Stability and the structure of the derived category of coherent sheaves on irreducible curves of genus one.

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1. Relative Fourier-Mukai transforms for singular genus one fibrations
   - The setting
   - The result
   - The proof
   - Duality

2. The structure of $D^b_{coh}(\mathbf{E})$ on irreducible singular curves of genus one
   - Comparison smooth – singular, I
   - Standard stability
   - Stability conditions
   - Comparison smooth – singular, II

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Motivation

$$X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$$

$$\mathcal{P} \in D^b_{coh}(X \times X)$$

Mukai: $\mathbf{R}\pi_{2*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_1^*(\,\cdot\,))$
FMT $\Leftrightarrow$ equivalence.

- moduli of vector bundles, $D^b_{coh}(X)$;
- In higher-dimensional birational geometry, $D^b_{coh}(X)$ is used to study MMP.
- Most results about FMT assume smoothness, but MMP leads to singular varieties.

Elliptic fibrations

k algebraically closed, characteristic zero,
X, S schemes

- reduced, connected;
- separated, of finite type over **k**.

$q : X \longrightarrow S$

- projective, flat;
- with section $\sigma : S \to X$, $\Sigma := \sigma(S)$
- fibers are integral Gorenstein curves of arithmetic genus one.

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Neither X nor S needs to be Gorenstein.

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The result

$$\begin{array}{ccc} X \times_S X & \xrightarrow{\pi_1} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{q} & S \end{array}$$

$$\mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_X(\Sigma) \otimes \pi_2^* \mathcal{O}_X(\Sigma)$$

$$\mathsf{FM}_{\mathcal{P}}(\mathcal{E}^\bullet) := \mathbf{R}\pi_{2*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_1^*(\mathcal{E}^\bullet))$$

Theorem

$\mathsf{FM}_{\mathcal{P}} : D^-_{coh}(X) \to D^-_{coh}(X)$ *equivalence.*

Same is true for D^b_{coh}, D^+_{coh}, D_{coh}.

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Theorem

$\mathrm{FM}_{\mathcal{P}} : \mathrm{D}^-_{\mathrm{coh}}(X) \to \mathrm{D}^-_{\mathrm{coh}}(X)$ *equivalence.*

Same is true for $\mathrm{D}^b_{\mathrm{coh}}, \mathrm{D}^+_{\mathrm{coh}}, \mathrm{D}_{\mathrm{coh}}$.

Proof – preliminaries

Starting point: $S = \mathrm{Spec}(\mathbf{k})$, X smooth elliptic curve

Mukai's result: $[1] \circ \mathrm{FM}_{\mathcal{P}} \circ \mathrm{FM}_{\mathcal{P}} \cong (-1)^*$

where $(-1) : X \to X$ is "taking the inverse".

Theorem (Burban, K.)

$[1] \circ \mathrm{FM}_{\mathcal{P}} \circ \mathrm{FM}_{\mathcal{P}} \cong i^*$

Assumptions: $S = \mathrm{Spec}(\mathbf{k})$ and $i : X \to X$ is "taking the inverse" on the smooth locus of X.

Involution $i : X \to X$ exists for elliptic fibrations with section.

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Proof

Idea: understand
$[1] \circ i^* \circ \mathrm{FM}_{\mathcal{P}} \circ \mathrm{FM}_{\mathcal{P}} = \mathrm{FM}_{\mathcal{Q}^\bullet}$
by restricting to fibers.

$$\begin{array}{ccc} X_s & \xrightarrow{j_s} & X \\ \downarrow & & \downarrow q \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

Lemma

With $\mathcal{P}^\bullet_s := \mathbf{L}(j_s \times j_s)^* \mathcal{P}^\bullet$*, we have* $j_{s*} \circ \mathrm{FM}_{\mathcal{P}^\bullet_s} \cong \mathrm{FM}_{\mathcal{P}^\bullet} \circ j_{s*}$.

$\Longrightarrow \quad \mathrm{FM}_{\mathcal{Q}^\bullet} \circ j_{s*} \cong j_{s*}$
$\Longrightarrow \quad \forall x \in X \quad \mathrm{FM}_{\mathcal{Q}^\bullet}(\mathbf{k}(x)) \cong \mathbf{k}(x)$
Bridgeland lemma $\Longrightarrow \quad \mathcal{Q}^\bullet \cong \delta_* \mathcal{L},\ \mathcal{L} \in \mathrm{Pic}(X)$
$\Longrightarrow \quad \mathrm{FM}_{\mathcal{P}} \circ \mathrm{FM}_{\mathcal{P}} \cong [-1] \circ i^* \circ (q^* \mathcal{N} \otimes \cdot\,)$
$\Longrightarrow \quad \mathrm{FM}_{\mathcal{P}}$ is an equivalence.

