Gromov-Witten theory in low dimensions

3

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dim 0 $\mathbb{P}^0$ KdV

dim 1 $\mathbb{P}^1$, E, , , ...

Exactly solvable

Toda, Virasoro constraints,

odd homology ...

3 paper series Okounkov - P

dim 2

↗ $p_g = 0$ Göttsche Conj $\tau_0(\text{point})$

Descendents ? $\tau_k(\gamma)$

unknown

↘ gen type Sketched outline of the theory

Expect both cases to be equivalent to descendent Donaldson Theory.

dim = 3

Most interesting dimension ...

I want to present dim 3 GW theory from the perspective of the

GW/DT Correspondence. [MNOP 1+2]

Hodge integrals
equivariant vertex

CY geom
melting crystal / top vertex

Andreini talk

Mirror Symmetry

GW/DT

Descendent Correspondence /
Virasoro Constraints

Relative Correspondence
Hilb Scheme of Points of a Surface

DT theory

Let X be a ns, proj, 3-fold.

for $\beta \in H_2(X, \mathbb{Z})$, let
$n \in \mathbb{Z}$

$I_n(X,\beta)$ = Hilbert scheme of 1 dim subschemes of X of class β and holomorphic Euler Char n.

$Y \subset X$

β = class Y
$n = \chi(\mathcal{O}_Y)$

$[Y] \in I_n(X,\beta)$

$I_n(X,\beta)$ is very bad: reducible, non reduced, mixed dimension

Dont worry...

Motivated by holomorphic Chern-Simons theory, R. Thomas constructed a virtual class on $I_n(X,\beta)$.

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

Hilbert scheme

Def $\quad \mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Y)$

$\quad\quad \mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{O}_Y)$

Obs complicated $\quad \mathrm{Ext}^2(\mathcal{I}_Y, \mathcal{O}_Y)$?

$\vdots$

Not what R. Thomas did.

Chern-Simons theory is about counting bundles.

Idea: view $I_n(X,\beta)$ as a moduli space of ideal sheaves

$$\mathcal{I}_Y \subset \mathcal{O}_X$$

rk 1, torsion free, det trivial

$$0 = \mathrm{Ext}^0_0(\mathcal{I}_Y, \mathcal{I}_Y)$$

$$\text{Def} \quad \mathrm{Ext}^1_0(\mathcal{I}_Y, \mathcal{I}_Y)$$

$$\text{Obs} \quad \mathrm{Ext}^2_0(\mathcal{I}_Y, \mathcal{I}_Y)$$

$\rightarrow$ 2 term Def theory!

$$0 = \mathrm{Ext}^3_0(\mathcal{I}_Y, \mathcal{I}_Y)$$

Richard's PhD: Construction of $[I_n(X,\beta)]^{vir}$ via obstruction theory
in '96

What is virtual dimension?

$$\text{vir dim } I_n(X,\beta) = \mathrm{Ext}^1_0(\mathcal{I}_Y,\mathcal{I}_Y) - \mathrm{Ext}^2_0(\mathcal{I}_Y,\mathcal{I}_Y)$$

$$\vdots \ \text{R-R}$$

$$\int_\beta c_1(X)$$ indep of n

$$\text{vir dim } \overline{M}_{g}(X,\beta) = \int_\beta c_1(X) + \cancel{(\dim X - 3)}(1-g)$$

$$= \int_\beta c_1(X)$$ indep of g

What are the Donaldson-Thomas invariants?

$$\begin{tikzcd}
 & \mathcal{I} \arrow[d] & \\
 & I_n(X,\beta)\times X \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
I_n(X,\beta) & & X
\end{tikzcd}$$

← universal ideal sheaf

for $k \geq 0$, $\gamma \in H^*(X, \mathbb{Z})$

$$\sigma_k(\gamma) : H_*(I_n(X,\beta)) \to H_*(I_n(X,\beta))$$

$$\sigma_k(\gamma)(\alpha) = \pi_*\Big(\underset{\substack{\cap \\ H^*(I\times X)}}{\mathrm{ch}_{k+2}(\mathbb{I}) \cdot p^*(\gamma)} \cap \underset{\substack{\cap \\ H_*(I\times X)}}{\pi^*(\alpha)}\Big)$$

