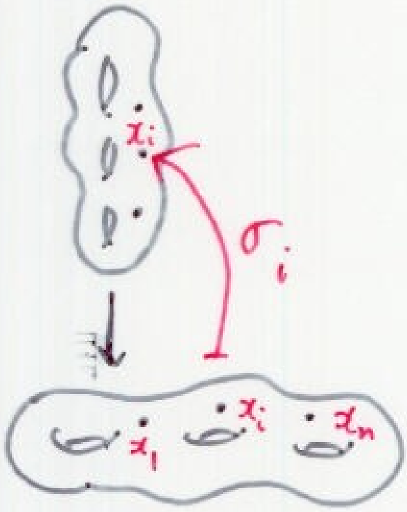
Witten–Kontsevich Theory Revisited

a survey of Recent Developments

- Notations.

$$
\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi"] \\
\overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "\sigma_i"']
\end{tikzcd}
\qquad \omega_\pi \to \overline{\mathcal{M}}_{g,n+1}
$$

$$\overline{\mathcal{M}}_{g,n} \ni [C, (x_1, \cdots, x_n)] =$$



$\overline{\mathcal{M}}_{g,n}$ = moduli of stable curves of genus g with n marked pts

ω_π = the relative dualizing sheaf

σ_i = canonical section $[C, (x_1, \cdots, x_n)] \mapsto x_i$

D_i = image of σ_i , divisor on $\overline{\mathcal{M}}_{g,n+1}$

Then

$$\psi_i = c_1(\mathcal{L}_i) = c_1(\sigma_i^* \omega_\pi) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

cotangent classes

$$\lambda_i = c_i(\pi_* \omega_\pi) \qquad \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

Hodge classes

$$K_i = \pi_*\left(\left[c_1\left(\omega_\pi\left(\sum_{i=1}^{n} D_i\right)\right)\right]^{i+1}\right) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

Question: How to compute

$$\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \quad ?$$

$$d_1 + \cdots + d_n = d = 3g - 3 + n.$$

Three different paths

Kontsevich Matrix Integral

$$\int_{\mathcal{H}_{N\times N}} e^{-\frac{1}{2}\operatorname{tr}(X^2\Lambda)} e^{\frac{i}{2}\cdot\frac{\operatorname{tr} X^3}{3}} \, dX$$

$\mathcal{H}_{N\times N} = \mathbb{R}^{N^2}$
$N\times N$ Hermit.
dX = Lebesgu
$\Lambda > 0$, diag

$$\downarrow \text{KdV} \mid \text{Virasoro}$$

$$\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle$$

Okounkov
Pandharipande
Kazarian
Lando

KdV — Special case (limit)

Virasoro — Special case

Mirzakhani
McShane formul

$$\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_1^{a_1} \cdots \lambda_g^{a_g} \rangle$$

Certain Hodge integrals are easier … !
ELSV formula

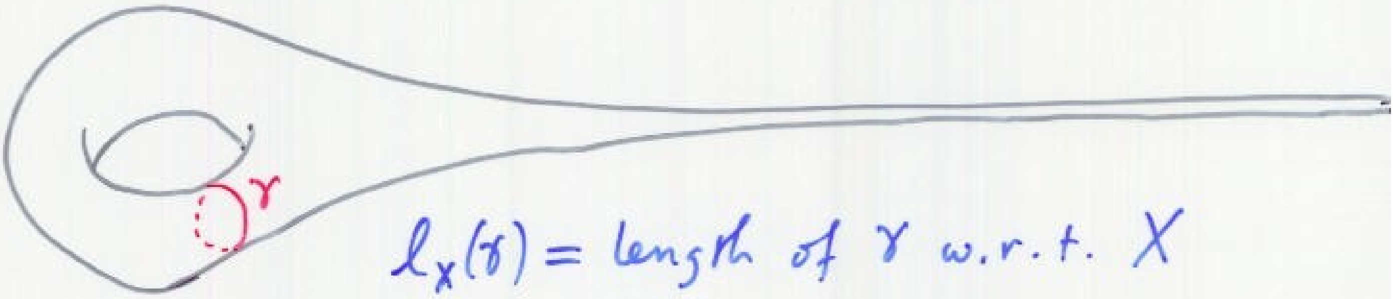
$$\langle \kappa_1^{d_0} \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle$$

Recursion formula for all these numbers!

Mirzakhani

- $\mathcal{M}_{1,1}$ = moduli of hyperbolic surfaces of genus 1 with one cusp

$= \{$  $\}$

$X =$ 

$\ell_X(\gamma)$ = length of γ w.r.t. X

Question. 1. Choose $X \in \mathcal{M}_{1,1}$. What are the lengths of Simple Closed Geodesics of X?

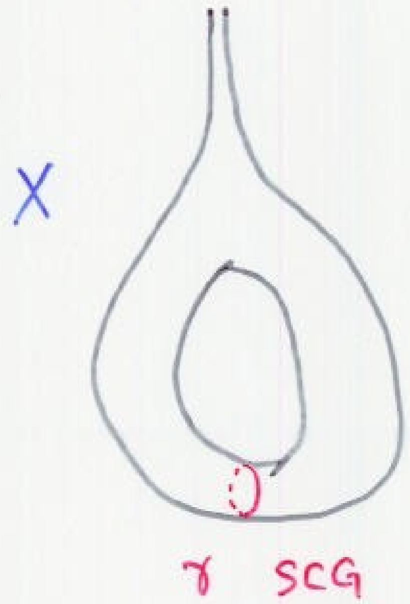
2. Are these lengths constrained?

$\ell_X(\gamma)$ = "period" ?

