Derived Categories and Birational Geometry

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X smooth projective variety $/\mathbb{C}$

$\mathrm{Coh}(X)$ abelian category of coherent sheaves.

biregular geometry

$D^b(\mathrm{Coh}(X))$ bounded derived cat.

birational geometry

triangulated

$\mathbb{C}$-linear

finite type $\sum_{k \in \mathbb{Z}} \dim \underbrace{\mathrm{Hom}^k(A,B)}_{\mathrm{Hom}(A,\,B[k])} < \infty$

explicit structure

(ex) $X = \mathbb{P}^n$

$$\Big[0 \to \Omega^n_X(n) \boxtimes \mathcal{O}_X(-n) \longrightarrow \cdots$$
$$\to \Omega^2(2) \boxtimes \mathcal{O}(-2) \to \Omega^1(1) \boxtimes \mathcal{O}(-1)$$
$$\to \mathcal{O}_{X\times X} \longrightarrow \Big] \mathcal{O}_\Delta \longrightarrow 0$$
$$\overset{\shortparallel}{E} \in D^b(\mathrm{Coh}(X \times X))$$

$$A \cong \Phi^E(A) = Rp_{2*}(p_1^* A \overset{L}{\otimes} E)$$

two-sided resolution of $A \in \mathrm{Coh}\, X$

$$\Big\{ 0 \to \bigoplus_{p=0}^{n} H^{p-n}(X, A \otimes \Omega^p_X(p)) \otimes \mathcal{O}_X(-p)$$
$$\longrightarrow \bigoplus_{p=0}^{n} H^{p-n+1}(X, A \otimes \Omega^p(p)) \otimes \mathcal{O}(-p)$$
$$\longrightarrow \cdots$$
$$\longrightarrow \bigoplus_{p=0}^{n} H^{p+n}(X, A \otimes \Omega^p(p)) \otimes \mathcal{O}(-p)$$
$$\longrightarrow 0 \Big\} \cong A$$

Beilinson

(ex) $A_i = \mathcal{O}(-n+i-1)$

exceptional collection $(A_1, \cdots, A_m)$

$$\mathrm{Hom}^i(A_j, A_j) = \begin{cases} \mathbb{C} & i \neq 0 \\ 0 & \text{other} \end{cases}$$

$$\mathrm{Hom}^i(A_j, A_k) = 0 \qquad j > k$$

strong : $\mathrm{Hom}^i(A_k, A_j) = 0 \qquad i \neq 0$

complete : generates as a smallest triangulated category

$$A = \bigoplus A_j, \quad R = \mathrm{Hom}(A, A)$$

$$\begin{array}{ccc} \Phi : D^b(X) & \xrightarrow{\sim} & D^b(\text{mod-}R) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ B & \longmapsto & R\mathrm{Hom}(A, B) \\ M \overset{L}{\underset{R}{\otimes}} A & \longleftarrow & M \end{array}$$

quiver

$$\underset{\mathcal{O}(-n)}{\bullet} \xrightarrow{n+1} \underset{\mathcal{O}(-n+1)}{\bullet} \xrightarrow{n+1} \bullet \to \cdots \to \underset{\mathcal{O}(-1)}{\bullet} \xrightarrow{n+1} \underset{\mathcal{O}}{\bullet}$$

[redacted]

Bondal-Kapranov

Semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{B}, {}^{\perp}\mathcal{B} \rangle = \langle \mathcal{B}^{\perp}, \mathcal{B} \rangle$$

$\mathcal{B}$ triangulated subcategory

$${}^{\perp}\mathcal{B} = \{ A \in \mathcal{D} \mid \mathrm{Hom}^{p}(A, B) = 0, B \in \mathcal{B} \}$$

$\forall A \in \mathcal{D}$

$$\begin{tikzcd} B \arrow[r] & A \arrow[r] & C \arrow[ll, bend left, dashed] \end{tikzcd} \qquad C \in {}^{\perp}\mathcal{B}.$$

(ex) $D(\mathbb{P}^n) = \langle \mathcal{O}(-n), \cdots, \mathcal{O}(-1), \mathcal{O} \rangle$

(rem) no orthogonal decomp.

$\mathcal{B}$ saturated $\Rightarrow$ SO decomp.

