Gromov-Witten Theory In Low Dimensions

0, 1, 2, 3

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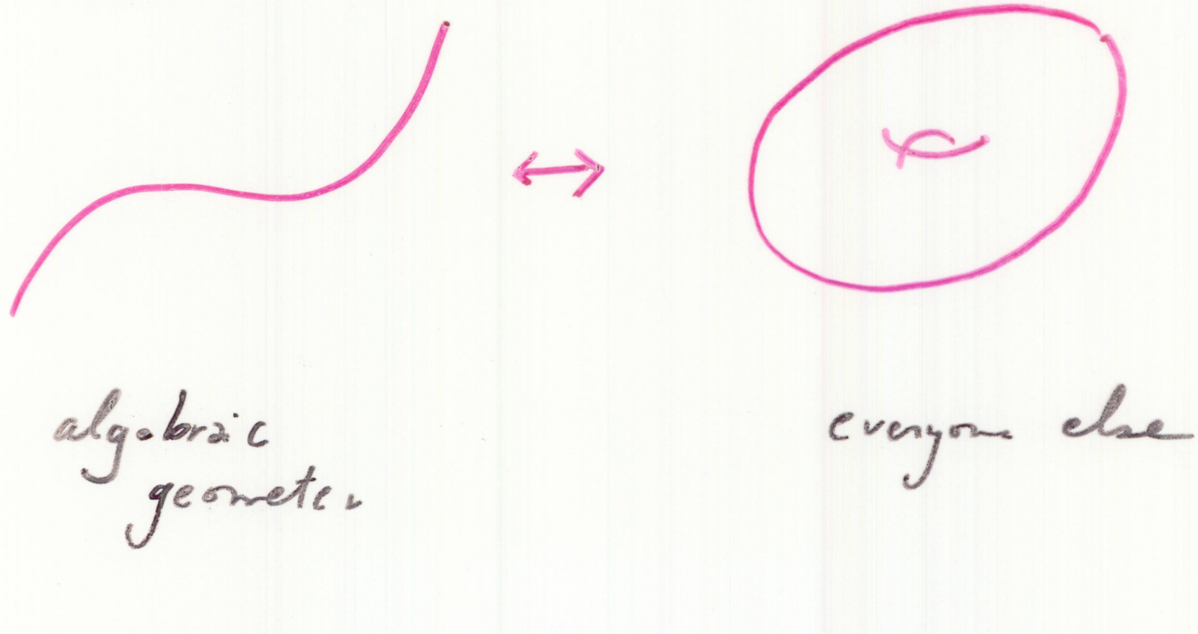
Gromov-Witten theory starts with the moduli space of curves : $\overline{\mathcal{M}}_{g,n}$

For the moduli problem, we consider complex algebraic curves

C

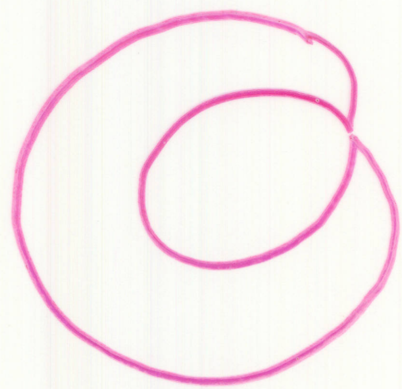
ex. 1



We allow curves with nodal singularities

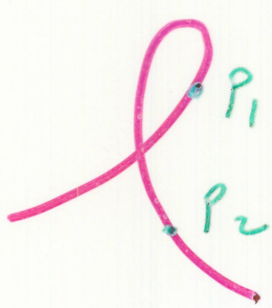
ex.| $\quad C$

$\leftrightarrow$



and marked point (not allowed to lie on the nodes)

ex.| $\quad (C, p_1, p_2)$



A pointed curve $(C, p_1, \ldots, p_n)$ is stable if there are only finitely many automorphisms

$$\sigma : C \to C$$

$$\sigma(p_i) = p_i$$

$\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable, n-pointed curves $(C, p_1, \ldots, p_n)$ of genus g

$$g(C) = h^1(C, \mathcal{O}_C)$$

First Remarks:

