Irreducible symplectic 4-folds which look like the Hilbert square of a $K3$

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A compact Kähler manifold M is *irreducible (holomorphic) symplectic* if:

(1) $\pi_1(X) = \{1\}$, and

(2) $H^{2,0}(X) = \mathbb{C}\sigma$ with σ a holomorphic symplectic form.

$dim = 2$: same as a $K3$ surface.

Examples: $S \subset \mathbb{P}^3$ a smooth quartic, $S = T/\langle -1 \rangle$ with T a 2-dim'l torus.

If dim > 2 the general theory developed mainly by Beauville ($\sim$ 1980) and Huybrechts ($\sim$ 2000) is very much like that of $K3$ surfaces; we think of higher-dimensional irreducible symplectic manifolds as higher dimensional $K3$'s.

Remark: As shown by Beauville any irreducible symplectic manifold can be deformed to a projective one.

Hilbert schemes:

S a $K3$ surface.

$$Hilb^n(S) = S^{[n]} := \{Z \subset S|\ \ell(\mathcal{O}_S/I_Z) = n\}. \tag{1}$$

Let σ_S be a non-zero 2-form on S. If

$$[\{p_1, \ldots, p_n\}] \in S^{[n]}, \quad p_i \neq p_j \text{ for } i \neq j \tag{2}$$

then

$$\Theta_{[Z]}S^{[n]} = \Theta_{p_1}S \oplus \cdots \oplus \Theta_{p_n}S \tag{3}$$

and hence σ_S defines a symplectic form on $\Theta_{[Z]}S^{[n]}$ giving a symplectic holomorphic form in a neighborhood of $[Z]$. One can prove that this form extends to a symplectic holomorphic form on all of $S^{[n]}$.

Theorem 1 (Beauville (Fujiki $n = 2$))**.** $S^{[n]}$ *is an irreducible symplectic manifold of dimension* $2n$*.*

$n = 1$: then $S^{[1]} = S$.

$n = 2$: then $S^{[2]}$ is the blow-up of $S^{(2)}$ along the diagonal.

In general: we have the *cycle map*

$$\begin{array}{ccc} S^{[n]} & \xrightarrow{c} & S^{(n)} \\ Z & \mapsto & \sum_{p \in S} \ell(\mathcal{O}_{p,Z})p \end{array} \tag{4}$$

which is birational with irreducible exceptional divisor

$$\Delta_n := \{[Z] |\ Z \text{ is non-reduced}\}. \tag{5}$$

Families of irreducible symplectic manifolds.

M is an irreducible symplectic manifold.

Theorem 2 (Bogomolov**).** *Deformations of M are unobstructed.*

Remark: There are examples with non-vanishing obstruction space $H^2(\Theta_M)$.

By Bogomolov $Def(M)$ is smooth and

$$\dim Def(M) = h^1(\Theta_M) = h^1(\Omega_M) = b_2(M) - 2. \tag{6}$$

Remark: We use the symplectic form to get an isomorphism $\Theta_M \cong \Omega_M$ and then the Hodge decomposition and $h^{2,0}(M) = 1$ to get the last equality.

Examples: $M = S^{[n]}$ with $S = K3$.

$n = 1$ i.e. $M = S$: by Noether's equality $b_2(M) = 22$ and hence $\dim Def(M) = 20$.

$n \geq 2$: by examining the cycle map (4) we get that $b_2(M) = b_2(K3) + 1 = 23$ and hence

$$\dim Def(S^{[n]}) = 21, \quad n \geq 2. \tag{7}$$

Thus the generic deformation of $S^{[n]}$ is not of the form $(K3)^{[n]}$: there is more to $S^{[n]}$ than $K3$'s.

Assume D is a divisor on M with $c_1(D) \neq 0$ (e.g. D effective). Let $Def(M,D) \subset Def(M)$ be "deformations that keep $c_1(D)$ of type $(1,1)$". Then $Def(M,D)$ is smooth,

$$\dim Def(M,D) = \dim Def(M) - 1 = b_2(M) - 3. \tag{8}$$

Problem Assume $D = H$ is ample: can we describe <u>explicitely</u> all varieties parametrized by $Def(M,H)$? (Here we are thinking also of deformations "in the large".)

$\dim M = 2$ i.e. M a $K3$: No in general but yes if $H \cdot H$ is small.

$H \cdot H = 2$: then $S \to \mathbb{P}^2$ double cover branched over a sextic.