(ex) $D^b(\mathrm{Coh}\, X)$ saturated

cf. Brown representability

(ex 1) projective space bundle

Mori fiber space

$$\phi : Y = \mathbb{P}(V) \longrightarrow X$$

$$D(X)_k = \phi^* D(X) \otimes \mathcal{O}_Y(k)$$

$$D(Y) = \langle D(X)_{-r+1}, \cdots, D(X)_{-1}, D(X)_0 \rangle$$

Orlov

(ex 2) blowing-up

$$\begin{tikzcd} Y \arrow[r, "\phi"] & X \\ F \arrow[u, phantom, "\cup", "j"'] \arrow[r] & E \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$D(Y) = \langle j_* D(E)_{-r+1}, \cdots, j_* D(E)_{-1}, \phi^* D(X) \rangle$$

divisorial contraction

(ex 3) flip

$$\begin{tikzcd} & F \subset Y \arrow[dl] \arrow[d, "f"'] \arrow[dr, "f^+"] \arrow[drr, bend left] & & \\ \mathbb{P}_G(V) = E \arrow[r, phantom, "\subset"] \arrow[dr] & X \arrow[d, "\phi"] & X^+ \arrow[dl, "\phi^+"] \arrow[r, phantom, "\supset"] & E^+ = \mathbb{P}_G(V^+) \arrow[dll, bend left] \\ & G \subset Z & & \end{tikzcd}$$

$$N_{E/X}|_{\phi^{-1}(g)} \simeq \mathcal{O}_{\mathbb{P}^{s-1}}(-1)^r$$

flip $s > r$

flop $s = r$

Bondal Orlov

$$D(X) = \langle j_* D(E)_{r-s}, \cdots, j_* D(E)_{-1}, \Phi(D(X^+)) \rangle$$

$$\Phi = f_* f^{+*} : D(X^+) \longrightarrow D(X) \quad \text{f.f.}$$

generalize to MMP ?

(ex) X Fano 3 fold $\rho = 1$, $K^3 = -12$

C moduli of v.b. on X

$c_1 = 1$, $c_2 = 5$

genus 7 curve

E univ. bundle on $X \times C$

$\Phi^E : D(C) \longrightarrow D(X)$ f.f.

U_+ : rk 5 spinor bdle on X

$$D(X) = \langle U_+, \mathcal{O}_X, D(C) \rangle$$ Kuznetsov

(ex) $H^i(X, \mathcal{O}_X) = 0 \quad i > 0$ Fano Enriques. ...

$\Rightarrow \mathcal{O}_X$ exceptional

$\Rightarrow D(X) = \langle \mathcal{O}_X, \mathcal{B} \rangle$

Serre functor (unique)

$S : D \longrightarrow D$

$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, S(A))^*$

smooth proj X. $S(A) = A \otimes \omega_X[\dim X]$

K can be recoverd.

(not $\mathrm{Coh}\,X$)

(ex) $K = 0 \Rightarrow$ no SO decomp

$\because$) SO $\Rightarrow \mathcal{O}$.

<u>rem</u> $f : X \to Y$ $Rf_*\mathcal{O}_X = \mathcal{O}_Y$

$\Rightarrow D(qcX) = \langle C, D(qcY) \rangle$

$C = \{A \mid Rf_*A = 0\}$

<u>rem</u> $\mathrm{Perf}(X) \underset{d}{\longrightarrow} D^b(\mathrm{Coh}\,X)$

for singular X

Minimal model program Log MMP

decreasing K (or $K+B$)

X normal variety (X,B) B: $\mathbb{Q}$-div

K_X $\mathbb{Q}$-Cartier $K+B$ $\mathbb{Q}$-Cartier

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$K_X > K_Y$

$\Leftrightarrow$ $f^*K_X - g^*K_Y$ effective

minimal model = minimal K

not unique, but K-equivalent

(ex) X = total space of $\mathcal{O}_E(-k)$ $k>0$

$$\begin{tikzcd} X \arrow[d] \arrow[ddl, "\phi"'] & \\ E = \mathbb{P}^{n-1} \arrow[d] & \\ Y \ni pt & \end{tikzcd}$$

$$K_X = \phi^* K_Y + \frac{n-k}{k} E$$

$$n > k \Leftrightarrow K_X > K_Y$$

<u>singularity</u> appears in dim ≥ 3

(1) Mori fiber space

$\phi : X \to Y \quad \dim Y < \dim X$

(2) divisorial contraction

$\phi : X \to Y$ birat morphism

contracts a prime divisor

(3) flip $\phi : X \dashrightarrow Y$ birat map

isomorphic in codim 1

(X, B) X : $\mathbb{Q}$-factorial

projective $/S$

B : $\mathbb{Q}$ (or $\mathbb{R}$)-divisor

minimalist : terminal (log terminal)

maximalist : canonical (log canonical)

