity of $\mu^2 = \mu^2_{\mathcal{C}}$, rewrite (3) as

$$C_{\kappa_0} \otimes C_{\kappa_1} \xrightarrow{\mu^2(c,\cdot)} C_{\kappa_0} \otimes C_{\kappa_1} \xrightarrow{\mu^2} C_{\gamma_1} \tag{4}$$

Graphically, this uses the coboundary axiom involving the splittings



Step 2: Consider the first map in (4). An application of the coboundary axiom for ladybugs shows that this



is homotopic to zero, proving our claim. Denote the resulting chain homotopy by h.

Triangle: The total complex of

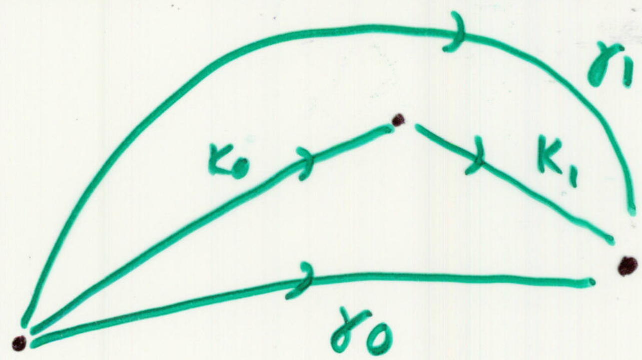
$$C_{\kappa_0} \otimes C_{\kappa_1} \xrightarrow{\mu^2} C_{\gamma_0} \xrightarrow{\mu^2(c,\cdot)} C_{\gamma_1}, \qquad h : C_{\kappa_0} \otimes C_{\kappa_1} \to C_{\gamma_1}$$

is acyclic.

This means that we have a long exact sequence

$$\cdots H(C_{\kappa_0}) \otimes H(C_{\kappa_1}) \longrightarrow H(C_{\gamma_0}) \longrightarrow H(C_{\gamma_1}) \cdots$$

intuitively drawn as a (categorified) "skein relation"; however, the acyclicity statement is more precise than the long exact sequence, since it includes the construction of h.

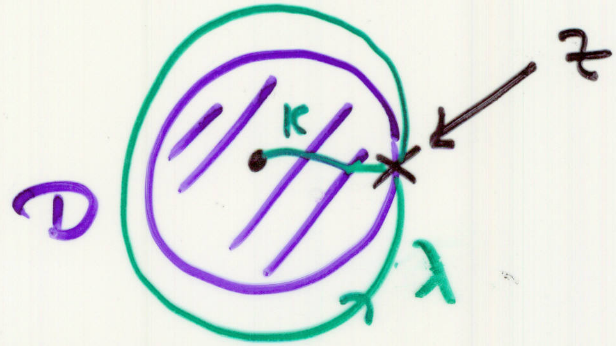


To interpret the triangle in categorical terms, we need to formally enlarge $\mathcal{C}$ by introducing "chain complexes". The outcome, denoted by $Tw(\mathcal{C}) \supset \mathcal{C}$, is the A_∞-category of twisted complexes. The underlying cohomological category $D(\mathcal{C}) = H^0(Tw(\mathcal{C}))$ is (mis-)called the derived category of $\mathcal{C}$; it is an ordinary triangulated $\mathbb{K}$-linear category. The same formal enlargement also works for functors and natural transformations. For any object $X \in Tw(\mathcal{C})$, one can introduce the twist functor $T_X : Tw(\mathcal{C}) \to Tw(\mathcal{C})$,

$$T_X(Y) = Cone\big(hom_{Tw(\mathcal{C})}(X,Y) \otimes X \to Y\big).$$

This comes with a natural transformation $id \to T_X$.

Take a disc D containing a single point of Σ. Choose a base point $z \in \partial D$. We then have:



the A_∞-category $Tw(\mathcal{C})$;
a preferred object X_κ in it;
the automorphism $Tw(\mathcal{G}_\lambda) : Tw(\mathcal{C}) \to Tw(\mathcal{C})$;
the natural transformation $Tw(N_D) : Tw(\mathcal{G}_\lambda) \to Id_{Tw(\mathcal{C})}$.

Corollary: (of the "triangle" axiom) $Tw(\mathcal{G}_\lambda)^{-1}$ is isomorphic to T_{X_κ} as an A_∞-functor, in a way which is compatible with the respective natural transformations from the identity functor to each.

Theorem: (1) The A_∞-category $\mathcal{C}$ can be reconstructed from the subcategory $\mathcal{B}$ given by any basis of vanishing cycles.