DT descendant invariant

$$\Big\langle \sigma_{k_1}(\gamma_1) \cdots \sigma_{k_\ell}(\gamma_\ell) \Big\rangle^X_{n,\beta} =$$

$$\text{ev } \sigma_{k_1}(\gamma_1) \cdots \sigma_{k_\ell}(\gamma_\ell) \, [I_n(X,\beta)]^{vir}$$

in $\mathbb{Q}$

Basic GW/DT

X ns, proj, 3-fold

let $\beta \in H_2(X, \mathbb{Z})$ be <u>fixed</u>

GW :

$$Z'_{GW}\left(\tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\right)_\beta$$

$$\parallel$$

$$\sum_g u^{2g-2}\left\langle \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\right\rangle^{\bullet X}_{g,\beta}$$

disconnected GW theory with <u>no</u> degree zero connected components

GW brackets denote

$$\int_{[\overline{M}_{g,\ell}(X,\beta)]^{vir}} \prod ev_i^*(\gamma_i)$$

$$\mathrm{vdim} = \int_\beta c_1(X)$$

DT :

$$Z_{DT}\left(\sigma_0(\gamma_1)\cdots\sigma_0(\gamma_\ell)\right)_\beta$$

$$\shortparallel$$

$$\sum_n q^n \left\langle \sigma_0(\gamma_1)\cdots\sigma_0(\gamma_\ell)\right\rangle^X_{n,\beta}$$

DT bracket denotes integration against $I_n(X,\beta)$

$$Z_{DT,0} = \sum_n q^n \int_{[I_n(X,0)]^{vir}} 1$$

Hilb scheme of n pts of X

$$Z'_{DT}\left(\sigma_0(\gamma_1)\cdots\sigma_0(\gamma_\ell)\right)_\beta$$

$$\shortparallel$$

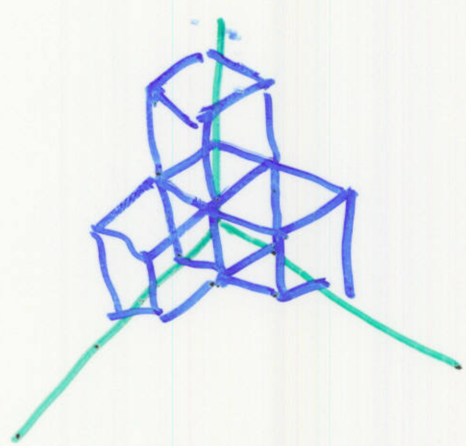
$$Z_{DT}(\sigma_0(\gamma_1)\cdots\sigma_0(\gamma_\ell))_\beta \,/\, Z_{DT,0}$$

GW/DT [MNOP 2]

Conj 1 : $Z_{DT,0} = M(-q)^{\int_X c_3(T_X \otimes K_X)}$

$$M(q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n}$$

gen fuction of 3-d partitions



Proven for Toric 3-folds [MNOP]

J. Li talk Today

Conj 2 [MNOP]

$$Z'_{DT}\left(\sigma_0(\gamma_1)\cdots\sigma_0(\gamma_\ell)\right)_\beta$$

is a rational function of q

Somewhat mysterious property...

Notation: $d =$ vir dim $= \int_\beta c_1(X)$

remember $Z'_{GW}(u)$

$Z'_{DT}(q)$

Conj 3 GW/DT

$$(-iu)^d \, Z'_{GW}\left(\tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\right)_\beta = (-q)^{-d/2}(-1)^\ell \, Z'_{DT}\left(\sigma_0(\gamma_1)\cdots\sigma_0(\gamma_\ell)\right)_\beta$$

under

$$-e^{iu} = q$$

Evidence

- Toric CY (Andrei's talk)
- local curves $\begin{array}{c} N \\ \downarrow \\ C \end{array}$ rk 2

total space of N

GW: J. Bryan's talk DT: [OP]

What about insertions?