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Duality

- Grothendieck-Verdier duality gives:
$$\mathbf{R}\mathcal{H}om_X(\mathrm{FM}_{\mathcal{P}}(\,\cdot\,), \mathcal{O}_X)$$
and
$$\mathrm{FM}_{\mathcal{P}}(\mathbf{R}\mathcal{H}om_X(\,\cdot\,, \mathcal{O}_X))$$
are equal up to $[1] \circ i^* \circ (q^*\mathcal{N} \otimes \,\cdot\,)$;
- S Gorenstein, then $\mathbb{D}_X := \mathbf{R}\mathcal{H}om_X(\,\cdot\,, \mathcal{O}_X)$ is a dualizing functor;
- $S = \mathrm{Spec}(\mathbf{k})$, then
$$\mathbb{D}_X \circ \mathrm{FM}_{\mathcal{P}} \cong [1] \circ i^* \circ \mathrm{FM}_{\mathcal{P}} \circ \mathbb{D}_X.$$

The smooth case

Let **E** be a smooth elliptic curve. For coherent sheaves, we have:

$$\text{locally free} \iff \text{torsion free}$$
$$\text{stable} \iff \text{simple}$$
$$\text{indecomposable} \implies \text{semi-stable}$$

- homological dimension of $\mathrm{Coh}(\mathbf{E})$ is 1, i.e. $\mathrm{Ext}^i(F, G) = 0 \quad \forall\, i > 1$;
- simple structure of $D^b_{coh}(\mathbf{E})$: any object is the direct sum of its cohomology sheaves;
- important tool: Serre Duality $\forall\, F, G \in \mathrm{Coh}(\mathbf{E}) : \quad \mathrm{Ext}^i(F, G) \cong \mathrm{Ext}^{1-i}(G, F)^*$

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Comparison

	smooth	singular
homological dim of $\mathrm{Coh}(\mathbf{E})$	1	∞
Serre duality holds	in general	if one object is perfect
torsion free implies locally free	yes	no
indecomposable coherent sheaves are semi-stable	yes	no
any indecomposable complex is isomorphic to a shift of a sheaf	yes	no

Stability and HNF

E irreducible singular curve of arithmetic genus one over **k**, algebraically closed, characteristic zero. $0 \neq \mathcal{F} \in \mathrm{Coh}(\mathbf{E})$

- slope $\mu(\mathcal{F}) = \deg(\mathcal{F})/\operatorname{rk}(\mathcal{F})$
- phase $\varphi(\mathcal{F}) \in (0, 1]$ such that $-\deg(\mathcal{F}) + i \cdot \operatorname{rk}(\mathcal{F}) \in \mathbb{R}_{>0} \cdot \exp(i\pi\varphi(\mathcal{F}))$
- $\varphi(\mathcal{O}) = 1/2, \quad \varphi(\mathbf{k}(x)) = 1$
- $\mathcal{F}$ semi-stable $\iff \forall\, 0 \neq \mathcal{G} \subset \mathcal{F} : \quad \varphi(\mathcal{G}) \leq \varphi(\mathcal{F})$.

Any $\mathcal{F} \in \mathrm{Coh}(\mathbf{E})$ has a HNF

$$0 \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}$$

such that the factors $\mathcal{A}_i = \mathcal{F}_i/\mathcal{F}_{i+1}$ are semi-stable and

$$\varphi(\mathcal{A}_n) > \varphi(\mathcal{A}_{n-1}) > \ldots > \varphi(\mathcal{A}_0).$$

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Stability on $D^b_{coh}(\mathbf{E})$

- $\varphi(\mathcal{F}[n]) := \varphi(\mathcal{F}) + n$ for $n \in \mathbb{Z}$ and $\mathcal{F} \in \mathrm{Coh}(\mathbf{E})$.
- slicing $\quad \mathrm{P}(\varphi) = \{$ semi-stable sheaves of phase $\varphi\}$

Theorem (Bridgeland; GKR)

$0 \neq X \in D^b_{coh}(\mathbf{E})$ *has a HNF, unique up to isomorphism,*

with $0 \neq A_i \in \mathrm{P}(\varphi_i)$ *and* $\varphi_+(X) = \varphi_n > \varphi_{n-1} > \ldots > \varphi_1 > \varphi_0 = \varphi_-(X)$.

