# Desingularizations of moduli spaces of rank 2 sheaves with trivial determinant

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- I. Vector Bundles over Curves
- II. Higgs Bundles over Curves
- III. Sheaves on K3 and Abelian Surfaces

#### I. Vector Bundles over Curves

#### 1. Moduli space of bundles $M_0$

- $X = \text{smooth proj. curve of genus } g \geq 3.$
- $F \to X$  rank 2 bundle with det  $F = \mathcal{O}_X$ .
- F is polystable if F is stable or  $F \cong L \oplus L^{-1}$  for  $L \in Pic^{0}(X) =: J$ .
- $M_0 := \{ \text{polystable } F \} / \text{isom}$  admits a scheme structure such that for any vector bundle  $\mathcal{F} \to S \times X$  with  $\mathcal{F}|_{\{s\} \times X}$  semistable for  $\forall \ s \in S$ , the obvious map  $S \to M_0$  which maps

$$s \mapsto [\operatorname{gr}(\mathcal{F}|_{\{s\} \times X})]$$

is a morphism of schemes.

## 2. Stratification of $M_0$

- A polystable bundle  $F \in M_0$  is one of the following;
  - (a) F stable

(b) 
$$F \cong L \oplus L^{-1}$$
 with  $L \ncong L^{-1}$ 

(c) 
$$F \cong L \oplus L$$
 with  $L \cong L^{-1}$ 

- $M_0 = M_0^s \sqcup (J/\mathbb{Z}_2 J_0) \sqcup J_0$ : stratification
  - (a)  $M_0^s$  = open subset of stable bundles

(b) 
$$J/\mathbb{Z}_2 = \{ L \oplus L^{-1} \mid L \in J \}$$

(c) 
$$J_0 = \mathbb{Z}_2^{2g} = \{ L \oplus L \mid L \cong L^{-1} \}$$

## 3. Singularities of $M_0$

• (Luna's slice theorem) For polystable F, the analytic type of singularity of  $F \in M_0$  is

$$H^1(\mathcal{E}nd_0(F))/\!/\mathsf{Aut}(F)$$

(a) If F is stable, then  $\operatorname{Aut}(\mathsf{F}) = \mathbb{C}^*$  acts trivially on  $H^1(\mathcal{E}nd_0(F))$ . Hence  $M_0$  is smooth at  $F \in M_0$  and

$$T_F M_0 = H^1(X, \mathcal{E}nd_0(F))$$

(b) If 
$$F=L\oplus L^{-1}$$
 with  $L\ncong L^{-1}$ , then 
$$\operatorname{Aut}(F)/\mathbb{C}^*=\mathbb{C}^* \quad \text{ and } \\ H^1(\mathcal{E}nd_0(F))\cong H^1(\mathcal{O}_X)\oplus H^1(L^2)\oplus H^1(L^{-2}) \\ \text{ where } \mathbb{C}^* \text{ acts with weight } 0,2,-2 \text{ respectively.}$$

 $\downarrow \downarrow$ 

 $M_0$  is singular at  $F \in M_0$  and the analytic type of the singularity is

$$H^{1}(L^{2}) \oplus H^{1}(L^{-2})/\!/\mathbb{C}^{*}$$

which is the affine cone over

$$\mathbb{P}\left(H^1(L^2) \oplus H^1(L^{-2})\right) / / \mathbb{C}^* = \mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$$

 $\Downarrow$ 

By blowing up at the vertex we get a desingularization

$$\mathcal{O}_{\mathbb{P}^{g-2} imes\mathbb{P}^{g-2}}(-1,-1)$$

(c) If 
$$F = L \oplus L$$
 with  $L \cong L^{-1}$ , then

$$\operatorname{Aut}(F)/\mathbb{C}^* = \mathbb{P}GL(2)$$
 and

$$H^1(\mathcal{E}nd_0(F)) \cong H^1(\mathcal{O}_X) \otimes \mathfrak{sl}(2)$$

where  $\mathbb{P}GL(2)$  acts by conjugation on  $\mathfrak{sl}(2)$ .



