# Lie Cylinders and higher obstructions to deforming submanifolds

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aims:

a) Give useful description of the obstructions to embedded deformations

b) Develop algebraic tools for Derived Deformation theory.

In this talk we consider only a)

Utility test = vanishing of obstructions under semiregularity

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Plan of the talk

1) Basic stuff (for non experts)

2) $(MC)^2$ (new formalism)

3) Obstructions and semiregularity.

# 1: Basics.

Work over $\mathbb{C}$, SET = category of sets

ART = category of local artinian $\mathbb{C}$-algebras, residue field $A/\mathfrak{m} = \mathbb{C}$

EXPERIMENTAL FACT 1: most deformation problems have lots in common e.g. Tangent space, obstructions, .....

(Easy but instructive) Example.

$$V = \bigoplus_{i \in \mathbb{Z}} V^i \quad \text{graded vector space}$$

Denote $\mathrm{Hom}^s(V,V) = \{ f : V \to V \mid f(V^i) \subset V^{i+s} \}$

Let $\bar{\partial} \in \mathrm{Hom}^1(V,V)$ a differential, $\bar{\partial}^2 = 0$

For $f \in \mathrm{Hom}^i(V,V)$, $g \in \mathrm{Hom}^s(V,V)$ define

$$[f,g] = f \circ g - (-1)^{is} g \circ f$$

$$df = [\bar{\partial}, f] = \bar{\partial} \circ f - (-1)^i f \circ \bar{\partial}$$

$d :: \mathrm{Hom}^*(V,V) \circlearrowleft$

The triple $\left(L = \bigoplus_n \mathrm{Hom}^n(V,V),\ d,\ [\,,\,]\right)$ satisfy the axioms of differential graded Lie algebra DGLA.

1) $[\,,\,]$ is graded skew-symmetric

$$[g,f] = -(-1)^{|f||g|}[f,g]$$

2) Leibniz $\quad d[f,g] = [df,g] + (-1)^{|f|}[f,dg]$

3) Jacobi

$$[[f,g],h] = [f,[g,h]] - (-1)^{|f||g|}[g,[f,h]]$$

Deformation problem:

Given $A \in \mathrm{ART}$, consider $A$-linear differentials $V \otimes A \to V \otimes A$ of the form $\partial + \xi$, $\xi \in \mathrm{Hom}^1(V,V) \otimes \mathfrak{m}_A$

this is a deformation of $\partial$ over $\mathrm{Spec}(A)$

max ideal of $A$

The condition $(\bar\partial+\xi)\circ(\bar\partial+\xi)=0$ is

$$\bar\partial^2+\bar\partial\xi+\xi\bar\partial+\xi\cdot\xi = d\xi+\frac{1}{2}[\xi,\xi]=0$$

$\boxed{\text{this is Maurer-Cartan (MC) eqn.}}$

Two deformations $\bar\partial+\xi$, $\bar\partial+\eta$ are equivalent if conjugated by an $A$-linear automorfism of $V$, i.e.

$$\bar\partial+\xi\sim\bar\partial+\eta \iff \exists\, a\in \mathrm{Hom}^0(V,V)\otimes m_A$$

$$\bar\partial+\eta = e^{a}\circ(\bar\partial+\xi)\circ e^{-a}$$

equivalentey $\bar\partial+\xi\sim\bar\partial+\eta$ iff

$$\eta = e^{a}\circ\bar\partial\circ e^{-a}-\bar\partial+e^{a}\circ\xi\circ e^{-a}, \text{ iff}$$

$$\eta = e^{a}*\xi := \xi+\sum_{n=0}^{\infty}\frac{[a,-]^n}{(n+1)!}([a,\xi]-da)$$

$\boxed{\text{This is the GAUGE ACTION}}$

FOR <u>EVERY</u> DGLA $L$ we can define

$\mathrm{Def}_L : \mathrm{ART} \longrightarrow \mathrm{SET}$ functor

$$\mathrm{Def}_L(A) = \frac{\{x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0\}}{\text{Gauge action of } \exp(L^0 \otimes \mathfrak{m}_A)}$$