- $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g-3+n$

  by a dimension count going back to Riemann in the 19th Century

- $\overline{\mathcal{M}}_{g,n}$ is a compact orbifold

  Deligne - Mumford - Knudsen ~ 1970

- $\overline{\mathcal{M}}_{g,n}$ is of general type for large g

  Harris - Mumford ~ 1980

Mumford's paper "Towards an enumerative Geometry of moduli space of curves" ~'84

in some sense anticipated GW theory

Philosophical parallel

$$S_K \downarrow Gr(k,r) \qquad \begin{array}{c} C \\ \downarrow \uparrow p_1, p_2, p_3 \\ [C, p_1, \ldots, p_n] \in \end{array} \overline{M}_{g,n} \xleftarrow{\pi} \mathcal{C}, \quad s_1, \ldots, s_n : \overline{M}_{g,n} \to \mathcal{C}$$

$$H^*(Gr) \ni c_i(S_k)$$

$\vdots$

Schubert Calculus

$$H^*(\overline{M}_{g,n}) \ni \psi_i$$

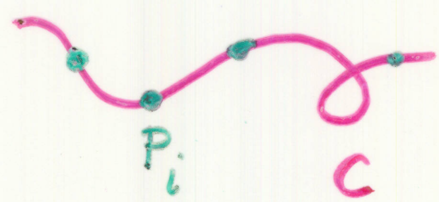
$$\psi_i = s_i^*(c_1 \omega_\pi)$$

$\vdots$

Gromov–Witten

Cotangent Lines :

$$\begin{array}{ccc} \mathbb{L}_i & & T^*_{p_i}C \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [\quad\quad] \end{array}$$



$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

First question in Gromov-Witten theory :

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} = \; ?$$

$$\sum \alpha_i = 3g-3+n$$

A good exercise in orbifolds is to compute

$$\int_{\overline{M}_{1,1}} \psi_1 = 1/24$$

P plane cubic thru 8 points

$P \longrightarrow \overline{M}_{1,1}$ degree 24

$$\int_P \psi_1 = 1.$$

$$1 + \sum_{g \geq 1} \left(\int_{\overline{M}_{g,1}} \psi_1^{3g-2} \right) z^g = \exp\left(\frac{z}{24} \right)$$

The question is extremely rich:

first answer is via Kontsevich's

Combinatorial model.

Define a generating function: $\overline{M}_{g,n}$

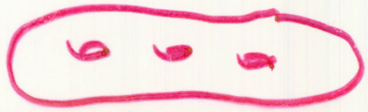
$$K_g(x_1, \dots, x_n) = \sum_{\substack{\alpha_1, \dots, \alpha_n \\ \sum \alpha_i = 3g-3+n}} \int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \prod_{i=1}^{n} \frac{(2\alpha_i - 1)!!}{x_i^{2\alpha_i+1}}$$

$$7!! = 7 \cdot 5 \cdot 3 \cdot 1$$

The combinatorial model expresses

$K_g(x_1, \dots, x_n)$ in terms of graph

enumeration on the topological

surface $\Sigma_g =$  of genus g

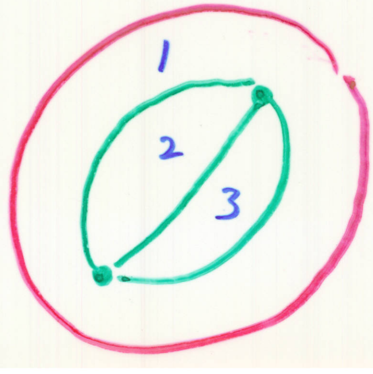
A map m on Σ_g with n cells consists of the following data:

(1) a graph Γ on Σ_g such that

- all vertices are 3-valent
- $\Sigma_g \setminus \Gamma$ is a disjoint union of n disks (called cells)

(2) a labelling of the cells with the integers $1, 2, \ldots, n$

ex! A genus 0, $n = 3$ map

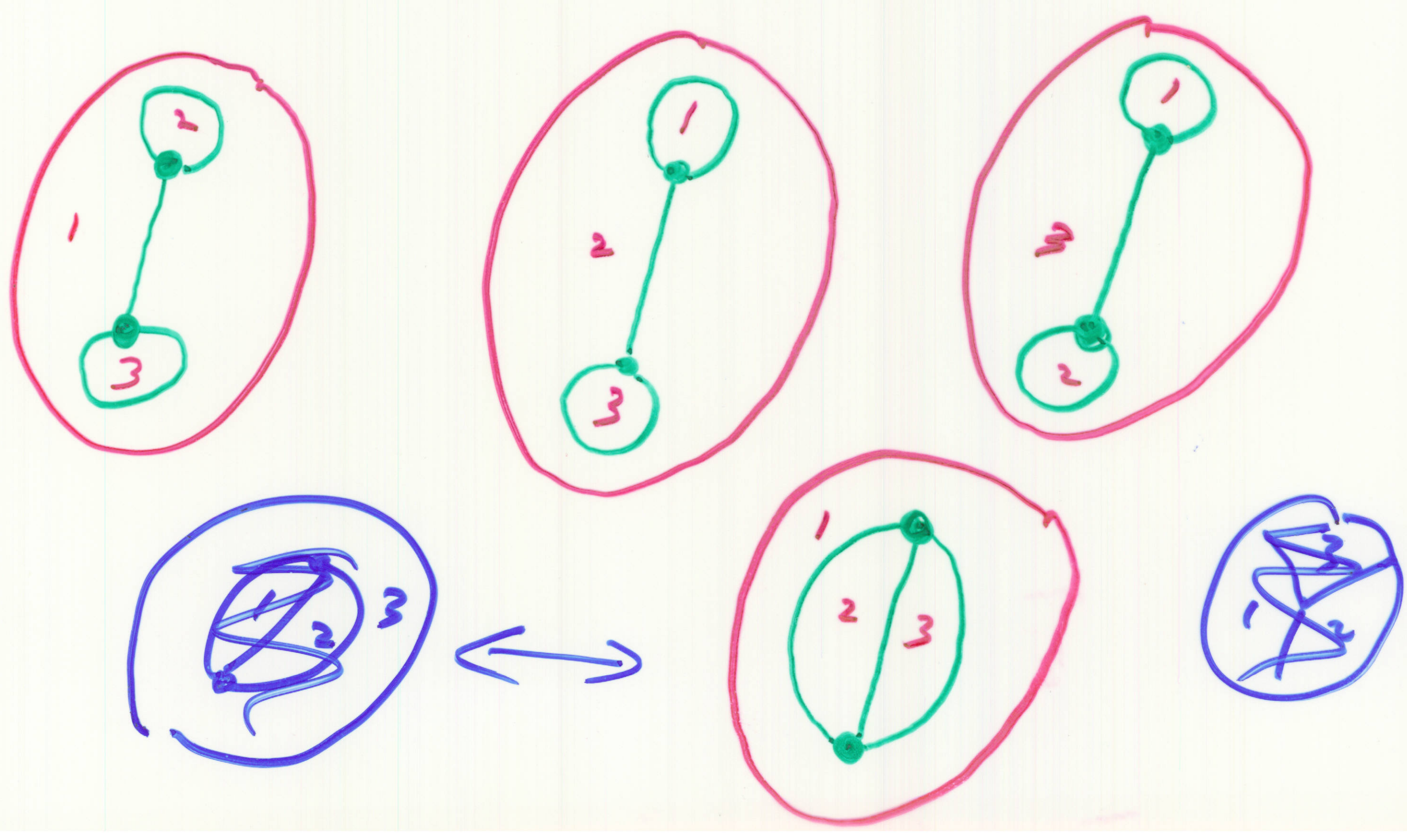


Let $\mathcal{M}ap(g,n)$ be the set of all maps (up to isom) on Σ_g with n cells

$\mathcal{M}ap(g,n)$ is a finite set.