 $M_0$  is singular at  $F \in M_0$  and the analytic type of the singularity is

$$H^1(\mathcal{O}) \otimes \mathfrak{sl}(2) // \mathbb{P}GL(2) = \mathbb{C}^g \otimes \mathfrak{sl}(2) // SL(2)$$



Need three blow-ups to desingularize

- Three blow-ups before quotient
  - $-W_0 = \mathbb{C}^g \otimes \mathfrak{sl}(2) \cong \operatorname{Hom}(\mathbb{C}^3, \mathbb{C}^g)$
  - $W_1$ = blow-up of  $W_0$  at 0 line bundle  $\mathcal{O}(-1)$  over  $\mathbb{P}\mathrm{Hom}(\mathbb{C}^3,\mathbb{C}^g)$
  - $W_2$ = blow-up of  $W_1$  along the proper transform of  $\operatorname{Hom}_1(\mathbb{C}^3,\mathbb{C}^g)$
  - $W_3$ = blow-up of  $W_2$  along the proper transform  $\Delta$  of  $\mathbb{P}\text{Hom}_2(\mathbb{C}^3,\mathbb{C}^g)$ .
- $W_3$  is a nonsingular quasi-projective variety acted on by SL(2)
- Locus of nontrivial stabilizers in  $W_3^{ss} = W_3^s$  is a divisor
- $W_3/\!\!/ SL(2)$  is nonsingular, i.e.  $\pi:W_3/\!\!/ SL(2) \to W_0/\!\!/ SL(2)$  is a desingularization

 $\bullet$   $\pi$  is the composition of three blow-ups

$$\pi: W_3/\!\!/ SL(2) \to W_2/\!\!/ SL(2) \to$$
 $\to W_1/\!\!/ SL(2) \to W_0/\!\!/ SL(2)$ 

- $D_i$ = proper transform of exceptional divisor of i-th blow-up in  $W_3/\!\!/SL(2)$ : smooth normal crossing divisors
- $\mathcal{A} \to Gr(2,g)$  tautological rank 2 bundle  $\mathcal{B} \to Gr(3,g)$  tautological rank 3 bundle
- $D_1$ = blow-up of projective bundle  $\mathbb{P}(S^2\mathcal{B})$  along the locus of rank 1 conics
- $D_3 = \mathbb{P}^2 \times \mathbb{P}^{g-2}$  bundle over Gr(2,g)
- $D_2 = [\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ -bundle over  $\text{bl}_0 \mathbb{C}^g]/\mathbb{Z}_2$

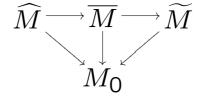
- normal bundle of  $D_3$ =  $\mathcal{O}(-1)$  along  $\mathbb{P}^2$ -direction
- $\Rightarrow$  can blow down along  $\mathbb{P}^2$ -direction of  $D_3$ 
  - $D_1$  becomes  $\mathbb{P}^5$ -bundle over Gr(3,g) normal bundle is  $\mathcal{O}(-1)$  along  $\mathbb{P}^5$
- $\Rightarrow$  can blow down along the  $\mathbb{P}^5$ -direction
  - ullet three desingularizations of  $W_0/\!\!/SL(2)$

#### 4. Kirwan's desingularization

ullet  $M_0$  can be desingularized by 3 blow-ups along

i) 
$$J_0 = \mathbb{Z}_2^{2g}$$

- ii) proper transform of  $J/\mathbb{Z}_2$
- iii) nonsingular subvariety  $\Delta$  lying in the exceptional divisor of the first blow-up.
- $\pi:\widehat{M}\to M_0$  Kirwan desingularization Explicit description of exceptional divisors
- ullet  $\widehat{M}$  can be blown down twice to give us three desingularizations of  $M_0$ :



#### 5. Applications

- Can compute the cohomology of  $\overline{M}$  and  $\widetilde{M}$  by using Kirwan's computation of  $H^*(\widehat{M})$ .
- discrepancy divisor :

$$\widehat{M} \xrightarrow{\mathbb{P}^2} \overline{M} \xrightarrow{\mathbb{P}^5} \widetilde{M} \xrightarrow{(g-2)\widetilde{D}_2} M_0$$

$$K_{\widetilde{M}} - \pi^* K_{M_0} = 2D_3 + 5D_1 + (g - 2)(D_2 + 3D_1 + 2D_3)$$
$$= (3g - 1)D_1 + (g - 2)D_2 + (2g - 2)D_3$$

Hence  $M_0$  has terminal singularities.