Answer to Question 2 (McShane 1998)

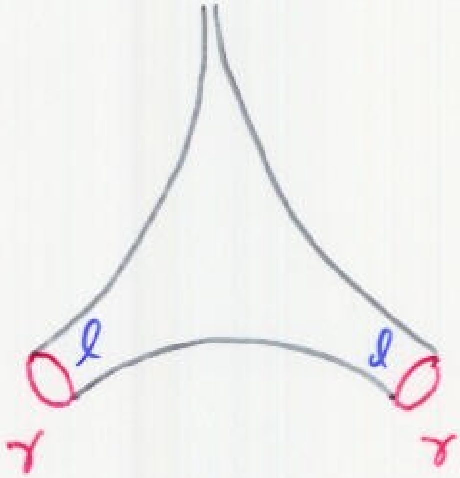
$$\boxed{2 \sum_{\gamma \text{ SCG on } X} \frac{1}{1 + e^{\ell_X(\gamma)}} = 1}$$

What is the use of this formula?



Cut X along a SCG γ

$\longmapsto$



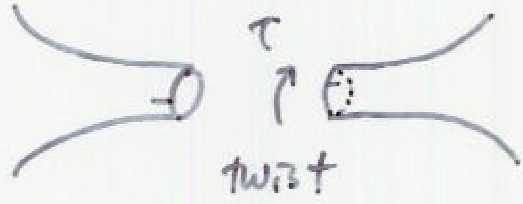
Pair of pants

$\ell = \ell_X(\gamma)$

No moduli once ℓ is chosen.

The moduli information of X is in $(\ell, \tau) \in \mathbb{R}_+ \times \mathbb{R}$

Fenchel - Nielsen Coord.



$\tau =$ twist parameter

$\mathcal{T}_{1,1} = \mathbb{R}_+ \times \mathbb{R} = \{(\ell, \tau)\}$

$\Gamma_{1,1} =$ mapping class group (Madsen's talk)

$\mathcal{M}_{1,1} = \mathcal{T}_{1,1} / \Gamma_{1,1}$ orbifold

$$\begin{array}{c} \mathcal{T}_{1,1} \\ \downarrow \\ \mathcal{M}_{1,1} \end{array} \quad \text{universal cover}$$

Let $\widetilde{\mathcal{M}}_{1,1} = \{(X, \gamma) \mid X \in \mathcal{M}_{1,1},\ \gamma \text{ SCG on } X\}$

$$= \{(\ell, \tau) \mid \ell \in \mathbb{R}_+,\ 0 \leq \tau \leq \ell\}$$

$$\begin{tikzcd}
\mathcal{T}_{1,1} = \mathbb{R}_+ \times \mathbb{R} \arrow[d] \arrow[dr] & \\
 & \widetilde{\mathcal{M}}_{1,1} = \mathbb{R}_+ \times S^1 \arrow[dl] \\
\mathcal{M}_{1,1} &
\end{tikzcd}$$

<u>Fact (True for $\mathcal{M}_{g,n}$)</u> W. Goldman

- $(\mathcal{T}_{1,1},\ \omega_{WP} = d\ell \wedge d\tau)$ symplectic manifold,
  $\omega_{WP} = d\ell \wedge d\tau$ invariant under $T_{1,1}$-action
  Weil-Petersson symplectic form

- Thus $\mathcal{M}_{1,1}$ and $\widetilde{\mathcal{M}}_{1,1}$ inherit a symplectic form, and we can talk about the Weil-Petersson volume of $\mathcal{M}_{1,1}$.

$$\mathrm{vol}_{WP}(\mathcal{M}_{1,1}) = \int_{\mathcal{M}_{1,1}} \omega_{WP}$$

$$= \int_{\mathcal{M}_{1,1}} 1 \cdot dl \wedge d\tau$$

$$= \int_{\mathcal{M}_{1,1}} 2 \sum_{\gamma \ \mathrm{SCG}} \frac{1}{1 + e^{l_x(\gamma)}} \, dl \wedge d\tau$$

$$= 2 \int_{\tilde{\mathcal{M}}_{1,1}} \frac{1}{1 + e^{l}} \, dl \, d\tau$$

$$= 2 \int_0^\infty \left[\frac{1}{1 + e^{l}} \int_0^{l} d\tau \right] dl$$

$$= 2 \int_0^\infty \frac{l}{1 + e^{l}} \, dl$$

$$= \zeta(2)$$

$$= \frac{\pi^2}{6}.$$

$$\mathrm{vol}_{WP}(\mathcal{M}_{1,1}) = \frac{\pi^2}{6}.$$

$\mathcal{M}_{1,1}(L) = \Big\{$  $\Big\}$

the moduli space of bordered hyperbolic surface of genus one with one SCG boundary of length $L \geq 0$.

$\Big\{$  $\Big\} = \widetilde{\mathcal{M}}_{1,1}(L)$

As before :

$$
\begin{aligned}
\mathcal{T}_{1,1}(L) &= \mathbb{R}_+ \times \mathbb{R} = \{(\ell, \tau)\} \\
&\downarrow \\
\widetilde{\mathcal{M}}_{1,1}(L) &= \mathbb{R}_+ \times S^1 \\
&\downarrow \\
\mathcal{M}_{1,1}(L) &= \mathcal{T}_{1,1}(L) \big/ \Gamma_{1,1}(L)
\end{aligned}
$$

MCG fixing the boundary pointwise
(Madsen's talk)

