Witten-Kontsevich Theory Revisited
a survey of Recent Developments

- Notations -

\( \Phi \to \overline{M}_{g,n+1} \xrightarrow{\pi} \overline{M}_{g,n} \)

\[ \exists [C, (x_1, \ldots, x_n)] = \sigma_i \]

\( \overline{M}_{g,n} \) = moduli of stable curves of genus \( g \) with \( n \) marked pts

\( \omega_\pi \) = the relative dualizing sheaf

\( \sigma_i \) = canonical section \([C, (x_1, \ldots, x_n)] \to x_i \)

\( D_i \) = image of \( \sigma_i \), divisor on \( \overline{M}_{g,n+1} \)

Then

\[ \psi_i = c_1(L_i) = c_1(\sigma_i^* \omega_\pi) \in H^2(\overline{M}_{g,n}, \mathbb{Q}) \]

cotangent classes

\[ \lambda_i = c_1(\pi_* \omega_\pi) \in H^{2i}(\overline{M}_{g,n}, \mathbb{Q}) \]

Hodge classes

\[ K_i = \pi_* \left( \left[ c_1(\omega_\pi \sum_{i=1}^{n} D_i) \right]^{i+1} \right) \in H^{2i}(\overline{M}_{g,n}, \mathbb{Q}) \]
Question: How to compute
\[
\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle = \int \frac{\psi_1^{d_1} \cdots \psi_n^{d_n}}{M_{g,n}}
\]
d_1 + \cdots + d_n = d = 3g - 3 + n.

Three different paths.

Kontsevich Matrix Integral
\[
\int \frac{e^{-\frac{1}{2} tr(X^2 \Lambda)}}{N_{2g-2, n}} e^{\frac{1}{3} \int X^3} \ dx
\]

Kähler

\[
\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle
\]

Kähler

\[
\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle
\]

Virasoro

Okounkov

Pandharipande

Kazarian

Lando

special case (limit)

\[
\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle
\]

Certain Hodge integrals are easier ...!

ELSV formula

Kähler

Virasoro

Mirzakhani

Möbius formula

\[
\langle K_1^{d_1} \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle
\]

Recursion formula

for all these numbers!
Mirzakhani

- $M_{1,1}$ = moduli of hyperbolic surfaces of genus 1 with one cusp

$X = \{ \}$

$\ell_X(\gamma) =$ length of $\gamma$ w.r.t. $X$

**Question 1.** Choose $X \in M_{1,1}$. What are the lengths of simple closed geodesics of $X$?

2. Are these lengths constrained?

$\ell_X(\gamma) =$ "period"?

**Answer to Question 2.** (McShane 1998)

$$2 \sum_{\gamma \in \text{SCG on } X} \frac{1}{1 + e^{\ell_X(\gamma)}} = 1$$
What is the use of this formula?

Cut $X$ along a SCG $\gamma$

$l = l_x(\gamma)$

Pair of pants
No moduli once $l$ is chosen.

The moduli information of $X$ is in $(l, \tau) \in \mathbb{R}_+ \times \mathbb{R}$
Fenchel-Nielsen coord.

$T = \text{twist parameter}$

$\mathcal{J}_{0,1} = \mathbb{R}_+ \times \mathbb{R} = \{(l, \tau)\}$

$\Gamma_{1,1} = \text{mapping class group (Madsen's talk)}$

$M_{1,1} = \mathcal{J}_{0,1} / \Gamma_{1,1}$ orbit folded

Universal cover
Let \( \tilde{M}_{1,1} = \left\{ (X, \gamma) \mid X \in M_{1,1}, \gamma \text{ SCG on } X \right\} \)
\[= \left\{ (l, \tau) \mid l \in \mathbb{R}_+, 0 \leq \tau \leq l \right\} \]

\[J_{1,1} = \mathbb{R}_+ \times \mathbb{R} \]

\[\tilde{M}_{1,1} = \mathbb{R}_+ \times S^1 \]

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Fact (True for \( M_{g,n} \))

\[ (J_{1,1}, \omega = dl \wedge d\tau) \text{ symplectic manifold,} \]

\[ \omega_{WP} = dl \wedge d\tau \text{ invariant under } T_{1,1} \text{- action} \]