$H \cdot H = 4$: then $S \hookrightarrow \mathbb{P}^3$ a smooth quartic or a "degenerate case".

etc.

What if $\dim M > 2$?

Beauville-Donagi: Let $Z \subset \mathbb{P}^5$ be a smooth cubic hypersurface and $F(Z)$ be the set of lines $\ell \subset Z$. Then $F(Z)$ is an irreducible symplectic manifold deformation equivalent to $(K3)^{[2]}$. (Why? If $sing Z_0 = \{p\}$ and Z_0 has an ordinary double point at p the set of lines $\ell \subset Z_0$ containing p is a $K3$ surface S_p; when $Z \to Z_0$ then $F(Z) \to S_p^{[2]}$.) We have the Plücker embedding

$$F(Z) \subset \mathbf{Gr}(1, \mathbb{P}^5) \hookrightarrow \mathbb{P}^{14} \qquad (9)$$

and hence the Plücker ample divisor class H on $F(Z)$. Varying Z we get all of $Def(F(Z), H)$.

Moduli of sheaves

S a projective $K3$ surface or an abelian surface, with choice of ample divisor D.

$M(r, c_1, s)$ is the moduli space of coherent pure D-semistable sheaves F on S with

$$rk(F) = r, \quad c_1(F) = c_1, \tag{10}$$

and

$$\chi(F) = \begin{cases} r + s & \text{if } S \text{ is a } K3, \\ s & \text{if } S \text{ is an abelian surface.} \end{cases} \tag{11}$$

If S is an abelian surface we have

$$\begin{array}{ccc} M(r, c_1, s) & \xrightarrow{\Phi} & S \times Pic^{c_1}(S) \\ [F] & \mapsto & (\Sigma(c), [\det F]) \end{array} \tag{12}$$

where c is the cycle map (4) and Σ is the "summation map"; Φ is a locally trivial fibration (except in pathological cases). Let

$$M(r, c_1, s)^0 := \Phi^{-1}(a, [L]). \tag{13}$$

Mukai: $M^{st}(r, c_1, s)$ and $M^{st}(r, c_1, s)^0$ are smooth,

$$\begin{aligned} \dim M^{st}(r, c_1, s) &= c_1^2 - 2rs + 2, & (14) \\ M^{st}(r, c_1, s)^0 &= c_1^2 - 2rs - 2. & (15) \end{aligned}$$

and they inherit from S a holomorphic symplectic form.

Mukai, Huybrechts-Göttsche, O'G, Yoshioka:

Suppose $M^{st}(r, c_1, s) = M(r, c_1, s)$.

(a) If S is a $K3$ then $M(r, c_1, s)$ is irreducible symplectic, a deformation of $(K3)^{[n]}$ in general <u>not</u> birational to $(K3)^{[n]}$.

(b) If S is an abelian surface then $M(r, c_1, s)^0$ is irreducible symplectic, a deformation of a generalized Kummer, $b_2(M(r, c_1, s)^0) = 7$.

Suppose $\dim M \geq 4$ ($\dim M^0 \geq 4$ if S ab. surf.). Then $NS(M)$ (respectively $NS(M^0)$) has rank at least 2; thus we do <u>not</u> get all of $Def(M, H)$ (respectively $Def(M^0, H)$) by varying (S, D).

Suppose that $M^{st} \neq M$ and $\dim M = 10$(and a technical genericity assumption on D). Let $S = K3$: a suitable desingularization $\widetilde{M}$ of M gives a *new deformation class* in dim $= 10$ with $b_2(\widetilde{M}) \geq 24$ (O'G). Let S be an abelian surface: a suitable desingularization $\widetilde{M}^0$ of M^0 gives a *new deformation class* in dim $=$ 6 with $b_2(\widetilde{M}) = 8$ (O'G). This construction can be carried out only in these dimensions (Kiem, Kaledin-Lehn-Sorger, Namikawa).

Deformation classes

dim = 2: Kodaira ($\sim$ 1960) proved that any two $K3$ surfaces are deformation equivalent.

Any dimension: Few deformation classes?

Let dim $M = 2n$. Topological restrictions:

(1) *Verbitsky:* Cup-product defines an injection

$$Sym^i H^2(M) \hookrightarrow H^{2i}(M), \quad i \leq n. \tag{16}$$

(2) *S. Salamon:* A non-trivial linear relation between $1 = b_0, b_2, \ldots, b_{2n}$.