Experimental fact 2: ( this is the Metatheorem of Kontsevich Lectures in deformation theory (1994))

a) Every deformation problem $/\mathbb{C}$ is represented by a functor $\mathrm{Def}_L$ for a suitable DGLA $L$.

b) The correct $L$ of a) is determined up to quasi-isomorphism

<u>PROBLEM</u>. In most cases is difficult to find the DGLA governing a deformations

Example. $X$ complex manifold compact.
$T_X$ = holomorphic tangent bundle
$A_X^{0,p}(T_X)$ = $(0,p)$ forms, $T_X$-valued.

$K_X := \bigoplus_{p \geq 0} A_X^{0,p}(T_X)$ is the Kodaira-Spencer DGLA.

Newlander-Nirenberg $\Rightarrow$ $\mathrm{Def}_{K_X}$ is the functor of infinitesimal deformations of $X$.

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Functoriality

$L \longrightarrow M$ DGLA-morphism $\rightsquigarrow$ $\mathrm{Def}_L \longrightarrow \mathrm{Def}_M$ natural transformation

THM. (Deligne - Goldman - Millson)
If $L \to M$ is quasi-isomorphism then $\mathrm{Def}_L \to \mathrm{Def}_M$ is ISOMORPHISM

(VERY-VERY USEFUL theorem)

# PART II. $(MC)^2$

$(MC)^2$ = Maurer-Cartan on Mapping cylinder

Assume $\chi : L \to M$ morphism of DGLA.
Want to define $\mathrm{Def}_\chi : \mathrm{ART} \to \mathrm{SET}$ such that $\mathrm{Def}_\chi = \mathrm{Def}_L$ when $M = 0$.

<u>Definition</u>: for $A \in \mathrm{ART}$

$$MC_\chi(A) = \left\{ (x, e^a) \in L^1 \otimes m_A \times \exp(M^0 \otimes m_A) \;\middle|\; \begin{cases} dx + \frac{1}{2}[x,x] = 0 \\ e^a * \chi(x) = 0 \end{cases} \right\}$$

= Maurer-Cartan on Mapping cylinder

$\underline{C_\chi}$. $\quad C_\chi^i = L^i \oplus M^{i-1}$

$$\delta : C_\chi^i \longrightarrow C_\chi^{i+1}$$

$$\delta(x,a) = (dx, \chi(x) - da)$$

<u>BEWARE</u>: $\not\exists$ NATURAL DGLA structure on $C_\chi$ (JACOBI FAILS)

Gauge action on $MC_\chi$

(for simplicity $M^i = 0 \ \forall i < 0$)

$$\mathrm{Def}_\chi(A) = \frac{MC_\chi(A)}{\exp(L^0 \otimes m_A)}$$

$$L^0 \otimes m \ni b \qquad e^b * (x, e^a) = (e^b * x, e^a \cdot e^{-\chi(b)})$$

PROPERTIES

FUNCTORIALITY:

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\chi"'] & I \arrow[d, "\eta"] \\ M \arrow[r] & K \end{tikzcd} \rightsquigarrow \begin{array}{l} \mathrm{Def}_\chi \to \mathrm{Def}_\eta \\ \text{natural trans.} \\ C_\chi \to C_\eta \\ \text{morphism of complexes} \end{array}$$

DGLA morphisms

**THM.** If $C_\chi \to C_\eta$ quasiisomorphism then $\mathrm{Def}_\chi \to \mathrm{Def}_\eta$ isomorphism.

## TANGENT SPACE

$$\mathrm{Def}_x\left(\frac{\mathbb{C}[t]}{t^2}\right) \simeq H^1(C_x)$$

EXAMPLE. $X$ complex manifold

$Z \subset X$ smooth submanifold closed

$N_{Z|X}$ = normal bundle.

$$L = \mathrm{Ker}\left(A_X^{0,*}(T_X) \longrightarrow A_Z^{0,*}(N_{Z|X})\right)$$