$f : Y \to X$

$K_Y = f^*(K_X + B) + \sum a_j E_j \quad a_j > 0$

$(a_j \geq 0)$

$a_j > -1 \ (a_j \geq -1)$

$$\mathrm{Perf}(X) \longrightarrow D(X) \longrightarrow D^b(\mathrm{Coh}\,X)$$

"homology" "cohomology"

Assumption :

(1) $B = \sum b_i B_i$, $b_i = 1 - \frac{1}{r_i}$, $r_i \in \mathbb{Z}_{>0}$

(2) $\exists$ smooth $U \xrightarrow{f} X$ quasi-finite

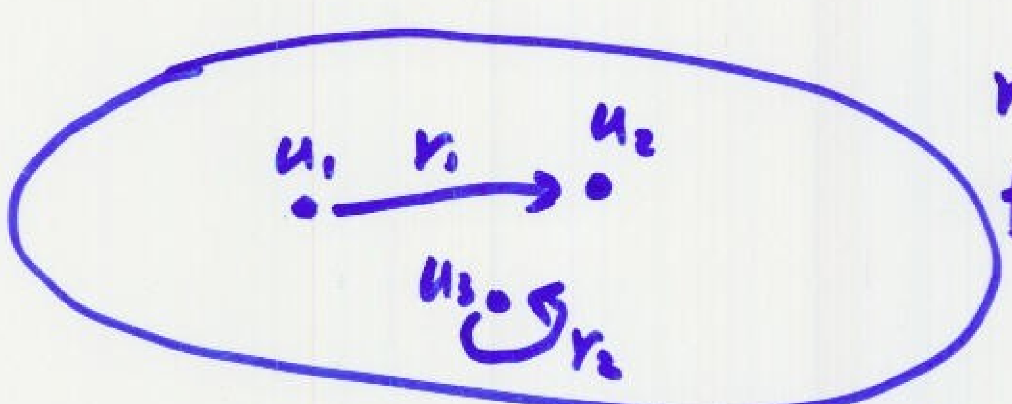
$f^*(K_X + B) = K_U$

$$R = (U \times_X U)^\sim \overset{p_1}{\underset{p_2}{\rightrightarrows}} U \quad \text{etale}$$

$\mathfrak{X} : (\mathrm{Sch}) \to (\mathrm{Groupoid})$ DM stack

$\mathrm{Obj}\,\mathfrak{X}(S) = \mathrm{Hom}(S, U)$

$\mathrm{Mor}\,\mathfrak{X}(S) = \mathrm{Hom}(S, R)$



$r = (u_1, u_2)$

$f(u_1) = f(u_2)$

$\mathfrak{X}(\mathrm{Spec}\,\mathbb{C})$

sheaf on $\mathfrak{X}$

= sheaf on U equivariant w.r.t. R

$\mathrm{Coh}(\mathfrak{X})$ abelian cat

$D(X,B) = D^b(\mathrm{Coh}\,\mathfrak{X})$

Serre functor : $\otimes\, \mathcal{O}_{\mathfrak{X}}(K_X+B)\,[\dim X]$.

(ex) $X = \mathbb{P}(a_0, \cdots, a_n)$. $B = 0$

$D(X) = \langle \mathcal{O}_{\mathfrak{X}}(-\sum a_i + 1), \cdots, \mathcal{O}_{\mathfrak{X}}(-1), \mathcal{O}_{\mathfrak{X}} \rangle$

strong exceptional collection

$\mathcal{O}_X(-P) = \pi_* \mathcal{O}_{\mathfrak{X}}(-P)$ not exceptional

<u>Conj</u> $(X, B), (Y, C), \mathfrak{X}, \mathfrak{Y}$ as above

$K_X + B \leq K_Y + C$

$\Longrightarrow \exists\, \Phi : D(X,B) \to D(Y,C)$ f.f.

equality $\Longrightarrow$ equivalence.