Tests can be done $\mathbb{P}^3$.

Method: virtual localization

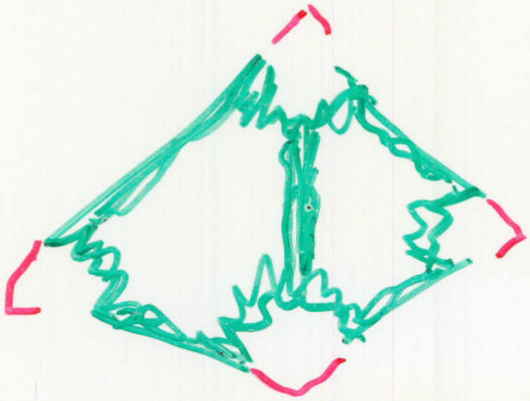
DT theory of $\mathbb{P}^3$

$\Downarrow$ virtual loc

$$\sum_{\substack{\text{monomial} \\ \text{ideals } \mathcal{I}}} w(\mathcal{I})$$

← highly non-trivial

evaluate by computer



We can now see why GW/DT is connected to integrals over $\overline{M}_{g,n}$

GW theory on $\mathbb{P}^3$

$\Downarrow$ virtual loc

$$\sum_{\Gamma} w(\Gamma) \longleftarrow \text{very nontrivial}$$

T-fixed loci $\overline{M}_{g,n}(X,\beta)$

Γ index

$$\int_{\overline{M}_{g,n}} \frac{\lambda_i \lambda_j \lambda_k}{\prod (1 - m_i \psi_i)}$$

general triple Hodge integral

Relative GW theory

X ns proj 3-fold
$\cup$
S ns proj surface

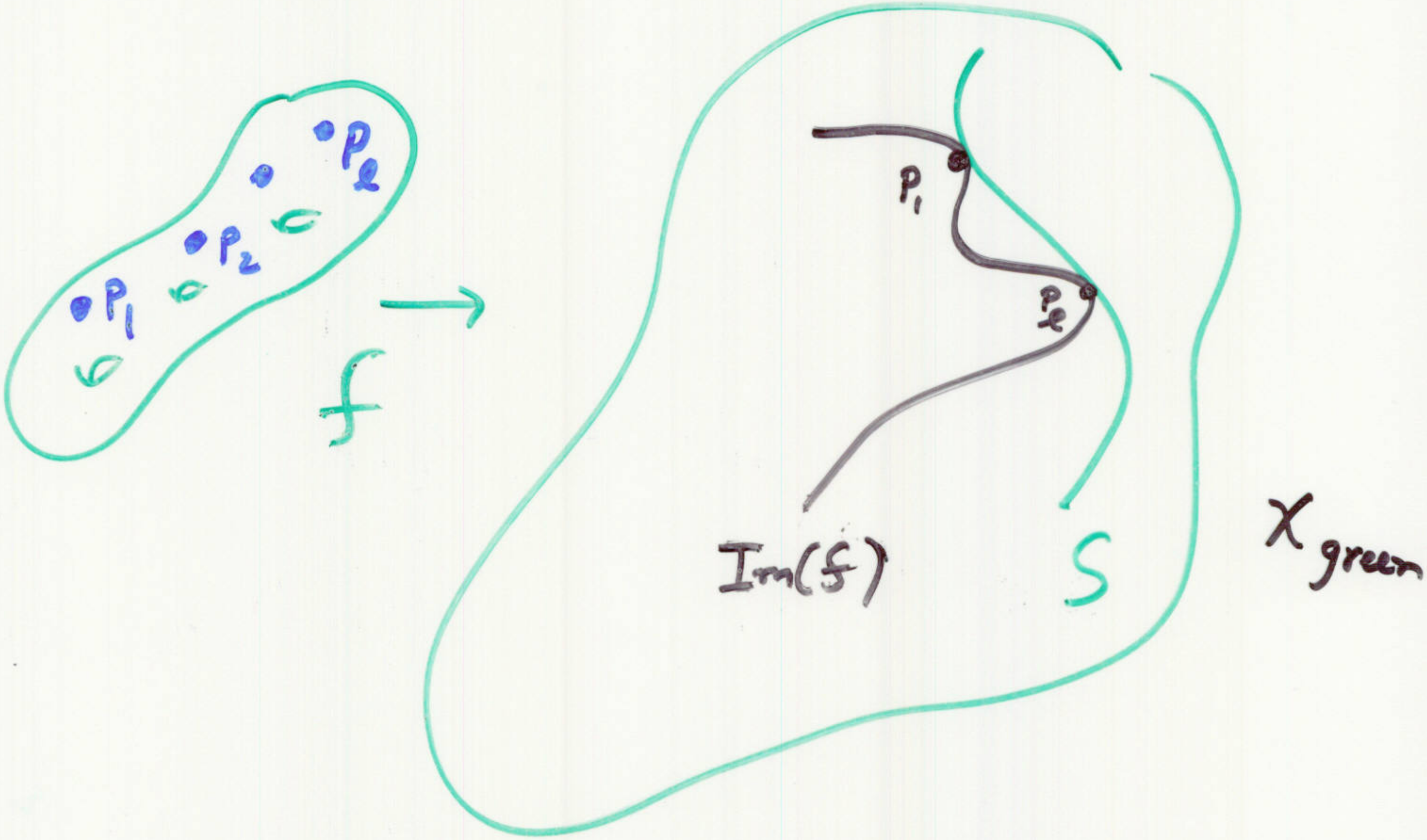
$\beta \in H_2(X, \mathbb{Z})$ $\quad \beta \cdot [S] \geq 0$