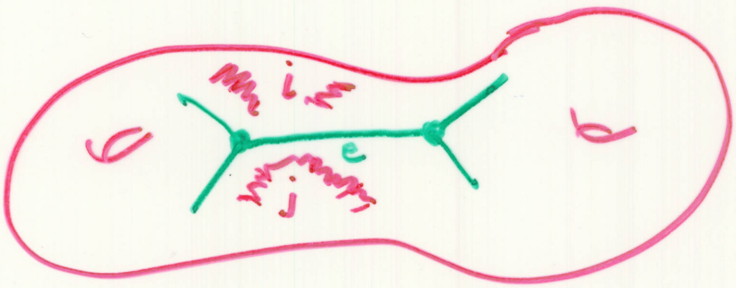
<u>Ex 1</u>

$\mathcal{M}ap(0,3)$



Comb model

$$K_g(x_1, \dots, x_n) = \sum_{m \in Map(g,n)} \frac{2^{2g-2+n}}{|Aut(m)|} \prod_{e \in Edges(\Gamma)} \frac{1}{x(e)}$$



$$x(e) = x_i + x_j$$

Kontsevich '92

Okounkov - P '00

Mirzakhani '03

Kazarian - Lando '05

$$F = \sum_{g\geq 0} \sum_{n\geq 1} \sum_{d_1,\dots,d_n} \int_{\overline{M}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} \frac{t_{d_1}\cdots t_{d_n}}{n!}$$

$F(t_0, t_1, t_2 \dots)$

$$u = \frac{\partial^2 F}{\partial t_0^2}$$

Then u satisfies KdV

$$\frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12} \frac{\partial^3 u}{\partial t_0^3}$$

also KdV hierarchy

proven from matrix integral representation of Kontsevich's comb model

The determination of $\displaystyle\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}$

is a first (numerical) step

towards a Schubert Calculus for $\overline{M}_{g,n}$

To go further, we would

like to understand $H^*(\overline{M}_{g,n})$

There are several directions

that can be pursued :

(1) Stable cohomology of $M_{g \to \infty}$

Mumford's Conjecture proven by

Madsen - Weiss

Tuesday Talk

(2) Cohomological study in low genus

$g = 2, 3, 4$

Getzler
Looijenga
Faber
Van der Geer
Bergström
Tommasi

Already in $g=1$, on $\overline{M}_{1,11}$

we find $H^{0,11}(\overline{M}_{1,11}) = \mathbb{C}$

(obtained from writing an explicit holomorphic diff form related to a cusp form)

In $g=2$, n large stranger fauna appear ... [Faber's talk]

(3) Tautological ring

$$R^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})$$

Minimal subring generated by the boundary and fiber geometry.

$$\overline{M}_{g_1, n_1+1} \times \overline{M}_{g_2, n_2+1} \longrightarrow \overline{M}_{g_1+g_2, n_1+n_2}$$

$$\overline{M}_{g-1, n+2} \longrightarrow \overline{M}_{g,n}$$

forgetful $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$

Formally, $R^*(\overline{M}_{g,n})$ is the smallest system of subring closed under push forward by all these maps.

ex 1

$\overline{M}_{g,1} \times \overline{M}_{0,3}$

$D = \Big[$  $\Big] \in R^2(\overline{M}_{g,2})$

Divisor

$$\begin{array}{c} \overline{M}_{g,2} \\ \downarrow \pi \\ \overline{M}_{g,1} \end{array}$$

$$-\pi_*\left[D^2\right] \in R^2(\overline{M}_{g,1})$$

$$\parallel$$

ψ_1 cotangent line class

Very many such calculations show the existence of a rich variety of classes in the tautological ring