• (Kiem-Li) Stringy E-function:

$$E_{st}(M_0) = \frac{(1 - u^2v)^g (1 - uv^2)^g - (uv)^{g+1} (1 - u)^g (1 - v)^g}{(1 - uv)(1 - (uv)^2)} - \frac{(uv)^{g-1}}{2} \left( \frac{(1 - u)^g (1 - v)^g}{1 - uv} - \frac{(1 + u)^g (1 + v)^g}{1 + uv} \right).$$

• (Kirwan) E-polynomial of  $IH^*(M_0)$ 

$$IE(M_0) = \sum_{k,p,q} (-1)^k h^{p,q} (IH^k(M_0)) u^p v^q$$

$$= \frac{(1 - u^2 v)^g (1 - uv^2)^g - (uv)^{g+1} (1 - u)^g (1 - v)^g}{(1 - uv)(1 - (uv)^2)}$$

$$- \frac{(uv)^{g-1}}{2} (\frac{(1 - u)^g (1 - v)^g}{1 - uv} + (-1)^{g-1} \frac{(1 + u)^g (1 + v)^g}{1 + uv}).$$

The stringy Euler number is

$$\frac{1}{4} \cdot \chi(J_0) = \frac{1}{4} \cdot 2^{2g}$$

#### 6. Seshadri's desingularization

- Fix  $x_0 \in X$ .  $E = \text{rank 4 bundle with } \det E \cong \mathcal{O}_X$   $0 \neq s \in E^*|_{x_0} \text{ quasi-parabolic structure}$   $0 < a_1 < a_2 \ll 1 \text{ parabolic weights.}$
- (Mehta-Seshadri)
   ∃ fine moduli space P of stable parabolic bundles of rank 4;
   P is a smooth projective variety.
- ullet Seshadri's desingularization  ${f S}$  is a nonsingular closed subvariety of P.

- Proposition (Seshadri)
  - (1)  $[\exists 0 \neq s \in E_{x_0}^* \text{ s.t. } (E, s) \text{ is stable}]$  $\Leftrightarrow [\nexists L \in \text{Pic}^0(X) \text{ s.t. } L \oplus L \hookrightarrow E]$
  - (2) Let  $(E_1, s_1), (E_2, s_2) \in P$ . Suppose dim  $\operatorname{End} E_1 = \dim \operatorname{End} E_2 = 4$ . Then  $(E_1, s_1) \cong (E_2, s_2) \Leftrightarrow E_1 \cong E_2$
- Corollary  $i: M_0^s \hookrightarrow P$ [: for  $F \in M_0^s$ ,  $E = F \oplus F$  does not contain  $L \oplus L$  for any  $L \in Pic^0(X)$  and  $End(F) = \mathfrak{gl}(2)$ .]
- Theorem (Seshadri)
  - (1)  $\mathbf{S} = \overline{\iota(M_0^s)}$  is the locus of (E,s),  $\det E = \mathcal{O}_X$  and  $\operatorname{End} E$  is a specialization of the algebra  $M(2) = \mathfrak{gl}(2)$  of  $2 \times 2$  matrices.
  - (2) **S** is a desingularization of  $M_0$ , i.e. **S** is smooth and  $\exists$  morphism  $\pi_S: \mathbf{S} \to M_0$  such that  $\pi_S = \imath^{-1}$  on  $M_0^s$ .

## **► Theorem** (Kiem-Li)

- (1)  $\exists$  birational morphism  $\rho_S:\widehat{M}\to \mathbf{S}$
- (2)  $\mathbf{S} \cong \widetilde{M}$  and  $\rho_S$  is the composition of two blow-ups  $\widehat{M} \to \overline{M} \to \widetilde{M}$ .

#### Remark

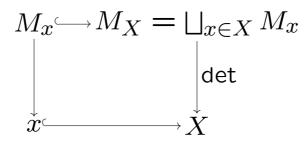
(1) is essential. (2) follows from Zariski's main theorem.

#### **▶** Strategy

Construct a suitable family of rank 4 semistable bundles near each point of  $\widehat{M}$ . Then use the universal property of  $\mathbf{S}$ .

#### 7. Moduli space of Hecke cycles

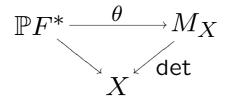
•  $M_x = \{ \text{stable } F \text{ of rank } 2, \det F \cong \mathcal{O}(-x) \} / \text{isom}$ 



ullet For  $F\in M_0^s$  and  $u\in \mathbb{P}F^*|_x$ , let

$$F^{\nu} := \ker(F \longrightarrow F_x \xrightarrow{\nu} \mathbb{C}) \in M_x$$

Define  $\theta_x : \mathbb{P}F_x^* \hookrightarrow M_x$  by  $\theta_x(\nu) = F^{\nu}$ 



•  $\Phi: M_0^s \to \mathsf{Hilb}(M_X), \quad \Phi(F) = \theta(\mathbb{P}F^*)$ 

Hilbert poly. P(n)=(4n+1)(4n-1)(g-1)  $\mathcal{O}_{M_X}(1)=K_{\det}^*\otimes(\det)^*K_X$  : ample on  $M_X$ 

**▶ Definition** (Narasimhan-Ramanan)

 $\mathbf{N} := \overline{\Phi(M_0^s)} = \text{irreducible component of } Hilb(M_X)$  containing  $\Phi(M_0^s)$ . A cycle in  $\mathbf{N}$  is called a Hecke cycle and  $\mathbf{N}$  is called the moduli of Hecke cycles.