Notice: $L = \oplus L^i$ is a DG LIE subalgebra of $A_X^{0,*}(T_X) = K_X$

Denote $x : L \to K_X$ inclusion

the mapping cylinder $C_x$ is quasi isomorphic to $A_Z^{0,*-1}(N_{Z|X})$ and then $H^i(C_x) = H^{i-1}(N_{Z|X})$

$$\Longrightarrow \mathrm{Def}_x\left(\frac{\mathbb{C}[t]}{t^2}\right) \simeq H^0(N_{Z|X})$$

**PROPOSITION.** $\exists$ an Isomorphism of functors

$$\mathrm{Def}_\chi \xrightarrow{\sim} \mathrm{Hilb}^Z_X = \left\{ \begin{array}{l} \text{Functor of} \\ \text{embedded deformations} \\ \text{of } Z \text{ in } X \end{array} \right.$$

**COROLLARY 1** the Hilbert functor $\mathrm{Hilb}^Z_X$ is governed by the DGLA

$$H = \left\{ x \in K_X \otimes \mathbb{C}[t, dt] \;\middle|\; \begin{array}{l} x \xrightarrow{t=0} 0 \\ x \xrightarrow{t=1} \in L \end{array} \right\}$$

**COROLLARY 2:** Let $a \in H^i(N_{Z|X})$,
$\Rightarrow$ a lifts to $\hat{a} \in A^{0,i}_X(T_X)$ such that $\bar{\partial} \hat{a} \in L$.

The "Whitehead" product

$$[\,,\,]_W : H^i(N_{Z|X}) \times H^j(N_{Z|X}) \to H^{i+j+1}(N_{Z|X})$$

$$[a, b]_W := \frac{1}{2}\left( [\hat{a}, \bar{\partial}\hat{b}] - (-1)^i [\bar{\partial}\hat{a}, \hat{b}] \right)$$

is a Lie structure on $H^{*-1}(N_{Z|X})$.

PART III: obstructions and semiregularity

Recall: a functor $\mathcal{F}: ART \to SET$ is UNOBSTRUCTED if

$$\underset{\text{surjective}}{A \to B} \Longrightarrow \underset{\text{surjective.}}{\mathcal{F}(A) \to \mathcal{F}(B)}$$

a <u>small extension</u> is an exact sequence of vector spaces

$$e: 0 \to I \to A \xrightarrow{\alpha} B \to 0$$

such that
i) $\alpha \in MOR_{ART}$
ii) $I \cdot m_A = 0$

an obstruction theory for $\mathcal{F}$ is given by a vector space $V$ and $\forall$ small extension $e$ of a map $v_e: \mathcal{F}(B) \to V \otimes_{\mathbb{C}} I$

with properties

i) $\partial(\mathcal{F}(A)) = \{ x \in \mathcal{F}(B) \mid v_e(x) = 0 \}$

ii) the maps $v_e$ are functorial in $e$

<u>Definition</u>: the obstruction space $O_{\mathcal{F}} \subset V$ is the minimal subspace such that $\operatorname{Im} v_e \subset O_{\mathcal{F}} \otimes I \quad \forall e$

Lemma. (Fantechi, -) $O_{\mathcal{F}}$ depends, up to isomorphism, only by $\mathcal{F}$.

$\mathcal{F}$ unobstructed $\Leftrightarrow O_{\mathcal{F}} = 0$

<u>BEWARE</u>: $O_{\mathcal{F}}$ may not be generated by obstructions arising from

$$0 \to \mathbb{C} \cdot t^n \to \frac{\mathbb{C}[t]}{(t^{n+1})} \to \frac{\mathbb{C}[t]}{(t^n)} \to 0$$

# EXAMPLES

1) $I \subset \mathbb{C}[[t_1, \dots, t_n]]$, $I \subset \mathfrak{m}^2$

$$R = \frac{\mathbb{C}[[t_1, \dots, t_n]]}{I} \qquad \mathfrak{m} = (t_1, \dots, t_n)$$

$$F(A) = \mathrm{Hom}(R, A)$$ prorepresentable functor.