Weil–Petersson symplectic form

Thus \( M_{1,1} \) and \( \tilde{M}_{1,1} \) inherits a symplectic form, and we can talk about the Weil–Petersson volume of \( M_{1,1} \).
\[ \text{vol}_{WP}(m_{1,1}) = \int_{m_{1,1}} \omega_{WP} \]

\[ = \int_{m_{1,1}} 1 \cdot d\lambda \cdot d\tau \]

\[ = \int_{m_{1,1}} 2 \sum_{\gamma \in SCG} \frac{1}{1 + e^{\lambda(\gamma)}} \cdot d\lambda \cdot d\tau \]

\[ = 2 \int_{m_{1,1}} \frac{1}{1 + e^{\lambda}} \cdot d\lambda \cdot d\tau \]

\[ = 2 \int_{m_{1,1}} \left[ \frac{1}{1 + e^{\lambda}} \int_{0}^{\infty} d\tau \right] d\lambda \]

\[ = 2 \int_{m_{1,1}} \frac{\lambda}{1 + e^{\lambda}} \cdot d\lambda \]

\[ = 5(2) \]

\[ = \frac{\pi^2}{6} \]

\[ \text{vol}_{WP}(m_{1,1}) = \frac{\pi^2}{6} \]
\[ m_{1,1}(L) = \{ \text{boundary of surface with one SCG} \} \]

the moduli space of bordered hyperbolic surface of genus one with one SCG boundary of length \( L > 0 \).

\[ \{ \text{boundary} \} = m_{1,1}^\sim(L) \]

As before:

\[ J_{1,1}(L) = \mathbb{R}_+ \times \mathbb{R} = \{(l, \tau)\} \]

\[ m_{1,1}^\sim(L) = \mathbb{R}_+ \times S^1 \]

\[ m_{1,1}(L) = J_{1,1}(L) / \Gamma_{1,1}(L) \]

\[ (m_{1,1}(L), \omega_{WP} = dl \wedge d\tau) \text{ symplectic} \]

\[ \text{orifold} \]

\[ \text{MCG fixing the boundary pointwise (Madsen's talk)} \]
$$\text{vol}_{WP}(m_{1,1}(L)) = \int m_{1,1}(L)$$

$$= \frac{1}{L} \int \sum_{\gamma \in \mathcal{X}} \left( \frac{1}{1 + e^{\frac{\ell_{x,0}}{L} + \frac{3}{2}}} + \frac{1}{1 + e^{\frac{\ell_{x,0}}{L} - \frac{3}{2}}} \right) dz \, d\tau$$

**Generalized McShane Identity (Mirzakhani)**

$$L = \int_{0}^{L} \sum_{\gamma \in \mathcal{X}} \left( \frac{1}{1 + e^{\frac{\ell_{x,0}}{L} + \frac{3}{2}}} + \frac{1}{1 + e^{\frac{\ell_{x,0}}{L} - \frac{3}{2}}} \right) dz \, d\tau$$

$$\left. \frac{d}{dL} \right|_{L=0} \Rightarrow \text{McShane}.$$
\[ S = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_n \]
\[ \exists S = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_n \]
\[ L = (L_1, \ldots, L_n) \in \mathbb{R}_+^n \]

\[ J_{g,n}(L) = \left\{ (X, f) \middle| \begin{array}{l}
X \text{ hyperbolic surface genus } g \\
\exists X = b_1 \cup \ldots \cup b_n \quad \text{scgs} \\
f : X \overset{\text{diffeo}}{\rightarrow} S \\
b_i \overset{\text{diffeo}}{\rightarrow} \beta_i
\end{array} \right\} \]

**Identification:**

\[ \exists f_t : [0, 1] \times X \rightarrow S \]
\[ \text{diffeo for } \forall t \in [0,1] \]
\[ f_0 = f, \quad f_1 = f' \circ \psi \]

\[ \text{Diff}^+(S, \partial S) = \text{orientation-preserving diffeo of } S \]
\[ \cup \]
\[ \text{Diff}^0(S, \partial S) = \text{connected component of } 1 \]

\[ \Gamma_{g,n}(\partial) = \text{Diff}^+(S, \partial S)/\text{Diff}^0(S, \partial S) \]
$M_{g,n}(L) = \frac{\mathcal{G}_{g,n}(L)}{\Gamma_{g,n}(\mathfrak{g})}$

differentiable orbifold

Fenchel-Nielsen Coordinates

Cut $X$ into $2g-2+n$ pairs of pants along $3g-3+n$ simple closed geodesics.

Each piece has no moduli: once length is fixed.

Thus $l_x = l_x(\gamma_x)$ and $\tau_x = \text{twist parameter along } \gamma_x$

are the moduli parameters.