$$\sum m_i = \beta \cdot [S]$$

$\overline{M}_g(X/S, \beta)_{m_1, \dots, m_\ell}$

moduli space of stable relative maps

$[f : C \to X]$



$$f^{-1}(S) = \sum^{X} m_i \, p_i$$

relative condition

$\overline{M}_{g,n}(X/S)_{m_1, \dots, m_\ell}$ has, in addition to $\tau_k(\gamma)$ insertion, relative constraints

$$\mathrm{rel}\ ev_j : \overline{M}_{g,n}(X/S)_{m_1,\dots,m_e} \longrightarrow S$$

index rel marking

$$ev_j^*(\delta) \in H^*(\overline{M}_{g,n}(X/S,\beta))$$

$$\delta \in H^*(S)$$

Full set of rel conditions

$$\left((m_1,\delta_1),\dots,(m_e,\delta_e) \right)$$

$$\sum m_i = \beta \cdot [S]$$

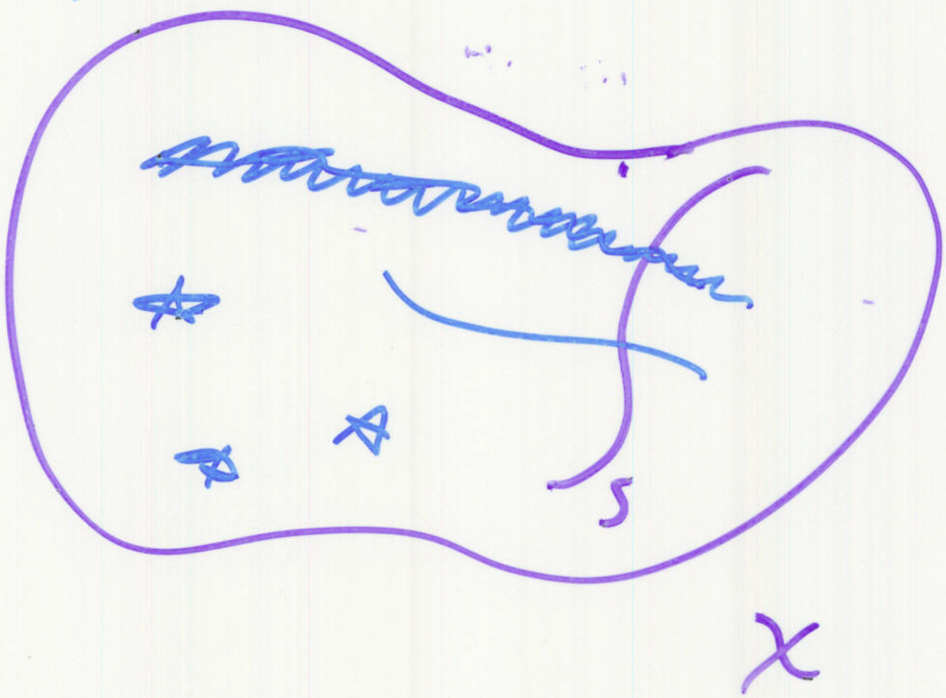
$$\delta_i \in H^*(S)$$

Virtual class $\dots>$ rel GW theory

Li-Ruan, Ionel-Parker, J. Li, EGH

What about Rel DT theory? J. Li

$I_n(X/S, \beta)$ Hilb scheme of X,
$\quad\; \cup$ Curves transverse
$\quad [Y]$ to S



equation of S not zero div in $\mathcal{O}_Y$

Hilb Scheme of points of S

Then

$$I_n(X/S,\beta) \xrightarrow{e} \mathrm{Hilb}_{\beta\cdot[S]}(S)$$

Relative Conditions

$$e^*(\phi) \qquad \phi \in H^*(\mathrm{Hilb}_{\beta\cdot[S]}(S))$$

Nakajima basis

$$\left((m_1,\delta_1),\dots,(m_\ell,\delta_\ell)\right)$$

$$\sum m_i = \beta\cdot[S] \qquad \delta_i \in H^*(S)$$

Cycle