► Theorem (Narasimhan-Ramanan)

 ${f N}$  is a nonsingular variety and  $\exists \ \pi_N : {f N} \to M_0$ , which is an isomorphism over  $M_0^s$ .

- ► Theorem (Choe-Choy-Kiem)
- (1)  $\exists$  birational morphism  $ho_N:\widehat{M} \to \mathbf{N}$
- (2)  $\mathbb{N} \cong \overline{M}$  and  $\rho_N$  is  $\widehat{M} \to \overline{M}$ .

#### **▶** Strategy

Construct a family of Hecke cycles near each point of  $\widehat{M}$ . Then use the universal property of  $\mathbf{N}$ .

#### II. Higgs Bundles over Curves

#### 1. Higgs pairs

- V= rank 2 bundle with  $\det V\cong \mathcal{O}_X$   $\phi\in H^0(\operatorname{End}_0V\otimes K_X)$   $(V,\phi)=$  an SL(2)-Higgs bundle
- $(V, \phi)$  is polystable if stable or  $(V, \phi) = (L, \psi) \oplus (L^{-1}, -\psi)$  for  $(L, \psi) \in T^*J$
- $\mathbf{M}=\{\text{polystable pairs }(V,\phi)\}/\text{isom}$  admits a structure of irreducible normal quasi-projective variety of dimension 6g-6
- stratification of M  $\mathbf{M} = \mathbf{M}^s \sqcup (T^*J/\mathbb{Z}_2 J_0) \sqcup J_0$

#### 2. Singularities of $\mathbf{M}$

- (a)  $\mathbf{M}^s$  is smooth, equipped with a (holomorphic) symplectic form, i.e.  $\mathbf{M}^s$  is hyperkähler.
  - (Kiem-Yoo) can compute  $E(\mathbf{M}^s)$  by carefully working out the subvarieties corresponding all possible types of V
- (b) (Simpson) Singularities along  $T^*J/\mathbb{Z}_2-J_0$   $\mathbb{H}^{g-1}\otimes_{\mathbb{C}}\mathbb{C}^2/\!/\!/\mathbb{C}^*$

where  $\mathbb{C}^*$  acts on  $\mathbb{C}^2$  with weights 1,-1

 desingularized by blowing up at the vertex of the cone:

$$\mathcal{O}(-1) \to \mathbb{P}(T^*\mathbb{P}^{g-2})$$

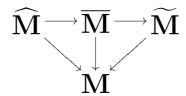
where  $\mathbb{P}(T^*\mathbb{P}^{g-2})$  is  $\mathbb{P}^{g-3}$ -bundle on  $\mathbb{P}^{g-2}$ ; a holomorphic contact manifold

# (c) (Simpson) Singularities along $J_0$ is $\mathbb{H}^g \otimes_{\mathbb{C}} \mathfrak{sl}(2) /\!/\!/ SL(2)$

- (O'Grady) desingularized by 3 blow-ups
- (O'Grady)
  - three exceptional divisors of the desingularization are smooth normal crossing
  - can describe the divisors and their intersections explicitly

#### 3. Desingularizations of ${f M}$

- M is desingularized by three blow-ups along
  - i)  $J_0$
  - ii) proper transform of  $T^*J/\mathbb{Z}_2$
  - iii) nonsingular subvariety lying in the exceptional divisor of the first blow-up
    - $\Rightarrow$  Kirwan desingularization  $\pi:\widehat{\mathbf{M}} \to \mathbf{M}.$
- $\bullet$  (O'Grady) can blow down  $\widehat{\mathbf{M}}$  twice to give three desingularizations of  $\mathbf{M}$



#### 4. Application

• The discrepancy divisor is  $(g \ge 3)$   $K_{\widehat{M}} = (6g-7)D_1 + (2g-4)D_2 + (4g-6)D_3.$ 

#### Question

Does there exist a (holomorphic) symplectic desingularization of  $\mathbf{M}$ ?