Deligne-Mumford Compactification $\overline{M}_{g,n}(L)$

limit

infinitely long cusps
Fenchel-Nielsen Coordinates
\[ \Gamma_{g,n}(L) = \mathbb{R}^d \times \mathbb{R}^d \quad d = 3g-3+n \]

Weil-Petersson symplectic form
\[
\omega_{WP} = \sum_{\alpha=1}^{d} d\alpha \wedge d\tau_{\alpha} = \frac{1}{2} \sum_{\alpha=1}^{d} d(l_{\alpha}^2) \wedge d\left(\frac{T_{\alpha}}{L_{\alpha}}\right)
\]

Thm. (Goldman)
\[ \omega_{WP} \text{ is invariant under } \Gamma_{g,n}(\mathbb{R})-\text{action} \]

Thm. (S. Wolpert)
\[ \omega_{WP} \text{ extends to a regular form on } \overline{M}_{g,n}(L) \]

\[ d\tau \wedge d\zeta = d\zeta \wedge d\left(\frac{T}{L} \right) \quad \tau/L = \phi \]

\[ dx \wedge dy = r dr \wedge d\theta \]

\[ : (\overline{M}_{g,n}(L), \omega_{WP}) \text{ is a compact symplectic orbifold diffeomorphic to } \overline{M}_g = \overline{M}_{g,n}(\mathbb{R}) \]
Symplectic Reduction

\[ \wedge M_{g,n} = \left\{ (x, \exists x, (p_1, \ldots, p_n)) \right\} \]  
\[ \exists x = b_1^{l_1} \cdots b_n^{l_n} \text{ s.g.s} \]  
\[ l_x(b_i) = 2\pi l_i \]  
\[ p_i \in b_i \]

\[ \dim_{\mathbb{R}} M_{g,n} = 2(3g-3+n) + n + n \]  
\[ = \dim_{\mathbb{R}} M_{g,2n} = \text{even!} \]

\[ \hat{\omega}_{WP} = \sum_{d=1}^{3g-3+n} \delta_{x} \wedge dt_{d} + \sum_{i=1}^{n} dL_{i} \wedge dt_{i} \]

\[ (\hat{M}_{g,n}, \hat{\omega}_{WP}) \text{ symplectic orbifold} \]

\[ T^n \text{-action moves } p_i \in b_i < \exists x \]

\[ L_i^2 = \text{moment map of the } S^1 \text{-action} \]

\[ \theta_i = \frac{T_i}{L_i} \in S^1 \]
\( M_{g,n} \)

\[ \begin{array}{c}
\mu \\
\downarrow \\
\text{moment map}
\end{array} \]

\[ \mathbb{R}^n = \text{Lie } T \]

\[ (L_1^2, \ldots, L_n^2) \]

\[ \mathbb{T}^n \rightarrow \hat{M}_{g,n} \]

\[ \omega_{WP} = \sum_{\alpha=1}^{3g-3+n} \frac{\partial}{\partial \alpha} + \sum_{i=1}^{n} d\ln dT_i \]

\[ \theta_i = \frac{\dot{T}_i}{L_i} \in S^1 \]

\[ \mu^{-1}(L_1^2, \ldots, L_n^2)/\mathbb{T}^n = (\overline{M}_{g,n}(L), \omega_{WP}(L)) \]

Sympctic reduction formula

\[ \omega_{WP}(L) = \omega_{WP}(0) + \sum_{i=1}^{n} 4\pi^2 L_i^2 dA_i \]

\[ \hat{M}_{g,n}/\mathbb{T}^n \cong \text{diffeo} \]

\[ \overline{M}_{g,n} \times \mathbb{R}_{\geq 0} \]

\[ \mathbb{T}^n \rightarrow \mu^{-1}(L^2) \]

\[ \overline{M}_{g,n}(L) \]

\[ [dA_i] = + c_i(L_i) = + \psi_i \]

\[ \rho_i \in \mathfrak{g}_i \]

- \( A_i \) = connection on the \( S^1 \)-bundle
\[ \omega_{\text{WE}}(L) = \omega_{\text{WE}}(0) + \sum 4\pi^2 L_i^2 \, dA \] 

on \( \tilde{M}_{g,n}/\Gamma_n \) as a form.

\[ 1 \leq \mathbf{L}_{g,n} \times 1 \mathbf{R}_{>0} \]