v_i dual to δ_i

GW/DT Correspondence for Rel

[MNOP2]

GW rel $\qquad\qquad\qquad$ DT rel

$$\big((m_1,\delta_1)\cdots(m_\ell,\delta_\ell)\big) \longleftrightarrow \big((m_1,\delta_1)\cdots(m_\ell,\delta_\ell)\big)$$

This brings the geometry of Hilb Scheme of points of S to the GW theory of 3-folds.

[Okounkov-P]

Equivariant Quantum Cohomology of the Hilb scheme of points of $\mathbb{C}^2$

Nakajima $\Rightarrow$

$$\bigoplus_{n \geq 0} H_T^*(\mathrm{Hilb}_n \mathbb{C}^2, \mathbb{Q})$$

$$\parallel$$

$$\mathrm{Fock}\ \mathbb{Q}[t_1, t_2]$$

$v_\emptyset$ vacuum

α_{-k} creation

α_{+k} annihilation

$$[\alpha_k, \alpha_l] = k\,\delta_{k+l}$$

$$\alpha_{+k} v_\emptyset = 0$$

basis of Fock

$$|\mu\rangle = \frac{1}{z(\mu)} \prod \alpha_{-\mu_i} v_\emptyset$$

$$\mu = \mu_1, \dots, \mu_m$$

Basic divisor $D \in H^2(\mathrm{Hilb}_n \mathbb{C}^2)$

$$D = -|2, 1^{n-2}\rangle$$

$-\frac{1}{2} \cdot$ [divisor where 2 points come together]

$D\star$: Fock $\to$ Fock

$$D\star : (t_1+t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k+1}{(-q)^k-1} \alpha_{-k}\alpha_k +$$

$$\frac{1}{2} \sum_{k,l>0} t_1 t_2 \, \alpha_{k+l}\alpha_{-k}\alpha_{-l} - \alpha_{-k-l}\alpha_k\alpha_l$$

$$-\frac{t_1+t_2}{2} \frac{(-q)+1}{(-q)-1} \sum_{k>0} \alpha_{-k}\alpha_k$$