$(\mathcal{M}_{1,1}(L), \, \omega_{WP} = d\ell \wedge d\tau)$ symplectic orbifold

$$\mathrm{vol}_{WP}(\mathcal{M}_{1,1}(L)) = \int_{\mathcal{M}_{1,1}(L)} \omega_{WP}$$

$$= \frac{1}{L}\int_{\mathcal{M}_{1,1}(L)} L \cdot d\ell \wedge d\tau$$

Generalized McShane Identity (Mirzakhani)

$$\boxed{L = \int_0^L \sum_{\substack{\gamma\subset X\\ SCG}} \left(\frac{1}{1+e^{\ell_X(\gamma)+\frac{z}{2}}}+\frac{1}{1+e^{\ell_X(\gamma)-\frac{z}{2}}}\right)dz}$$

$$\left.\frac{d}{dL}\right|_{L=0} \Rightarrow \text{McShane.}$$

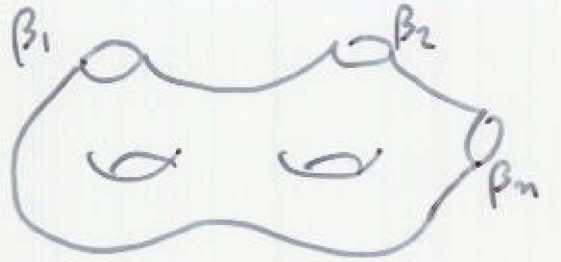
$$= \frac{1}{L}\int_{\mathcal{M}_{1,1}(L)} \int_0^L \sum_{\gamma} \left(\frac{1}{1+e^{\ell_X(\gamma)+\frac{z}{2}}}+\frac{1}{1+e^{\ell_X(\gamma)-\frac{z}{2}}}\right)dz\,d\ell\,d\tau$$

$$= \frac{1}{L}\int_{\tilde{\mathcal{M}}_{1,1}(L)} \int_0^L \left(\frac{1}{1+e^{\ell+\frac{z}{2}}}+\frac{1}{1+e^{\ell-\frac{z}{2}}}\right)dz\,d\ell\,d\tau$$

$$= \frac{1}{L}\int_0^L\int_0^\infty \ell\left(\frac{1}{1+e^{\ell+\frac{z}{2}}}+\frac{1}{1+e^{\ell-\frac{z}{2}}}\right)d\ell\,dz$$

$$= \zeta(2) + \frac{1}{24}L^2$$

$$= \mathrm{vol}_{WP}(\mathcal{M}_{1,1}) + \overset{\langle\tau_1\rangle\,\shortparallel}{\langle\psi_1\rangle_{1,1}} \cdot L^2$$

$S =$ 

oriented topological surface

$\partial S = \beta_1 \sqcup \beta_2 \sqcup \cdots \sqcup \beta_n$ $\quad n$ disjoint circles

$L = (L_1, \cdots, L_n) \in \mathbb{R}_+^n$

$$\mathcal{T}_{g,n}(L) = \left\{ (X, f) \;\middle|\; \begin{array}{l} X \text{ hyperbolic surface genus} = g \\ \partial X = b_1 \sqcup \cdots \sqcup b_n \quad \text{SCGs} \\ f: \begin{array}{ccc} X & \xrightarrow{\text{diffeo}} & S \\ \cup & & \cup \\ b_i & \longrightarrow & \beta_i \end{array}, \quad \ell_X(b_i) = 2\pi L_i \end{array} \right.$$

Identification:

$$\begin{array}{ccc} X & \xrightarrow[\text{isometry}]{\varphi} & X' \\ & f \searrow \quad \swarrow f' & \\ & S & \end{array}$$

(X, f) and (X', f') are identified if

$\exists\, f_t : \underset{\overset{\Psi}{t}}{[0,1]} \times X \longrightarrow S$ isotopic

diffeo for $\forall t \in [0,1]$

$f_0 = f, \quad f_1 = f' \circ \varphi$

$\mathrm{Diff}^+(S, \partial S)$ = orientation-preserving diffeo of S fixing ∂S point-wise

$\cup$

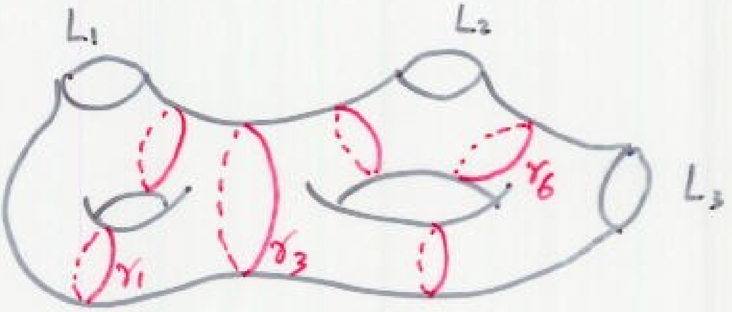
$\mathrm{Diff}^0(S, \partial S)$ = connected component of 1

$\Gamma_{g,n}(\partial) = \mathrm{Diff}^+(S, \partial S) / \mathrm{Diff}^0(S, \partial S)$