$$\Longrightarrow \quad O_F \simeq \left(\frac{I}{\mathfrak{m} \cdot I}\right)^{\vee}$$

2) $\chi : L \longrightarrow M$ morphism of DGLA

$\mathrm{Def}_\chi : \mathrm{ART} \to \mathrm{SET}$

$\exists$ "explicit" obstruction theory.

with $V = H^2(C_\chi)$

take small extension

$$0 \to \mathbb{C} \to A \longrightarrow B \to 0$$

and $(x, e^a) \in \mathrm{Def}_\chi(B)$

Recall
$x \in L^1 \otimes \mathfrak{m}_B$
$a \in M^0 \otimes \mathfrak{m}_B$

choose liftings

$$L^1 \otimes \mathfrak{m}_A \ni y \longmapsto x$$

$$H^0 \otimes \mathfrak{m}_A \ni b \longmapsto a$$

Then $\left(dy + \frac{1}{2}[y,y],\ e^b * \chi(y)\right)$ is a cocicle in $C_\chi$. Its class in $H^2(C_\chi)$ is the obstruction.

<u>Example</u>. For $\mathcal{F} = Hilb^2_X$ and the primary extension

$$0 \to \mathbb{C}t^2 \to \frac{\mathbb{C}[t]}{t^3} \to \frac{\mathbb{C}[t]}{t^2} \to 0$$

the obstruction map is

$$q:\ H^0(N_{Z|X}) \longrightarrow H^1(N_{Z|X}) \quad \left(= H^2(C_\chi)\right)$$

$$q(x) = \frac{1}{2}[x,x]_W$$

Whitehead product.

# SEMIREGULARITY

$X$ smooth complex manifold $\dim_X = n$

$Z \subset X$ smooth submanifold of pure codimension $p$

have commutative diagram $\quad n-p = \dim Z$

$$\begin{array}{ccc} T_X \otimes \Omega_X^{n-p+1} & \xrightarrow{\lrcorner} & \Omega_X^{n-p} \\ \downarrow & & \downarrow \\ N_{Z|X} \otimes \Omega_X^{n-p+1} & \xrightarrow{\lrcorner} & \Omega_Z^{n-p} = \omega_Z \end{array}$$

Take cohomology

$$\theta_i : H^i(N_{Z|X}) \longrightarrow \bigoplus_j \mathrm{Hom}\left(H^j(\Omega_X^{n-p+1}), H^{i+j}(\omega_Z)\right)$$

Serre duality

$$\theta_i : H^i(N_{Z|X}) \to \bigoplus_j \mathrm{Hom}\left(H^j(\mathcal{O}_Z), H^{p+i+j}(\Omega_X^{p-1})\right)$$

Defn: $\theta_1$ = semiregularity map

(Severi–Kodaira for $Z$ divisor, Bloch in general)

<u>theorem</u> If $X$ is Kähler, then the obstruction space $O_{Hilb^Z_X} \subset H^2(N_{Z|X})$ is contained in the kernel of $\theta_1$.

<u>Idea of proof</u>: consider the graded vector space $W = \oplus W^i$

$$W^i = \bigoplus_s \mathrm{Hom}(H^s(\mathcal{O}_Z), H^{p+i+s}(\Omega^{p-1}_X)).$$

assume there exists commutative diagram of DGLA morphisms

$$\begin{array}{ccc} L & \to & 0 \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \eta} \\ K_X & \xrightarrow{f} & W \end{array}$$

such that $f$ induces $\theta_1$ on cohomology

$\leadsto$ $Hilb^Z_X = Def_\chi \to Def_\eta$ ← IS UNOBSTRUCTED

$\leadsto$ $f$ annihilates obstructions.

PROBLEM: $\nexists$ such $f$ as DGLA morphism but only (If $X$ Kähler) as $L_\infty$-morphism.