Geometry and physics of localization sums

Equivariant localization (Atiyah & Bott, Duistermaat & Heckman, Berline & Vergne, ...) and its generalization to virtual classes (Graber & Pandharipande) may be the single most powerful tool currently available in enumerative geometry (..., Ellingsrud & Strømme, Kontsevich, ...).
It reduces computations in $T$-equivariant cohomology to contributions of torus-fixed loci.

In the case when the torus-fixed points are isolated, the result is a finite sum.
Albeit finite, this sum may be complicated and extracting useful information from it may not be easy.
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Adopting a probabilistic viewpoint may help.
This will be illustrated by a single example, namely ...
Localization in Donaldson-Thomas theory
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Localization sums arising in instanton counting (i.e. from moduli of framed sheaves on \( \mathbb{P}^2 \)) form an important special case.
Donaldson-Thomas theory is an enumerative theory of curves in a nonsingular projective 3-fold $X$.

Its relationship to the Gromov-Witten theory of the same 3-fold $X$ is the subject of conjectures proposed in [MNOP].
The double life of a curve

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- a parameterized curve, i.e., the image of a map from an abstract curve to $X$. Or, ...

- it may be described by its equations, i.e. by an ideal sheaf on $X$

plot(...) vs. implicitplot(...)
The natural moduli space on two sides are Kontsevich’s moduli space of stable maps $\overline{M}_g(X; \beta)$ and the Hilbert scheme of curves $\text{Hilb}(X; \beta, \chi)$, respectively. Here

$\beta \in H_2(X)$ is the degree of curve

$g =$ domain genus of a map

$\chi =$ Euler char of an ideal sheaf $\approx 1 - g$

The geometry of these moduli spaces is very different.
One feature $\overline{M}_g(X; \beta)$ and Hilb$(X; \beta, \chi)$ share is the existence of a virtual fundamental class (Li & Tian, Behrend & Fantechi, Thomas) of dimension

$$\dim [\ ]^\text{vir} = -\beta \cdot K_X.$$ 

GW and DT invariants of $X$ are defined by evaluating natural cohomology classes on $[\ ]^\text{vir}$. 
Today, we will focus on the Calabi-Yau case $K_X = 0$. In this case, $\dim \vir = 0$ and there is one invariant, “the number of curves”, in every degree and genus.

The case of general $X$ will, probably, be discussed in Rahul’s lecture . . .
GW/DT partition function

Form the following generating function

\[ Z_{GW}(t, u) = \sum_{\beta, g} u^{2g - 2} t^\beta \int \left[ \overline{\mathcal{M}}_g(X; \beta) \right]_{\text{vir}} 1 \]

- multi-index
- disconnected
**GW/DT partition function**

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\[ Z_{GW}(t, u) = \sum_{\beta, g} u^{2g-2} t^\beta \int_{\mathcal{M}_g(X; \beta)^{\text{vir}}} 1 \]

and its reduced version

\[ Z'_{GW}(t, u) = \frac{Z_{GW}(t, u)}{Z_{GW}(0, u)} \]

multi-index

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known
GW/DT partition function

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and its reduced version

\[ Z'_{GW}(t, u) = Z_{GW}(t, u) / Z_{GW}(0, u) \]

Define \( Z_{DT} \) and \( Z'_{DT} \) by the same formulas with

\[ u^{2g-2} \Rightarrow q^x \]
Conjecture ([MNOP])

\[ Z'_{GW}(t, u) = Z'_{DT}(t, -e^{iu}) \]

has a generalization to Descendent, relative, etc. invariants of an arbitrary smooth projective 3-fold \( X \).
**Known cases:**

- $X =$ canonical bundle of a smooth toric surface ([ORV] + [MNOP] + Liu-Liu-Zhou)

- $X =$ any rank 2 bundle over a smooth curve (Bryan-Pandharipande + [OP])
In this talk, we will explore the case $X = \text{toric CY}$, for which:

- $GW = DT$ is “almost proven”,
- we can compute $Z_{DT}$ by localization as a sum over torus-fixed points in $\text{Hilb}(X; \beta, \chi)$
How to visualize the $T$-fixed points $\text{Hilb}(X; \beta, \chi)^T$?
Warm-up: \( \text{Hilb}(\mathbb{C}^2; d, n) \)

By definition, it is formed by ideals \( I \subset \mathbb{C}[x, y] \) such that

\[
\text{codim } I_{\leq k} = dk + n. \quad k \gg 0
\]

The torus \( (\mathbb{C}^*)^2 \) acts on \( \text{Hilb}(\mathbb{C}^2; d, n) \) by rescaling \( x \) and \( y \).
Monomials \( x^i y^j \) are eigenvectors of the torus action with distinct eigenvalues.
Torus-fixed ideals \( I \) are spanned by monomials.
Example:

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$\chi$ = renormalized area ( = 9 here).