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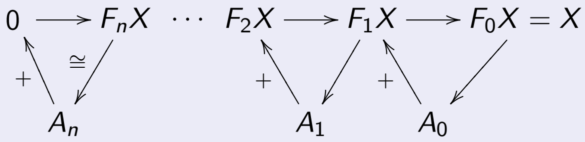

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- general: $\varphi_-(X) > \varphi_+(Y) \implies \mathrm{Hom}(X,Y) = 0$
- curve case:
  $\varphi_-(X) < \varphi_+(Y) < \varphi_-(X) + 1 \Rightarrow \mathrm{Hom}(X,Y) \neq 0$

The main tool to prove this are Seidel-Thomas twists:

$$T_E : D^b_{coh}(\mathbf{E}) \to D^b_{coh}(\mathbf{E})$$

- given by $\mathbf{R}\mathrm{Hom}(E,F) \otimes E \to F \to T_E(F) \xrightarrow{+}$
- T_E is an equivalence, if E is spherical, i.e. perfect and
  $\mathrm{Hom}(E, E[i]) = \begin{cases} \mathbf{k} & \text{if } i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$

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$$\mathbb{F} = T_{\mathbf{k}(p_0)} T_{\mathcal{O}} T_{\mathbf{k}(p_0)}$$

$\mathbb{F} \cong$ FMT with kernel $\mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}(p_0) \otimes \pi_2^* \mathcal{O}(p_0)[1]$

Theorem (Bruzzo et al.)

$\mathbb{F}$ *preserves semi-stability.*

Our proof uses

- the degree zero case (`math.AG/0401437`);
- $\mathbb{F}\mathbb{F} = i^*[1]$ (`math.AG/0401437`);
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$\widetilde{\mathrm{SL}}(2,\mathbb{Z})$-action

$\widetilde{\mathrm{SL}}(2,\mathbb{Z}) = \langle A, B, T \mid ABA = BAB, (AB)^6 = T^2, AT = TA, BT = TB\rangle$, a central extension of $\mathrm{SL}(2,\mathbb{Z})$, acts on $D^b_{coh}(\mathbf{E})$ by

$$A \mapsto T_{\mathcal{O}},\ B \mapsto T_{\mathbf{k}(p_0)},\ T \mapsto [1].$$

Corollary

$\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ *acts transitively on the set of non-zero slices* $\mathrm{P}(\varphi)$.

$$\begin{array}{ccccl} \mathrm{P}(\varphi) & \xrightarrow{\cong} & \mathrm{P}(1) & = & \{\text{ coherent torsion sheaves}\} \\ \uparrow & & \uparrow & & \\ \mathrm{P}(\varphi)^s & \xrightarrow{\cong} & \mathrm{P}(1)^s & = & \{\mathbf{k}(x) \mid x \in \mathbf{E}\} \quad \text{stable objects} \end{array}$$

Structure of $\mathsf{P}(\varphi)$

- Objects in $\mathsf{P}(\varphi)$ have JHF with stable JH-factors;
- Indecomposable objects have a single JH-factor;
- We call the unique non-perfect element in $\mathsf{P}(\varphi)^s$ the *extreme* stable element of phase φ;
- E.g. $\mathbf{k}(s) \in \mathsf{P}(1)^s$, if $s \in \mathbf{E}$ is the singular point;
- stable objects are either perfect or extreme;
- $\mathbb{F}(\mathsf{P}(\varphi)) = \mathsf{P}(\varphi + \frac{1}{2})$.

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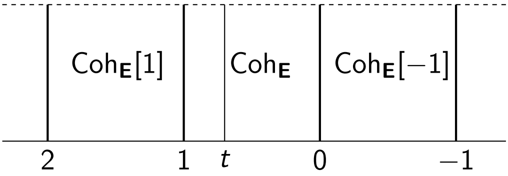
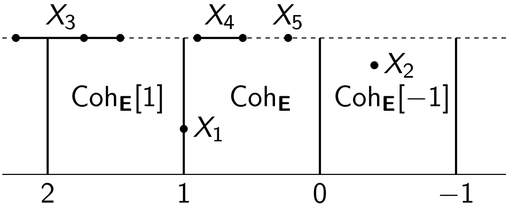


Shadows

The shadow of an indecomposable object is the set of all JH-factors of HN-factors, connected by line segments.