$$\mathcal{M}_{g,n}(L) = \mathcal{T}_{g,n}(L) \big/ \Gamma_{g,n}(\partial)$$

differentiable orbifold

Fenchel-Nielsen Coordinates



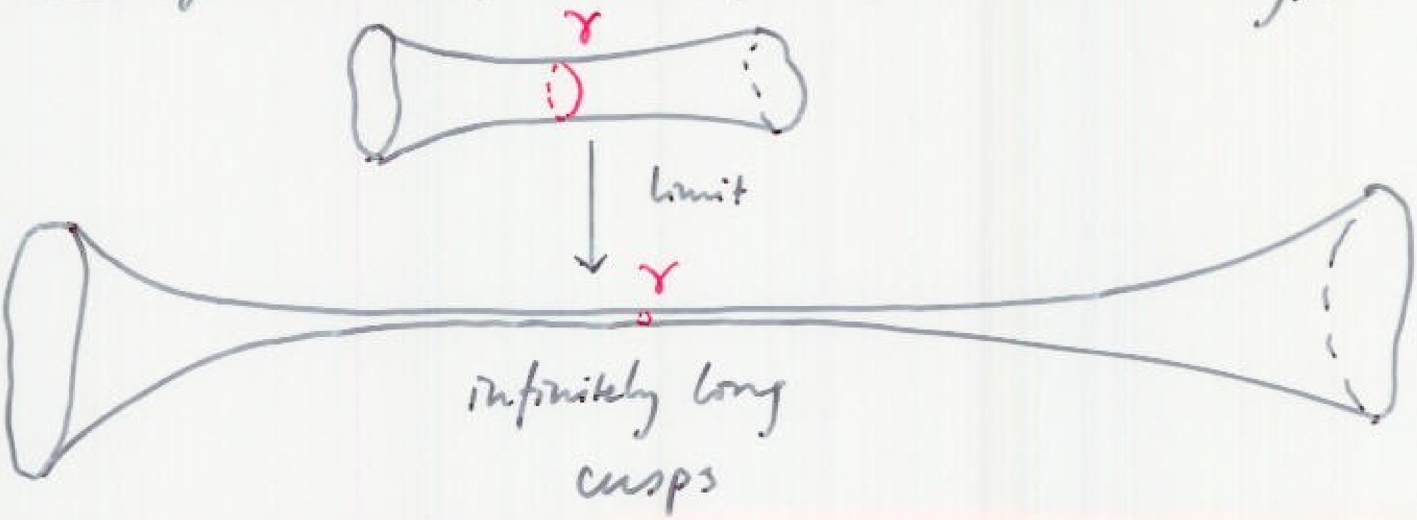
Cut X into $2g-2+n$ pairs of pants along $3g-3+n$ simple closed geodesics.

Each piece has no moduli once length is fixed.

Thus $\ell_\alpha = \ell_X(\gamma_\alpha)$ and τ_α = twist parameter along γ_α

are the moduli parameters.

Deligne-Mumford Compactification $\overline{\mathcal{M}}_{g,n}(L)$



Fenchel - Nielsen Coordinates

$$\mathcal{T}_{g,n}(L) = \mathbb{R}_+^d \times \mathbb{R}^d \qquad d = 3g-3+n$$

<u>Weil - Petersson symplectic form</u>

$$\omega_{WP} = \sum_{\alpha=1}^{d} dl_\alpha \wedge d\tau_\alpha$$

$$= \frac{1}{2} \sum_{\alpha=1}^{d} d(l_\alpha^2) \wedge d\left(\frac{\tau_\alpha}{l_\alpha}\right)$$

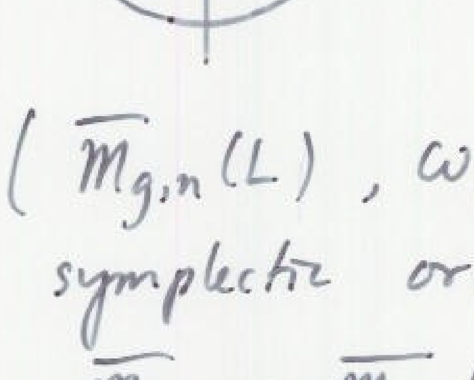
<u>Thm</u> (Goldman)

ω_{WP} is invariant under $\Gamma_{g,n}(\partial)$-action.

<u>Thm</u> (S. Wolpert)

ω_{WP} extends to a regular form on $\overline{\mathcal{M}}_{g,n}(L)$.

$$dl \wedge d\tau = l\, dl \wedge d\left(\frac{\tau}{l}\right) \qquad \tau/l = \theta$$



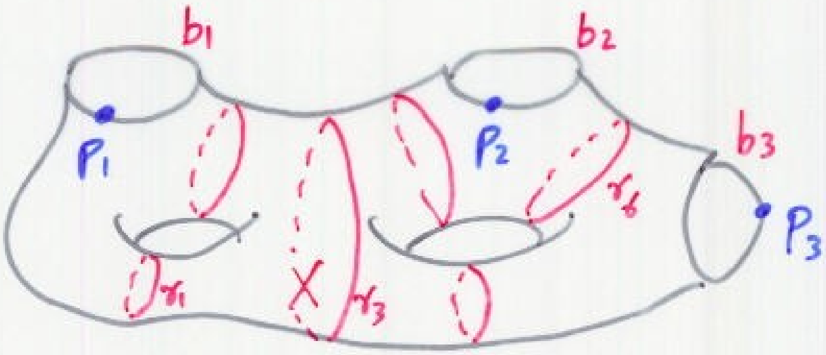
$$dx \wedge dy = r\, dr \wedge d\theta$$

$\therefore \left(\overline{\mathcal{M}}_{g,n}(L), \omega_{WP}\right)$ is a compact symplectic orbifold diffeomorphic to

$\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}(0)$

Symplectic Reduction

$$\widehat{\mathcal{M}}_{g,n} = \left\{ (X, \partial X, (p_1, \dots, p_n)) \;\middle|\; \begin{array}{l} X \in \overline{\mathcal{M}}_{g,n}(L), \; L_i \geq 0 \\ \partial X = b_1 \sqcup \dots \sqcup b_n \quad \text{SCGs} \\ \ell_X(b_i) = 2\pi L_i \\ p_i \in b_i \end{array} \right\}$$



$$\dim_{\mathbb{R}} \widehat{\mathcal{M}}_{g,n} = \underset{\substack{\uparrow \\ X}}{2(3g-3+n)} + \underset{\substack{\uparrow \\ (L_1, \dots, L_n)}}{n} + \underset{\substack{\uparrow \\ (p_1, \dots, p_n)}}{n}$$