As a class, \( \left[ \omega_{\text{WE}}(L) \right] = 2\pi^2 K_1 + 4\pi^2 \sum_{i=1}^{n} L_i^2 \psi_i \).

\[ \mathbf{V}_{g,n}(L) = \left[ \frac{1}{d!} \left( \frac{\omega_{\text{WE}}(L)}{2\pi^2} \right)^d \right]_{M_{g,n}(L)} \]

\[ = \sum_{d_0 + d_1 + \cdots + d_n = d} \frac{2^{d_0 + \cdots + d_n}}{d_0! \cdot d_1! \cdots d_n!} \left\langle K_1 \psi_{i_1} \cdots \psi_{i_n} \right\rangle_{L_1, L_2, \cdots, L_n} \]

\[ \in \mathbb{Q}[L_1, L_2, \cdots, L_n] \]

The generalized McShane identity gives a recursion relation for \( \mathbf{V}_{g,n}(L) \)'s:
Three ways of cutting off a pair of pants:

(c.f. C.C.M. Liu's talk)

\[ L_1 \rightarrow L_1 \]

\[ (g, n) \rightarrow (g-1, n+1) \]

\[ (g_1, n_1) \rightarrow (g_1, n_1+1) \]

\[ g = g_1 + g_2 \]

\[ n = n_1 + n_2 + 1 \]

\[ (g, n) \rightarrow (g, n-1) \]
\[ K(x, y) = \frac{1}{1 + e^\pi (x + y)} + \frac{1}{1 + e^\pi (x - y)} \]

\[ \mathcal{V}_g, n(L) = \frac{4}{L_1} \int_0^{L_1} dx \int_0^\infty dy \int_0^\infty dz \ x y K(x+y,z) \mathcal{V}_{g-1, n+1}(x, y, L_2, \ldots, L_n) \]

\[ + \frac{4}{L_1} \sum_{g_1 + g_2 = g} \sum_{m_1 + m_2 = n-1} \int_0^L dx \int_0^\infty dy \int_0^\infty dz \ x y K(x+y,z) \mathcal{V}_{g_1, n+1}(x, L, L_2, \ldots, L_{m_2}) \mathcal{V}_{g_2, m_2+1}(y, L_{m_2}) \]

\[ + \frac{1}{L_1} \sum_{j=2}^{\infty} \int_0^L dx \int_0^\infty dz \ x \left[ K(x, z + L_j) + K(x, z - L_j) \right] \mathcal{V}_{g, n-1}(x, L_2, \ldots, L_j, \ldots, L_n) \]

\[
\begin{cases}
\mathcal{V}_{0,3}(L) = 1 \\
\mathcal{V}_{1,1}(L) = \frac{1}{12} (1 + L^2)
\end{cases}
\]

\[ \mathcal{V}_{g, n}(L) \text{ completely determined by the recursion formula} \]
Remarks:

1. The Mirzakhani recursion restricted to the top degree terms produces an algebraic relation among the $Y$-class intersections. This relation is exactly the Virasoro Constraint.

2. Mondello recently established that the combinatorial cycles (Witten classes) on $\text{M}_{g,n}$ are polynomials of $K_i$-classes. This implies that the generating function of the Mirzakhani intersections (volumes) is a generalized Kontsevich integral

$$\int e^{-\frac{1}{2} \text{tr}(X^2 \Lambda)} \prod_{j=1}^{\infty} \frac{A_j}{2j+1} e^{tr X^{2j+1}}$$

for a particular choice of the coupling constants $A_1, A_2, \ldots$.

This gives a different solution to the KdV equations. (Actual formula being finalized as of now.)
Purely Algebraic Proof of Witten - Kontsevich

Kazarian and Lando, based on the works of Okounkov, Okounkov-Pandharipande, etc...

Starting point: ELSV formula

\[ \text{# of Hurwitz covers of } \mathbb{P}^1 \]
\[ = \text{Intersection of } \nu \text{ and } \lambda \text{ classes on } \overline{M}_{g,n} \]

\[ h^g, \mu_d = \frac{(d+n+2g-2)!}{|\text{Aut}(\mu_1, \ldots, \mu_n)|} \prod_{i=1}^{n} \frac{\mu_i!}{\mu_i!} \int_{\overline{M}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g}{(1-\mu_1 \lambda_1) \cdots (1-\mu_n \lambda_n)} \]

\[ \mu_1 \]
\[ \mu_2 \]
\[ \vdots \]
\[ \mu_n \]

\[ \infty \]

\[ \rightarrow \]

\[ \mathbb{P}^1 \]

\[ r \text{ simple ramification profile of ramification at } \infty \]

\[ (\mu_1, \mu_2, \ldots, \mu_n) = \] 

\[ \text{ramified covering of degree } d \]
\[ d = \mu_1 + \mu_2 + \cdots + \mu_n \]
\[ r = d + n + 2g - 2 \]