Explicitely:

$$b_2 = 22, \quad n = 2. \tag{17}$$

$$b_4 = 46 + 10b_2 - b_3, \quad n = 2. \tag{18}$$

Exercise: Let $\dim M = 4$. Using (16)-(18) show that $b_2(M) \leq 23$ and that if $b_2(M) = 23$ then

$$b_3(M) = 0, \qquad Sym^2 H^2(M;\mathbb{Q}) \cong H^4(M;\mathbb{Q}). \tag{19}$$

Notice: $b_2((K3)^{[2]}) = 23$.

Idea: imitate Kodaira's proof in dim = 4 (thank *Claire* for this approach).

Need to fix some discreet invariants. An irreducible symplectic 4-fold M is a *numerical* $(K3)^{[2]}$ if for S a $K3$ there exists an isomorphism of abelian groups

$$\psi\colon H^2(M;\mathbb{Z}) \xrightarrow{\sim} H^2(S^{[2]};\mathbb{Z}) \tag{20}$$

such that

$$\int_M \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4, \quad \alpha \in H^2(M;\mathbb{Z}). \tag{21}$$

Project: classify numerical $(K3)^{[2]}$'s up to deformation of complex structure (and determine the degree of period map).

We deform M to X with $H^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}h$ with

$$\int_X h^4 = 12, \tag{22}$$

i.e. $(h,h) = 2$. Then $\pm h$ is ample by *Huybrecht's Projectivity criterion*, so h ample. We may assume that

$$h \wedge h \in H^4(X;\mathbb{Z})/Tors \text{ is indivisible.} \tag{23}$$

Furthermore we may assume that the Hodge structure on $H^\bullet(X)$ is generic among those subject to (22)-(23). Let H be a divisor with $h = c_1(H)$. One has $h^0(\mathcal{O}_X(H)) = 6$ and hence a rational map

$$X \dashrightarrow |H|^\vee \cong \mathbb{P}^5. \tag{24}$$

Theorem 3. *Let X, H be as above. One of the following holds:*

(a) There exist an anti-symplectic involution $\phi\colon X \to X$ with quotient map $f\colon X \to Y$ and an inclusion $j\colon Y \hookrightarrow |H|^\vee$ such that $j \circ f$ is Map (24).

(b) Map (24) is birational onto Y with $6 \leq \deg Y \leq 12$. *(We can exclude* $\deg Y \leq 8$.*)*

Conjecture 4. *Case (b) never occurs. Evidence: if X, H satisfies (a) any small deformation in $Def(X, H)$ satisfies (a).*

Sketch of proof?

Problem: How do we describe the X in Case (a)? (Thank *Adrian*.)

Let $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \eta$ be the decomposition into eigen-spaces. Look for a "symmetric" resolution of $j_*\eta$.

It turns out that the symmetric resolution was written down by Eisenbud-Popescu-Walter (without realizing the connection with irreducible symplectic 4-folds).

EPW sextics: Let V be a 6-dimensional vector space. Wedge product defines a symplectic form on $\wedge^3 V$ (we trivialize $\wedge^6 V$); thus $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ is a symplectic vector-bundle of rank 20. Let F be the sub-vector-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ with fiber over $\ell \in \mathbb{P}(V)$ equal to

$$F_\ell := Im\left(\ell \otimes \wedge^2(V/\ell) \hookrightarrow \wedge^3 V\right). \qquad (25)$$

Then F is a Lagrangian sub-bundle of $\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$.

For $A \in \mathbb{LG}(\wedge^3 V)$ we let

$$\lambda_A \colon F \longrightarrow (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \qquad (26)$$

be the obvious map. Let $Y_A \subset \mathbb{P}(V)$ be

$$Y_A := div(\det(\lambda_A)). \qquad (27)$$

If $Y_A \neq \mathbb{P}(V)$ then Y_A is a sextic: this is an *EPW-sextic*.

Theorem 5. *Let* (X, H) *be as in (a) of Theorem (3). Then* $Y = X/\langle\phi\rangle$ *is a (generic) EPW-sextic. Conversely if* Y *is a generic EPW-sextic and* $f \colon X \to Y$ *is the natural double cover then* X *is a deformation of* $(K3)^{[2]}$ *and letting* $H := f^*\mathcal{O}_Y(1)$ *the couple* (X, H) *satisfies (a) of Theorem (3).*

Remark: The parameter space for generic EPW-sextics is irreducible; thus if Conjecture (4) holds any numerical $(K3)^{[2]}$ is a deformation of $(K3)^{[2]}$.