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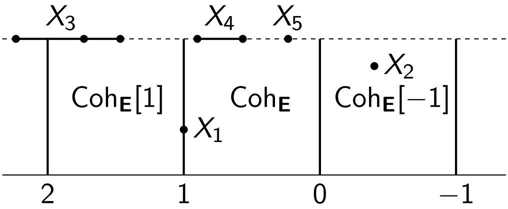

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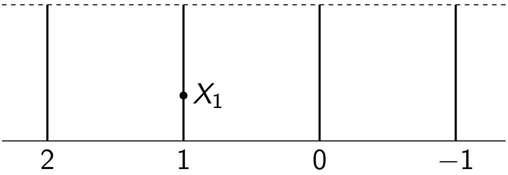
Indecomposable Objects

Corollary

There exist four types of indecomposable objects in $\mathrm{Coh}(\mathbf{E})$*:*

1. *semi-stable with perfect JH-factor;*
2. *semi-stable, perfect but with extreme JH-factor;*
3. *semi-stable and not perfect;*
4. *not semi-stable, all HN-factors without a perfect summand.*



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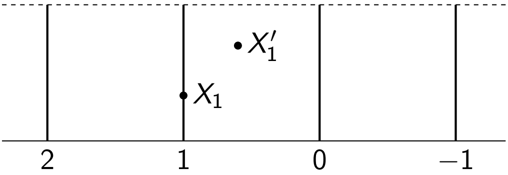

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There exist four types of indecomposable objects in Coh(**E**):

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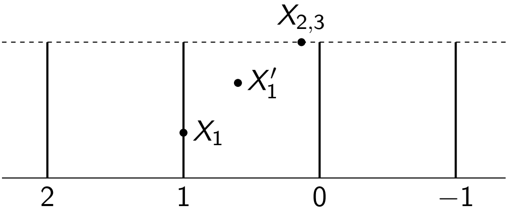

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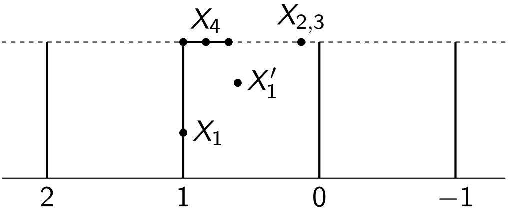

Stability Conditions – Definition

Definition (Bridgeland)

(W, R) stability condition

- $W : \mathrm{K}(\mathbf{E}) \to \mathbb{C}$ homomorphism of groups;
- R is a compatible slicing, i.e.
  - $\mathrm{R}(t+1) = \mathrm{R}(t)[1]$,
  - $\mathrm{Hom}(A_1, A_2) = 0$, if $t_1 > t_2$ and $A_i \in \mathrm{R}(t_i)$,
  - $0 \neq X \in D^b_{coh}(\mathbf{E})$ has a HNF with $A_i \in \mathrm{R}(\varphi_i)$,
  - $W(A) \in \mathbb{R}_{>0} \exp(i\pi t)$, if $A \in \mathrm{R}(t)$.

Stability Conditions – Classification

Theorem

$\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ *acts simply transitive on* $\mathrm{Stab}(\mathbf{E})$.

$\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ = all pairs (A, f) with $A \in \mathrm{GL}^+(2,\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ such that f is strictly increasing, $f(t+1) = f(t) + 1$ and A and f induce the same mapping on $(\mathbb{R}^2 \setminus \{0\})/\mathbb{R}^* = S^1 = \mathbb{R}/2\mathbb{Z}$.

$$(A, f) \cdot (W, \mathrm{R}) := (A^{-1} \circ W, \mathrm{R} \circ f)$$

Main tool: we describe all t-structures on $D^b_{coh}(\mathbf{E})$.

Common features

1. $F \in \mathrm{Coh}(\mathbf{E})$ is stable if and only if $\mathrm{End}(F) \cong \mathbf{k}$.
2. Any spherical object is a shift of a stable vector bundle or of a structure sheaf $\mathbf{k}(x)$ at a smooth point $x \in \mathbf{E}$.
3. The category of semi-stable sheaves of a fixed slope is equivalent to the category of coherent torsion sheaves. Such an equivalence is induced by an auto-equivalence of $D^b_{coh}(\mathbf{E})$.
4. With $\mathrm{Aut}^0 = \langle \mathrm{Aut}(\mathbf{E}), \mathrm{Pic}^0(\mathbf{E}), [2] \rangle$, there is an exact sequence of groups

$$0 \to \mathrm{Aut}^0 \to \mathrm{Aut}(D^b_{coh}(\mathbf{E})) \to \mathrm{SL}(2, \mathbb{Z}) \to 0.$$

5. $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ acts transitively on $\mathrm{Stab}(\mathbf{E})$.
6. $\mathrm{Stab}(\mathbf{E})/\mathrm{Aut}(D^b_{coh}(\mathbf{E})) \cong \mathrm{GL}^+(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$.