(K-equivalence) (D-equivalence)

BKR : $G \subset SL(n, \mathbb{C})$ finite

$X = \mathbb{C}^n/G$, $(B = 0)$

$\mathfrak{X} = [\mathbb{C}^n/G]$

$Y = G\text{-Hilb}(\mathbb{C}^n) \xrightarrow{f} X$

assume $\dim Y \times_X Y \leq n+1$

$\Rightarrow$ (1) Y smooth, $K_Y = f^* K_X$

(2) $D(Y) \simeq D(X)$

(ex) $n = 3$, $G \subset Sp(2m, \mathbb{C})$

(ex)
$$\begin{tikzcd}
X \arrow[r] & Y & \mathcal{Y} \arrow[l] \\
\mathbb{P}^{n-1} \simeq E \arrow[u, phantom, "\cup" description] \arrow[r] & pt \arrow[u, phantom, "\rotatebox{90}{$\in$}" description] & Q \arrow[l] \arrow[u, phantom, "\rotatebox{90}{$\in$}" description]
\end{tikzcd}$$

$$N_{E/X} \simeq \mathcal{O}(-k)$$

$n > k$ $\quad D(X) \simeq \langle \mathcal{O}_E(-n+k), \cdots, \mathcal{O}_E(-1), D(Y) \rangle$

$n = k$ $\quad D(X) \simeq D(Y)$

$n < k$ $\quad D(Y) \simeq \langle \mathcal{O}_Q(-n), \cdots, \mathcal{O}_Q(-k+1), D(X) \rangle$

$\mathcal{O}_Q$ skyscraper of length 1

stabilizer $\mathbb{Z}_k$ acts by $t \to t^k$.
(μ_k)

(ex)
$$\begin{tikzcd}
 & & Y = X \times_Z X^+ \arrow[dl, "f"'] \arrow[dr, "f^+"] & & \\
E \subset & X \arrow[dr, "\phi"] & & X^+ \arrow[dl, "\phi^+"'] & \supset E^+ \\
& & Z & & \\
& & Q \in Z \arrow[from=uull, bend right] \arrow[from=uurr, bend left] & &
\end{tikzcd}$$

$\dim X = 4$ smooth, $E \simeq \mathbb{P}^2$,

$N_{E/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2)$

$E^+ \simeq \mathbb{P}^1$. $\mathrm{Sing}\, X^+ = \{P\} \subset E^+$

$P \in \mathcal{X}^+$ stabilizer μ_2.

$K_X = K_{X^+}$

$$\mathcal{Y} = X \times_Z \mathcal{X}^+ \quad \begin{matrix} \xrightarrow{\tilde f} X \\ \xrightarrow[\tilde f^+]{} \mathcal{X}^+ \end{matrix}$$

$$\Phi = R\tilde f^+_* L\tilde f^* : D(X) \to D(X^+)$$

but $Rf^+_* Lf^*(\Omega_E(-1)) = 0$

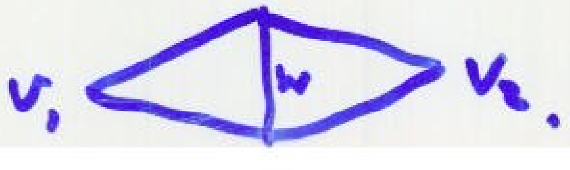
$$\Phi(\Omega_E(-1)) = \mathcal{O}_P(1).$$

(X, B) $\mathbb{Q}$-factorial toric
B: invariant $\mathbb{Q}$-divisor.