Riemann-Hurwitz.
Okounkov generating function

\[ H(\rho; t_1, t_2, t_3, \ldots) \]

\[ = \sum_{g \geq 0} \sum_{n \geq 0} \sum_{\mu_1, \mu_2, \ldots, \mu_n > 0} h_{g; \mu_1, \mu_2, \ldots, \mu_n} \frac{\rho^{\beta}}{t_{\mu_1} t_{\mu_2} \cdots t_{\mu_n}} \]

\[ d = \mu_1 + \ldots + \mu_n \]

Okounkov:
Theorem. \[ \tau = e^H \]
satisfies the KP equations as \( t_1, t_2, \ldots \) being KP-time variables.

KP Theory

\[ \Lambda_{\infty}^2(V) \sim \text{linear isom.} \rightarrow C[[t_1, t_2, t_3, \ldots]] \]

Bosonization (A kind of supersymmetry)

\[ V = C(z^-) \]

\[ \mathcal{P}(\Lambda_{\infty}^2(V)) = B \rightarrow \mathcal{P}(C[[t_1, t_2, t_3, \ldots]]) \]

\[ \text{Plücker} \rightarrow \text{KP} \text{ in Hirota form} \]

\[ \text{Gr}(\infty) = B \rightarrow \{ \tau - \text{functions} \} \]
Kazarian – Lando’s trick.

Throw away $\lambda_i$-classes from

\[
\prod_{i=1}^{n} \frac{1 - \lambda_i + \lambda_2 - \cdots \pm \lambda_n}{(1 - \mu_1 \lambda_1) \cdots (1 - \mu_k \lambda_k)}.
\]

Recall that

\[
\frac{1}{1-64} = 1 + 64 + 64^2 + \cdots.
\]

Claim. For every $d \geq 0$, \( \exists \ C_id, \ i=1,2,\ldots,d+1, \) such that

\[
\sum_{i=1}^{d+1} \frac{C_id}{1 - 64^i} = 4^d + \text{higher}.
\]

Proof.

\[
\sum_{i=1}^{d+1} C_id \left(1 + 64 + 64^2 + 64^3 + \cdots\right) = 4^d + \text{higher}.
\]
\[
\begin{aligned}
\langle \psi_1^{d_1} \ldots \psi_n^{d_n} \rangle &= \sum_{\mu_1=1}^{d_1+1} \cdots \sum_{\mu_n=1}^{d_n+1} C_{\mu_1 \ldots \mu_n} d_1 \cdots d_n \\
&\quad \cdot \frac{1-\lambda_1 + \lambda_2 - \cdots + \lambda_n}{(1-\mu_1 t_1) \cdots (1-\mu_n t_n)} M_{\mu_1 \cdots \mu_n}
\end{aligned}
\]

There is just no room for \( \lambda \)-classes to come in!

\[
\therefore \langle T_{d_1} \ldots T_{d_n} \rangle = \text{linear combination of Hurwitz numbers}
\]

\[
\downarrow \text{Kazarian-lando}
\]

\[
F(t_0, t_1, \ldots) = \langle e^\sum_{j=0}^\infty t_j T_j \rangle \text{ is expressible in terms of Okounkov's } H \text{-function}
\]

Using the explicit formula,

\[
\text{KdV on } H \Rightarrow \text{KdV on } F_t.
\]

\[
\therefore F_t \text{ satisfies the KdV}!
\]
Kontsevich's original proof is indeed correct.

- One should read the space in between lines.
- Often one needs to erase the lines first, and then let the space in between lines shine.
- Then the true proof comes up.

The three approaches together give a "clear" picture of the generalized Kontsevich integral.
- Virasoro = cut and join
- KdV/KB = graph enumeration

Limitation: $\overline{M_{g,n}}$ is a nice smooth orbifold. Therefore, coordinatization was possible to do integral analysis on it. This kind of analysis does not seem to be done on $\overline{M_{g,n}}$.

Analysis on $\overline{M_{g,n}}$ may be useful in other ways.