(2) $Tw(\mathcal{B}) \to Tw(\mathcal{C})$ is an equivalence. In particular, $D(\mathcal{B})$ is independent of the choice of basis of vanishing cycles.

(3) If $\mathcal{A} \subset \mathcal{B}$ is the directed subcategory, then $D(\mathcal{A})$ is also independent of the choice of basis.

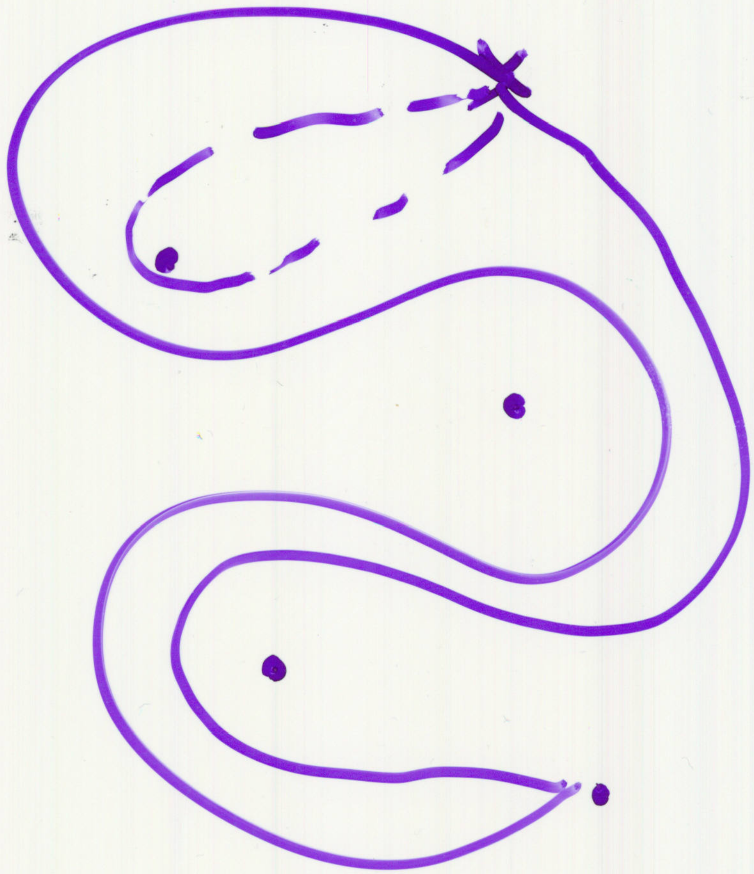
Some geometry: By a Lefschetz fibration we (officially) mean

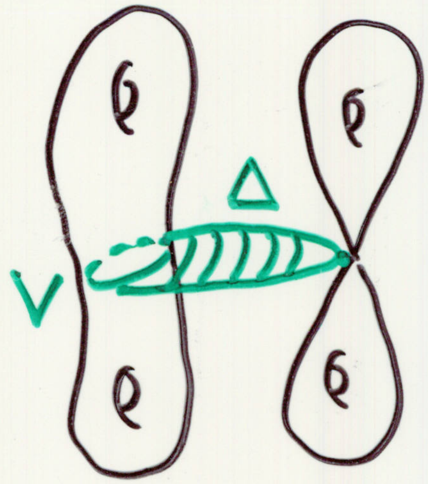
X^{n+1}	smooth projective variety
$\mathcal{L} \to X$	ample line bundle
$s_0, s_\infty \in H^0(\mathcal{L})$	linearly independent sections

$$X_0 = s_0^{-1}(0),\ \ X_\infty = s_\infty^{-1}(0),\ \ M = X \setminus X_\infty,$$
$$\pi = s_0/s_\infty : M \longrightarrow \mathbb{C}$$

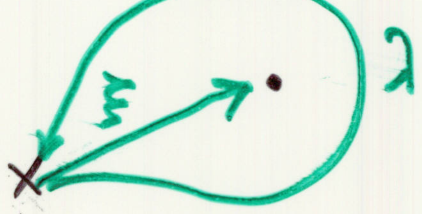
with the following additional properties: X_∞ is a normal crossing divisor; X_0 should be smooth near $X_0 \cap X_\infty$, and should intersect each stratum of X_∞ transversally; most importantly, π should have only nondegenerate critical points (locally modelled on $x_1^2 + \cdots + x_{n+1}^2$), at most one in each fibre. Finally, to get $\mathbb{Z}$-gradings on the algebraic structures, one should assume that $\mathcal{L} = \mathcal{K}_X^{\otimes d}$ for some d, which makes M "Calabi-Yau".