$$= \dim_{\mathbb{R}} \overline{\mathcal{M}}_{g,2n} = \text{even!}$$

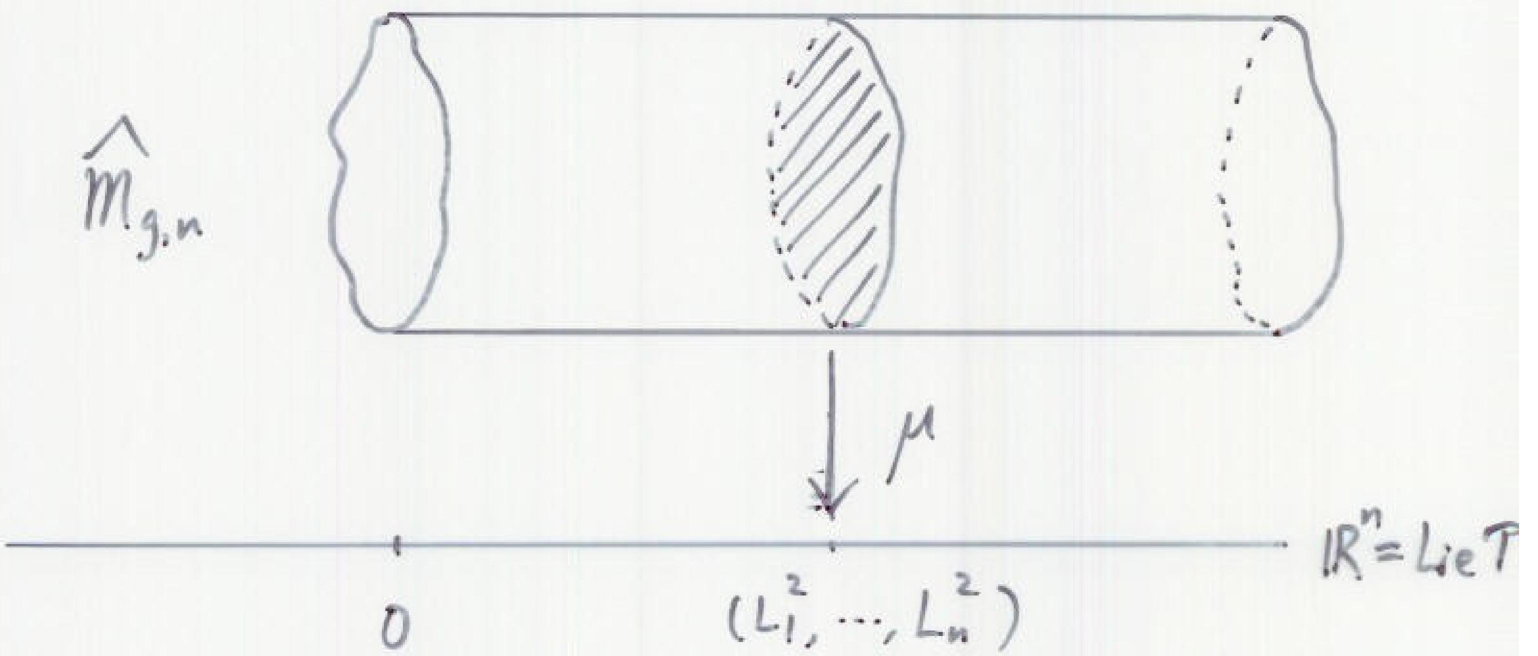
$$\hat{\omega}_{WP} = \sum_{\alpha=1}^{3g-3+n} d\ell_\alpha \wedge d\tau_\alpha + \sum_{i=1}^{n} dL_i \wedge d\underbrace{T_i}_{\text{coord of } p_i}$$

$(\widehat{\mathcal{M}}_{g,n}, \hat{\omega}_{WP})$ symplectic orbifold

T^n-action moves $p_i \in b_i \subset \partial X$

L_i^2 = moment map of the S^1-action

$$\theta_i = \frac{T_i}{L_i} \in S^1$$



$$
\begin{tikzcd}
T^n \curvearrowright \widehat{\mathcal{M}}_{g,n} \arrow[d, "\mu"] \\
\mathbb{R}^n
\end{tikzcd}
$$

moment map

$$\widehat{\omega}_{WP} = \sum_{\alpha=1}^{3g-3+n} dl_\alpha \wedge d\tau_\alpha + \sum_{i=1}^{n} dL_i \wedge dT_i$$

$$\theta_i = \frac{T_i}{L_i} \in S^1$$

$$\mu^{-1}(L_1^2, \cdots, L_n^2)\big/_{T^n} = \left(\overline{\mathcal{M}}_{g,n}(L),\ \omega_{WP}(L) \right)$$

Symplectic Reduction formula

$$\boxed{\omega_{WP}(L) = \omega_{WP}(0) + \sum_{i=1}^{n} 4\pi^2 L_i^2 dA}$$

$$\widehat{\mathcal{M}}_{g,n}/T^n \underset{\text{diffeo}}{\cong} \overline{\mathcal{M}}_{g,n} \times \mathbb{R}^n_{\geq 0}.$$

$$
\begin{tikzcd}
T^n \curvearrowright \mu^{-1}(L^2) \arrow[d] \\
\overline{\mathcal{M}}_{g,n}(L)
\end{tikzcd}
$$

$-A_i$ = connection on the S^1-bundle

$$[dA_i] = + c_1(\mathcal{L}_i) = + \psi_i$$

$$\omega_{WP}(L) = \omega_{WP}(0) + \sum 4\pi^2 L_i^2 \, dA_i$$

on $\widehat{\mathcal{M}}_{g,n}/_{T^n}$ as a form.

$$\begin{array}{c} \widehat{\mathcal{M}}_{g,n}/_{T^n} \\ \text{\rotatebox{90}{$\cong$}} \\ \overline{\mathcal{M}}_{g,n} \times \mathbb{R}_{\geq 0}^n \end{array}$$