- <u>Kontsevich's theorem</u>: If there is a crepant (=symplectic) resolution of M,  $E_{st}(M)$  is a polynomial with integer coefficients.
- (Kiem-Yoo) can give an explicit formula of  $E_{st}(\mathbf{M})$  and prove that it is not a polynomial with integer coefficients for  $g \geq 3$ .  $\Rightarrow \sharp$  symplectic desingularization for  $g \geq 3$
- (O'Grady) For g=2,  $\exists$  symplectic desingularization

#### III. Sheaves on K3 and Abelian Surfaces

#### 1. Moduli space of rank 2 sheaves

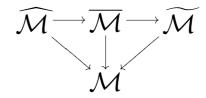
- S= K3 or Abelian surface, generic  $\mathcal{O}_S(1)$
- F = rank 2 torsion-free sheaf with $c_1(F) = 0 \text{ and } c_2(F) = 2n \text{ for } n \ge 2$
- $\mathcal{M} = \mathcal{M}_S(2,0,2n) = \{\text{polystable sheaves } F\}/\sim$  admits a structure of irreducible normal projective variety of dimension 8n-6 (K3) or 8n+2 (Abelian)
- stratification of  $\mathcal{M}$   $\mathcal{M} = \mathcal{M}^s \sqcup (\Sigma - \Omega) \sqcup \Omega$ where  $\Omega = S^{[n]}$ ,  $\Sigma = \operatorname{Sym}^2(S^{[n]})$  (K3 case) or  $\Omega = S^{[n]} \times \widehat{S}$ ,  $\Sigma = \operatorname{Sym}^2(S^{[n]} \times \widehat{S})$  (Abelian)

#### 2. Singularities of $\mathcal{M}$

- (a) (Mukai)  $\mathcal{M}^s$  is smooth, equipped with a (holomorphic) symplectic form, i.e.  $\mathcal{M}^s$  is hyperkähler.
- (b) (O'Grady) Singularities along  $\Sigma-\Omega$   $\mathbb{H}^{n-1}\otimes_{\mathbb{C}}\mathbb{C}^2/\!/\!/\mathbb{C}^*$  where  $\mathbb{C}^*$  acts on  $\mathbb{C}^2$  with weights 1,-1
- (c) (O'Grady) Singularities along  $\Omega$  is  $\mathbb{H}^n \otimes_{\mathbb{C}} \mathfrak{sl}(2)/\!/\!/SL(2)$ 
  - desingularized by 3 blow-ups
  - three exceptional divisors of the desingularization are smooth normal crossing
     can describe the divisors and their intersections explicitly

# 3. Desingularizations of ${\mathcal M}$

- $\mathcal{M}$  is desingularized by three blow-ups  $\Rightarrow$  Kirwan desingularization  $\pi:\widehat{\mathcal{M}}\to\mathcal{M}$ .
- $\bullet$  can blow down  $\widehat{\mathcal{M}}$  twice to give three desingularizations of  $\mathcal{M}$



- (O'Grady) When  $\dim \mathcal{M}=10$ ,  $\widetilde{\mathcal{M}}$  is a symplectic desingularization of  $\mathcal{M}$ .
  - ⇒ 2 new irreducible symplectic manifolds!
- Question (O'Grady) Does there exist a symplectic (or crepant) desingularization of  $\mathcal M$  when  $\dim \mathcal M > 10$ ?

- (Choy-Kiem) can give an explicit formula of  $E_{st}(\mathcal{M}) E(\mathcal{M}^s)$  and prove that  $E_{st}(\mathcal{M})$  is not a polynomial when dim  $\mathcal{M} > 10$ .  $\Rightarrow \#$  symplectic desingularization when dim  $\mathcal{M} > 10$  by Kontsevich's theorem.
- Kaledin-Lehn-Sorger proved this nonexistence result by showing  $\mathbb{Q}$ -factoriality of  $\mathcal{M}$ .

#### IV. Questions

• Are the desingularizations

$$\overline{\mathbf{M}}, \widetilde{\mathbf{M}}$$
 of  $\mathbf{M}$  and  $\overline{\mathcal{M}}, \widetilde{\mathcal{M}}$  of  $\mathcal{M}$ 

moduli spaces of some natural classes of objects as in the curve case?

[Choy proved that  $\widetilde{\mathcal{M}}$  is the moduli space analogous to Seshadri's.]

- When does the stringy E-function  $E_{st}(Y)$  of a projective (singular) variety Y coincide with the E-polynomial IE(Y) of intersection cohomology  $IH^*(Y)$ ?
- What is the equivariant version  $E_{st}(Y,G)$  of stringy E-function when a reductive group G is acting on a (singular) variety Y?

Thank you!!