Equip $\mathcal{L}$ with a metric that gives rise to a Kähler form on X, and restrict that to M. Then, any two smooth fibres $M_z = \pi^{-1}(z)$, $z \in \mathbb{C} \setminus \Sigma$, are symplectically isomorphic. In fact, parallel transport along a path $\alpha : [0;1] \to \mathbb{C} \setminus \Sigma$ gives an isomorphism $\phi_\alpha : M_{\alpha(0)} \to M_{\alpha(1)}$, which varies continuously with α. If $\xi : [0;1] \to \mathbb{C}$ is an embedded vanishing path, so $\xi^{-1}(\Sigma) = \{1\}$, then the limiting behaviour of parallel transport gives rise to the Lefschetz thimble Δ_ξ, which is a Lagrangian $D^{n+1} \subset M$, $\pi(\Delta_\xi) = \xi([0;1])$, $\partial \Delta_\xi = \Delta_\xi \cap M_{\xi(0)}$.





$V_\xi = \partial\Delta_\xi$ is called the associated vanishing cycle, and is a Lagrangian sphere in $M_{\xi(0)}$ (this is also defined if ξ is not embedded). If ξ is a vanishing path, and λ a loop doubling around it, then the monodromy ϕ_λ is the Dehn twist (Picard-Lefschetz transformation) along V_ξ, written as τ_{V_ξ}.



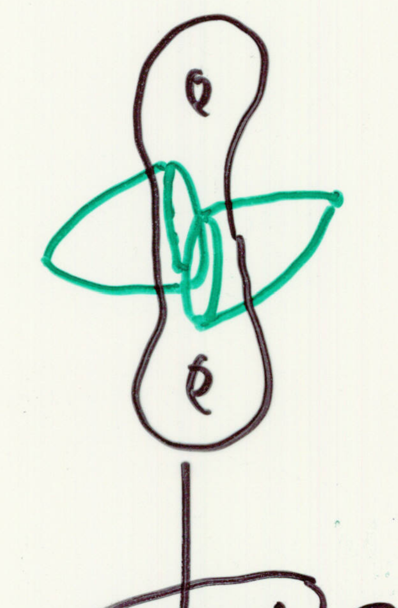
A basis $\mathcal{X} = (\xi_1, \ldots, \xi_m)$ of vanishing paths gives rise to a basis $\mathcal{V} = (V_1, \ldots, V_m)$ of vanishing cycles. Symplectically,

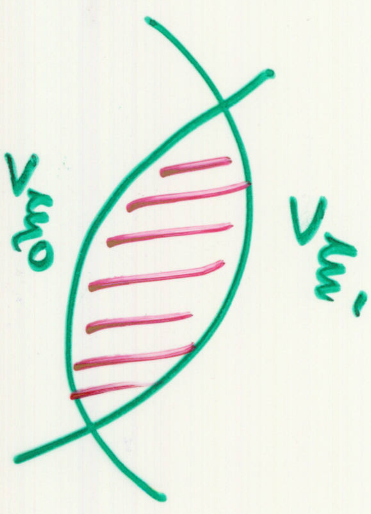
(fibre M_z, with basis $\mathcal{V}$)

$\iff$ (Lefschetz fibration π, with basis $\mathcal{X}$).

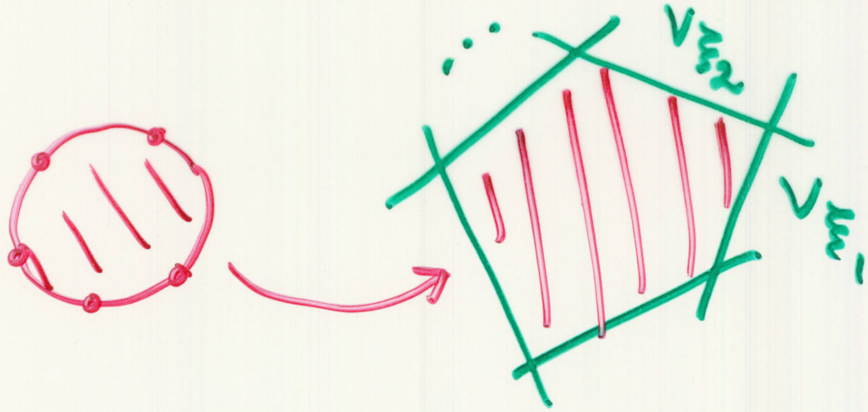
In particular, one can reconstruct M from $(M_z, \mathcal{V})$.