As a class, $[\omega_{WP}(L)] = 2\pi^2 \kappa_1 + 4\pi^2 \sum_{i=1}^n L_i^2 \psi_i$.

$$\therefore \quad V_{g,n}(L) \underset{\text{def}}{=} \int_{\overline{\mathcal{M}}_{g,n}(L)} \frac{1}{d!} \left(\frac{\omega_{WP}(L)}{2\pi^2} \right)^d$$

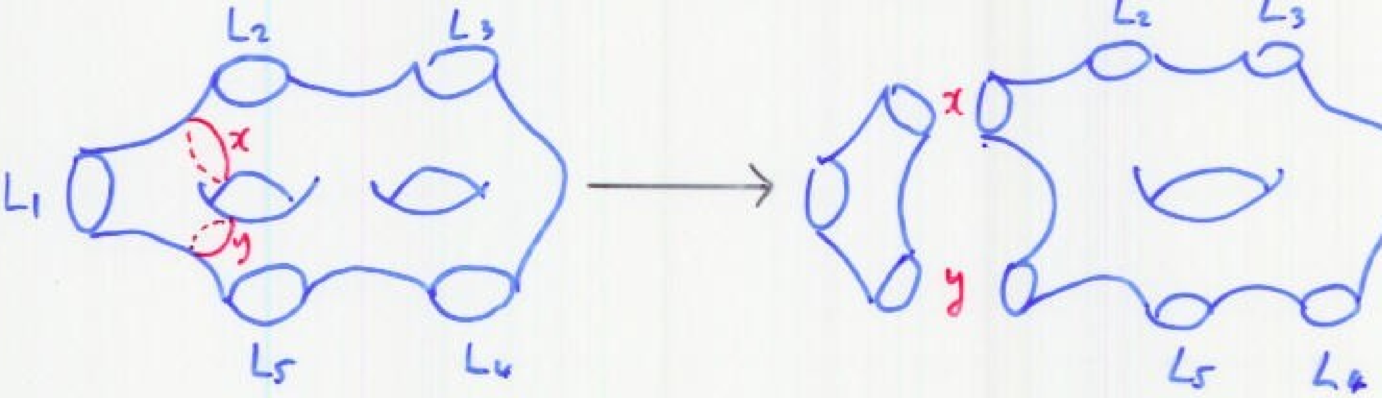
$$= \sum_{d_0 + d_1 + \cdots + d_n = d} \frac{2^{d_1 + \cdots + d_n}}{d_0! \, d_1! \cdots d_n!} \langle \kappa_1^{d_0} \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle L_1^{2d_1} L_2^{2d_2} \cdots L_n^{2d_n}$$

$$\in \mathbb{Q}[L_1, L_2, \cdots, L_n]^{S_n}$$

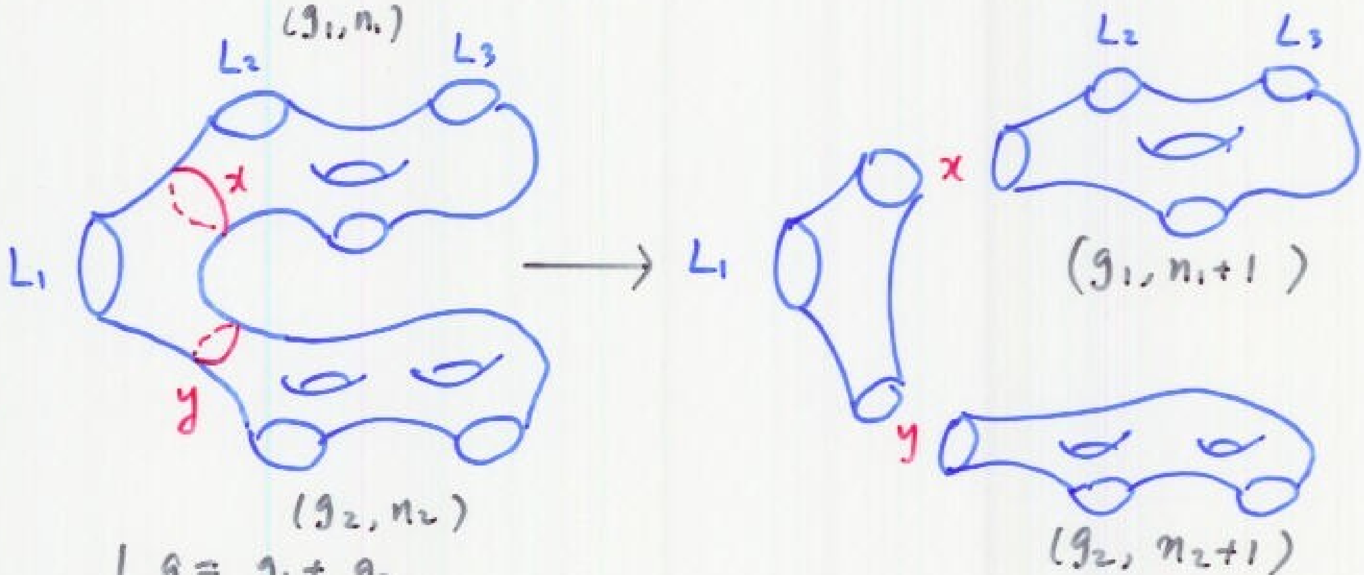
- The generalized McShane identity gives a recursion relation for $V_{g,n}(L)$'s :

Three ways of cutting off a pair of pants:

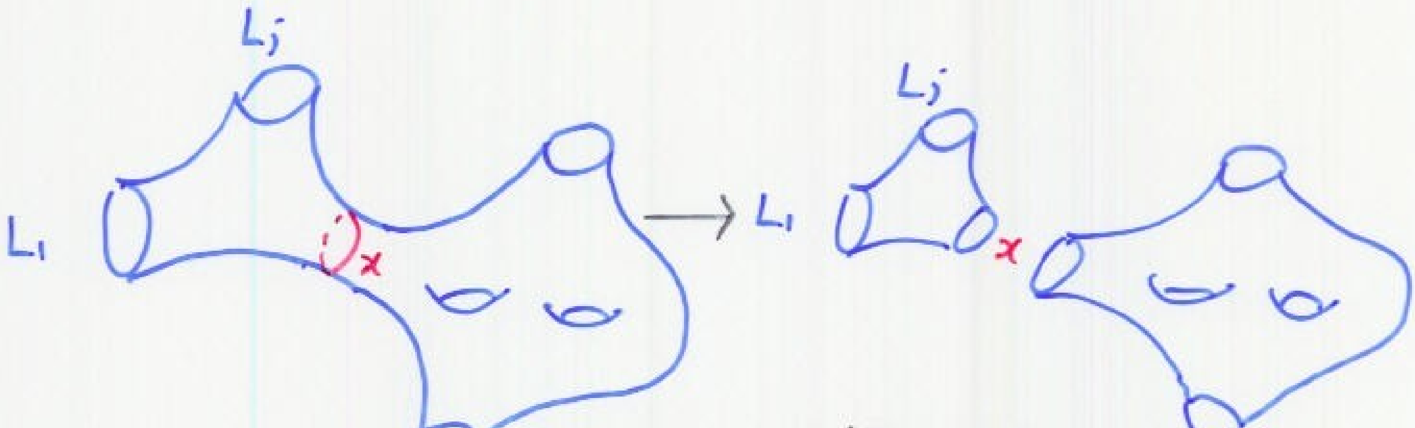
(c.f. C.C.M. Liu's talk)



$$(g, n) \longmapsto (g-1, n+1)$$



$$\begin{cases} g = g_1 + g_2 \\ n = n_1 + n_2 + 1 \end{cases}$$



$$(g, n) \longmapsto (g, n-1)$$

$$K(x,y) \underset{\mathrm{def}}{=} \frac{1}{1+e^{\pi(x+y)}} + \frac{1}{1+e^{\pi(x-y)}}$$

$$v_{g,n}(L)$$

$$= \frac{4}{L_1} \int_0^{L_1} dz \int_0^{\infty} dx \int_0^{\infty} dy \; xy\, K(x+y,z)\, v_{g-1,n+1}(x,y,L_2,\cdots,L_n)$$

$$+ \frac{4}{L_1} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n-1}} \int_0^{L} dz \int_0^{\infty} dx \int_0^{\infty} dy \; xy\, K(x+y,z)\, v_{g_1,n_1+1}(x,L^{(n_1)})\, v_{g_2,n_2+1}(y,L^{(n_2)})$$

$$+ \frac{1}{L_1} \sum_{j=2}^{n} \int_0^{L} dz \int_0^{\infty} dx \; x\left[K(x,z+L_j) + K(x,z-L_j)\right] \times$$

$$v_{g,n-1}(x,L_2,\cdots,\widehat{L_j},\cdots,L_n) .$$

$$\begin{cases} v_{0,3}(L) = 1 \\ v_{1,1}(L) = \frac{1}{12}(1+L^2) \end{cases}$$ initial data.

$v_{g,n}(L)$ completely determined by the recursion formula.

Remarks.

1. The Mirzakhani recursion restricted to the top degree terms produces an algebraic relation among the ψ-class intersections.

   This relation is exactly the Virasoro Constraint

2. Mondello recently established that the combinatorial cycles (Witten classes) on $\mathcal{M}_{g,n}$ are polynomials of κ_i-classes.

   This implies that the generating function of the Mirzakhani intersections (volumes) is a generalized Kontsevich integral

$$\int_{\mathcal{H}_{N\times N}} e^{-\frac{1}{2}\operatorname{tr}(X^2\Lambda)}\, e^{\frac{i}{2}\sum_{j\geq 1}\frac{s_j}{2j+1}\operatorname{tr} X^{2j+1}}\, dX$$

   for a particular choice of the coupling constants $s_1, s_2, \cdots$.

   This gives a different solution to the KdV equations. (Actual formula being finalized as of now.)

Purely Algebraic Proof of Witten – Kontsevich

Kazarian and Lando, based on the works of

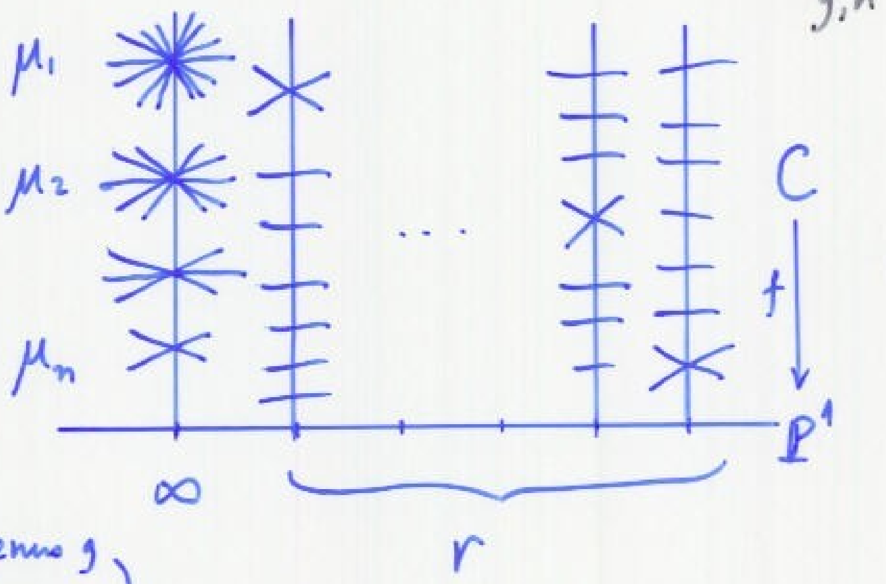
Okounkov, Okounkov – Pandharipande, etc...