$d$ = total width of infinite rows and columns ( = 2 here)
For \( \text{Hilb}(\mathbb{C}^3; d, \chi) \), torus-fixed points correspond to 3D partitions, with possibly infinite legs along the coordinate axes:

Here \( d \) is the total cross-section of the infinite legs and \( \chi \) is the renormalized volume.
Now for a picture of a torus-fixed point in $\text{Hilb}(X; \beta, \chi)$ assemble 3D partitions according to toric combinatorics of $X$.

Here is an example for $X = (\mathbb{P}^1)^3$. 

\begin{center}
\end{center}
Here is a CY example: \( X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \)

\[ \beta = 2[\mathbb{P}^1] \]
What about the localization weight?
What about the localization weight?

The CY condition implies it always equals $\pm 1$, in fact,

$$Z_{DT} = \sum_{\text{fixed points}} (-q)^x (\pm t)^\beta.$$
For example, for $X = \mathbb{C}^3$, $\beta = 0$, we get

$$Z_{DT} = \sum_{\text{3D partitions } \pi} (-q)^{\text{vol}(\pi)}$$

$$= \prod_{n > 0} \frac{1}{(1 - (-q)^n)^n}$$

McMahon
For example, for $X = \mathbb{C}^3$, $\beta = 0$, we get

$$Z_{DT} = \sum_{\text{3D partitions } \pi} (-q)^{\text{vol}(\pi)}$$

$$= \prod_{n > 0} (\gamma)$$

*Excuse me, but this looks like my ensemble!*
And, indeed, it is time for a

Stat Mech 101 interruption
Bottle duality takes 3D partitions to dissolving crystal corners
Gibbs ensemble

In equilibrium, the probability of a given configuration decays exponentially with its energy.
For a model of crystal, we can take

\[ \text{Energy} = -\mu_c \text{ Volume} + \mu_s \text{ Surface} \]

Chemical potential is the energy bill for removing an atom.

When \( \mu_s \gg 0 \) (low temperature), only configuration that minimize surface (= 3D partitions) survive.
The shape of our dissolving crystal is given by the moment polytope of $X$. For example, for local $\mathbb{P}^1$

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$$

the atoms are arranged like this:
The shape of our dissolving crystal polytope of $X$. For example, for local $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ the atoms are arranged like this:
Removing a whole row of atoms changes: degree by 1, surface by $-2$, and volume by $\approx L$, where $L$ is the edge length.

For $\mu_s, L \gg 0$ (large cold crystal), approximate balance occurs when

$$\mu_c L \approx 2 \mu_s .$$
Conclusion:

For $X = \text{local } \mathbb{P}^1$, the DT/GW partition function $Z$ is the large size, low temperature expansion of the Seagram crystal partition function with

$$u = i\mu_c \quad \text{genus} \leftrightarrow \text{chem potential}$$

$$t = -\exp(-L\mu_c + 2\mu_s) \quad \text{degree} \leftrightarrow \text{edge length}$$
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\end{align*}

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Known random partition/matrix models for GW invariants are special/limit cases of this.
Did we lose sight of geometry? ...
Did we lose sight of geometry? ... Geometry reappears as the Limit shape
The connected GW invariants are the coefficients in the expansion of $\ln Z$ in powers of $u$.

The everyday life limit

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**Thermodynamic limit**
Law of large numbers

A nonrandom macroscopic limit shape forms *

*Rigorously: as the parameter $\mu_c$ goes to 0, our random surface (=measure on Lipschitz surfaces), scaled by $\mu_c$ in all directions, converges weakly to the $\delta$-measure on a single surface — the limit shape
Here is a computer simulation of the limit shape formation near a corner.
For our random surfaces (and all dimer models in general), the limit shapes can be computed exactly.
For local $\mathbb{P}^1$, we get the Ronkin function $R$ of

$$P(z, w) = 1 + z + w + tz w$$

$$R(x, y) = \frac{1}{(2\pi i)^2} \int\int_{|z|=e^x \atop |w|=e^y} \log P(z, w) \frac{dz}{z} \frac{dw}{w}$$
It may look surprising that the limit shape is an algebraic curve in disguise, but, in fact, this happens for any “polygonal” boundary conditions [KO]
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Here we see a cardioid.

*Theorems about real curves are hidden here ...*
For a general CY toric $X$, the limit shape is the Ronkin function of a plane curve $P$.

This plane curve $P$ it the Hori-Vafa mirror of $X$.
By basic probability theory

genus 0 GW invariants of $X = \lim_{u \to 0} u^2 \ln Z$

$= \text{surface tension of the limit shape}$

\[ \downarrow \]

\[ \text{periods of } P \]
In the setting of instanton counting, the limit shape is the Seiberg-Witten curve. This is how the probabilistic proof of Nekrasov conjecture works.

For pure gauge theory, different, nonprobabilistic proofs given by Nakajima-Yoshioka and Braverman.
All orders in the $u \to 0$ asymptotics of $\ln Z$ (= GW invariants of all genera) should be computable in terms of the limit shape.

This has been worked out for random matrices (Eynard), which is a limit case of our random surfaces.
How to do without a torus action?
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Not a rhetorical question ...