Q1) Does $R^*(\overline{M}_{g,n}) = H^*(\overline{M}_{g,n})$

ANS: No, $\overline{M}_{1,11}$ has odd cohomology

Another example

$$\begin{array}{c} \mathcal{C} \\ \downarrow \pi \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

$$\begin{array}{c} \mathbb{E} = \pi_* \omega \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

$\mathbb{E}$ rk g bundle of holomorphic diff forms.

$$\begin{array}{ccc} \mathbb{E} & & H^0(C, \omega) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [\ C\ ] \end{array}$$

$$\lambda_i \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \qquad \lambda_i = c_i(\mathbb{E})$$

$ch_i(\mathbb{E})$ determined by GRR in terms
ω, boundary geometry (Mumford)

$$\Downarrow$$

$$\lambda_i \in R^*(\overline{\mathcal{M}}_{g,n})$$

Q2) Does $R^*(\overline{M}_{g,n})$ equal the subring of $H^*(\overline{M}_{g,n})$ generated by the algebraic cycle classes

Ans: No

$$\overline{M}_{1,12} \times \overline{M}_{1,12} \xrightarrow{i} \overline{M}_{2,22}$$

$$\cup$$

$$\Delta$$

$$\|$$

$$\{E \times E\}$$

$$i_*[\Delta] \notin R^*(\overline{M}_{2,22})$$

Graber - P

Q3) Does $R^*(\overline{M}_{g,n})$ have any structure?

Ans: Yes (conjecturally)

Conjecture : $R^*(\overline{M}_{g,n})$ satisfies

Poincaré duality

$$\int : R^k(\overline{M}_{g,n}) \times R^{3g-3-k+n}(\overline{M}_{g,n}) \to \mathbb{Q}$$

Perfect

Evidence

- $\overline{M}_{0,n}$ true since $R^*(\overline{M}_{0,n}) = H^*(\overline{M}_{0,n})$
- $\overline{M}_{1,n}$ true by Getzler's work
- $\overline{M}_{2,\text{small}}$ true
- $\overline{M}_3$, $\overline{M}_{3,1}$
- $\overline{M}_4$

$\Big\}$ inspection

Not an overwhelming amount of evidence.

Faber $\Longrightarrow$ Hain-Looijenga

Faber-P

One of the reasons we want Poincaré duality to hold in the following:

If $R^*(\overline{M}_{g,n})$ satisfies duality, then the entire ring structure is determined by the pairings

$$\int : R^k(\overline{M}_{g,n}) \times R^l(\overline{M}_{g,n}) \times R^{3g-3+n-k-l}(\overline{M}_{g,n}) \to \mathbb{Q}$$

which can be seen to be determined by

$$\int_{\overline{M}_{h,m}} \psi_1^{d_1} \cdots \psi_m^{d_m}$$

$\overline{M}_{g,n}$

$\cup$

D^k D

Then the entire tautological ring structure would be accessible.

Study of $R^*(\overline{M}_{g,n})$ is Gromov-Witten theory in dim $= 0$

Before we proceed, I would like to point out a different perspective on $R^*(\overline{M}_{g,n})$

One can consider

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n})$$

Chow ring of cycle classes

Then duality makes a nontrivial prediction

$$R^{3g-3+n}(\overline{M}_{g,n}) \cong \mathbb{Q}$$

True Faber - P
Graber - Vakil

$$[C] \in \overline{M}_{g,n} \longrightarrow \in A^{3g-3+n}(\overline{M}_{g,n})$$

When is $[C] \in R^{3g-3+n}(\overline{M}_{g,n})$?

Speculation: If C is defined over $\overline{\mathbb{Q}}$, then

$$[C] \in R^{3g-3+n}(\overline{M}_{g,n})$$