Starting point : ELSV formula

# of Hurwitz covers of $\mathbb{P}^1$

= Intersection of ψ and λ classes on $\overline{\mathcal{M}}_{g,n}$.

$$h_{g,\mu\vdash d} = \frac{(d+n+2g-2)!}{|\mathrm{Aut}(\mu_1,\cdots,\mu_n)|}\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{1-\lambda_1+\lambda_2-\cdots+(-1)^g\lambda_g}{(1-\mu_1\psi_1)\cdots(1-\mu_n\psi_n)}$$



r simple ramification

$(\mu_1, \mu_2, \cdots, \mu_n) =$ profile of ramification at ∞

genus g

$f : C \longrightarrow \mathbb{P}^1$ ramified covering of degree d

$d = \mu_1 + \mu_2 + \cdots + \mu_n$ partition of d of length n

$r = d + n + 2g - 2$ Riemann - Hurwitz.

Okounkov generating function

$$H(\beta; t_1, t_2, t_3, \cdots)$$

$$= \sum_{g \geq 0} \sum_{n \geq 0} \sum_{\mu_1, \mu_2, \cdots, \mu_n > 0} h_{g;\mu_1, \cdots, \mu_n} \frac{\beta^{d+n+2g-2}}{(d+n+2g-2)!} t_{\mu_1} t_{\mu_2} \cdots t_{\mu_n}$$

$$d = \mu_1 + \cdots + \mu_n$$

Okounkov:

<u>Thm</u>. $\tau = e^H$ satisfies the KP equations as $t_1, t_2, \cdots$ being KP-time variables.

<u>KP Theory</u> The same as Pandharipande's talk.

$$\Lambda^{\frac{\infty}{2}}(V) \xrightarrow[\text{linear isom.}]{\sim} \mathbb{C}[[t_1, t_2, t_3, \cdots]]$$

$V = \mathbb{C}((z^{-1}))$ Bosonization (A kind of supersymmetry) $e^{\sum_{j=1}^{\infty} t_j z^{+j}}$

$$\begin{array}{ccc} \mathbb{P}(\Lambda^{\frac{\infty}{2}}(V)) & \overset{B}{=\!=} & \mathbb{P}(\mathbb{C}[[t_1, t_2, t_3, \cdots]]) \\ \bigcup \text{Plücker} & & \text{KP} \bigcup \text{ in Hirota form} \\ Gr(\infty) & \overset{B}{=\!=} & \{\tau\text{-functions}\} \end{array}$$

Kazarian – Lando's trick.

Throw away λ_i - classes from

$$\prod_{i=1}^{n} \frac{1-\lambda_1+\lambda_2-\cdots\pm\lambda_g}{(1-\mu_1\psi_1)\cdots(1-\mu_n\psi_n)} .$$

Recall that

$$\frac{1}{1-b\psi} = 1 + b\psi + b^2\psi^2 + \cdots .$$

<u>Claim.</u> For every $d \geq 0$, $\exists$ C_{id}, $i=1,2,\cdots,d+1$, such that

$$\sum_{i=1}^{d+1} \frac{C_{id}}{1-i\psi} = \psi^d + \text{higher}.$$

proof.

$$\sum_{i=1}^{d+1} C_{id}\left(1 + i\psi + i^2\psi^2 + i^3\psi^3 + \cdots\right) = \psi^d + \text{higher}$$

$$\Longleftrightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & d+1 \\ 1 & 4 & 9 & \cdots & (d+1)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2^d & 3^d & \cdots & (d+1)^d \end{bmatrix}}_{d+1} \begin{bmatrix} C_{1d} \\ C_{2d} \\ C_{3d} \\ \vdots \\ C_{d+1\,d} \end{bmatrix} = \left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right\} d+1$$

$$\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle = \sum_{\mu_1=1}^{d_1+1} \cdots \sum_{\mu_n=1}^{d_n+1} C_{\mu_1 d_1} \cdots C_{\mu_n d_n} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1-\lambda_1+\lambda_2-\cdots\pm\lambda_g}{(1-\mu_1\psi_1)\cdots(1-\mu_n\psi_n)}$$

There is just no room for λ-classes to come in!

$\therefore \ \langle \tau_{d_1} \cdots \tau_{d_n} \rangle =$ linear combination of Hurwitz numbers

$$\Big\Downarrow \text{ Kazarian–Lando}$$

$F(t_0, t_1, \cdots) = \left\langle e^{\sum_{j=0}^{\infty} t_j \tau_j} \right\rangle$ is expressible in terms of Okounkov's H-function

Using the explicit formula,

KP on $H \implies$ KdV on F.

$\therefore$ F satisfies the KdV!

Kontsevich's original proof is indeed correct.

- One should read the space in between lines.
- Often one needs to erase the lines first, and then let the space in between lines shine.
- Then the true proof comes up.

- The three approaches together give a "clear" picture of the generalized Kontsevich integral.
- Virasoro = Cut and join
- KdV / KP = Graph enumeration

Limitation : $\overline{\mathcal{M}}_{g,n}$ is a nice smooth orbifold. Therefore, coordinatization was possible to do integral analysis on it.

This kind of analysis does not seem to be done on $[\ \ ]^{vir}$ ...

Analysis on $\overline{\mathcal{M}}_{g,n}$ may be useful in other ways.