Obtaining the algebraic structure: Take a worm γ and cut it into two, so that it consists of a pair (ξ_0, ξ_1) of vanishing paths starting at the same point z. From that, one gets two Lagrangian spheres $V_{\xi_0}, V_{\xi_1} \subset M_z$. The chain complex C_γ is the associated Floer complex $CF^*(V_{\xi_0}, V_{\xi_1})$, generated by intersection points $x \in V_{\xi_0} \cap V_{\xi_1}$ (after perturbation to general position), with the differential given by counting holomorphic "bigons" on M_z with boundaries on our Lagrangian spheres (mod translation).

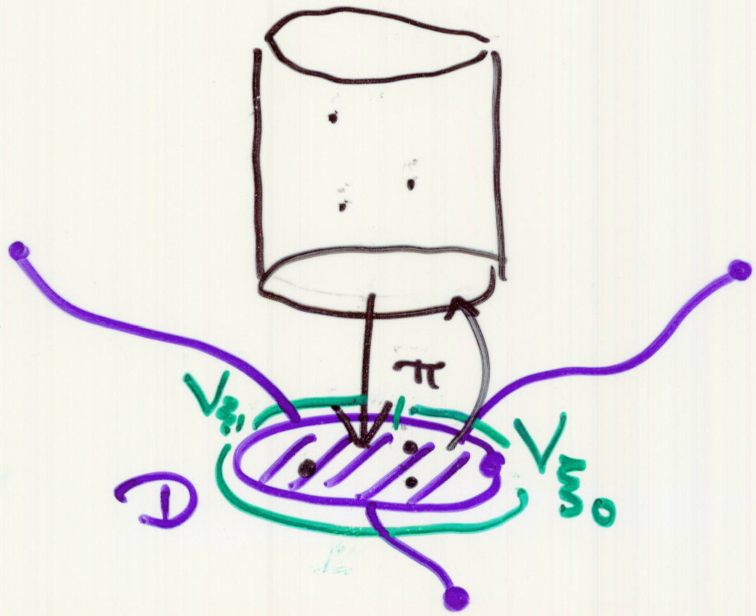




Similarly, the legs of a spider σ give rise to a cyclically ordered collection of vanishing cycles $V_{\xi_1}, \ldots, V_{\xi_{d+1}} \subset M_z$, and we define c_σ by counting holomorphic $(d+1)$-gons (where the domain can have any complex structure).



Finally, given a ladybug β with body D, we consider $\pi^{-1}(D)$ and take holomorphic sections of this, with boundary conditions in the vanishing cycles indicated by the legs of β. The moduli spaces of such sections provide c_β.



Dictionary: We will now translate our terminology into conventional symplectic topology terms. Given a symplectic manifold Q with $c_1(Q) = 0$, there is a cyclic A_∞-category $\mathcal{F}(Q)$, the Fukaya category, whose objects are closed Lagrangian submanifolds $L \subset Q$. Morphism spaces and composition maps are again given by Floer complexes $CF^*(L_0, L_1)$ and holomorphic polygons. Hence, our previous $\mathcal{C}$ is the full A_∞-subcategory of $\mathcal{F}(M_z)$ whose objects are vanishing cycles V_ξ for the given Lefschetz fibration $\pi : M \to \mathbb{C}$.

Any symplectic automorphism ϕ of Q gives rise to an automorphism ϕ_* of $\mathcal{F}(Q)$. In our case, we had a loop λ in $\mathbb{C} \setminus \Sigma$, $\lambda(0) = \lambda(1) = z$; the automorphism of $\mathcal{F}(M_z)$ induced by the monodromy map ϕ_λ preserves the subcategory $\mathcal{C}$. Restrict it and take the inverse, to get the previously mentioned functor $\mathcal{G}_\lambda$.

What about natural transformations N_D? One possible way to understand this is to also consider the "closed string" theory. For every automorphism ϕ of Q, there is a chain complex $CF(\phi)$, whose cohomology is called fixed point Floer cohomology $HF(\phi)$. For instance, $HF(id_Q) = H^*(Q; \mathbb{K})$. A general construction, the open-closed string map, says that any element of $HF(\phi)$ induces a natural transformation $id \to \phi_*$ of A_∞-functors.

If we have a disc $D \subset \mathbb{C}$, and λ parametrizes ∂D, then we first get a natural element of $HF(\phi_\lambda^{-1})$, which then gives rise to the natural transformation N_D between the functors $\mathcal{G}_\lambda, Id : \mathcal{C} \to \mathcal{C}$.