$\leftrightarrow$ simplicial fan $\Delta_X \subset N_X \otimes \mathbb{R}$

$\varphi : X \to Z$ extremal contraction

$\leftrightarrow$ wall $w = \langle v_3, \cdots, v_{n+1} \rangle$
corresponding to an extr. curve.



$$a_1 v_1 + a_2 v_2 + \cdots + a_{n+1} v_{n+1} = 0$$

$$a_i \in \mathbb{Z}, \quad \gcd(a_i) = 1$$

$$\begin{cases} a_i > 0 & 1 \le i \le \alpha \\ a_i = 0 & \alpha+1 \le i \le \beta \\ a_i < 0 & \beta+1 \le i \le n+1 \end{cases}$$

$$2 \le \alpha \le \beta \le n+1$$

$$\begin{cases} \text{Mori fiber space} & \beta = n+1 \\ \quad (\dim Z = n+1-\alpha) & \\ \text{divisorial contraction} & \beta = n \\ \text{flipping contraction} & \beta < n \end{cases}$$

D_i prime divisor $\longleftrightarrow v_i$

$$B = \sum \frac{r_i - 1}{r_i} D_i$$

$$\rightsquigarrow \pi^* D_i = r_i \mathcal{D}_i \qquad \pi : \mathcal{X} \to X$$

$K_X + B$ φ-negative

$$\longleftrightarrow \sum_{i=1}^{n+1} \frac{a_i}{r_i} > 0$$

⓪ Fano, $\rho = 1$ case $(\alpha = n+1)$

$$\begin{tikzcd}
\mathbb{P}(a_1 : \cdots : a_{n+1}) \arrow[r] & X & \text{etale in codim 1} \\
\widetilde{\mathbb{P}}(a_1, \cdots, a_{n+1}) \arrow[r] \arrow[u] & \mathcal{X} \arrow[u] & \text{etale}
\end{tikzcd}$$

$$\left\langle \mathcal{O}_{\mathcal{X}}\left(\sum_i k_i D_i\right) \,\middle|\, 0 \geq \sum_i \frac{a_i k_i}{r_i} > -\sum_i \frac{a_i}{r_i} \right\rangle$$

strong complete exceptional collection

① Mori fiber space

$$\begin{array}{ccc}
N_X & \longrightarrow & N_Z = N_X / \left(\bigoplus_{i=1}^{\alpha} \mathbb{R} v_i \cap N_X \right) \\
\cup & & \cup \\
v_i & \longrightarrow & a_i \bar{v}_i , \quad \alpha+1 \leq i \leq n+1
\end{array}$$

$$C = \sum \left(1 - \frac{1}{r_i a_i}\right) E_i \quad \mathbb{Q}\text{-div. on } Z$$

$$E_i := \varphi(D_i)$$

$$\rightsquigarrow \quad \begin{tikzcd}
X \arrow[r, "\varphi"] & Z \\
\mathcal{X} \arrow[r, "\psi", "\text{etale}"'] \arrow[u] & \mathcal{Z} \arrow[u]
\end{tikzcd}$$

$$D(X) = \left\langle \psi^* D(\mathcal{Z}) \otimes \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{\alpha} k_i D_i\right) \,\middle|\, 0 \geq \sum_{i=1}^{\alpha} \frac{a_i k_i}{r_i} > \sum^{\alpha} \frac{a_i}{r_i} \right\rangle$$

SO decomp.

② divisorial contraction

$D = D_{n+1}$ exc. div, $E_i = \varphi_* D_i \quad 1 \le i \le n$

$$\begin{tikzcd}
& \mathcal{W} = (\mathfrak{X} \times \mathfrak{Y})^{\sim} \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\psi\downarrow"] & \\
\mathfrak{X} \arrow[d] & & \mathfrak{Y} \arrow[d] \\
X \arrow[rr] & & Y = Z
\end{tikzcd}$$

$$f_* g^* \mathcal{O}_{\mathfrak{Y}}\Big(\sum_{i=1}^{n} k_i E_i\Big) = \mathcal{O}_{\mathfrak{X}}\Big(\sum_{i=1}^{n+1} k_i D_i\Big)$$

$$k_{n+1} = \left\lfloor \frac{r_{n+1}}{b_{n+1}} \sum_{i=1}^{n} \frac{a_i k_i}{r_i} \right\rfloor, \quad b_{n+1} = -a_{n+1}$$

$\bar{\varphi} : D \to F = \varphi(D)$ Mori fiber space

$$N_X \longrightarrow N_Y = N_X / \mathbb{Z} v_{n+1}$$
$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_i & \longmapsto & t_i \bar{v}_i \end{array} \quad 1 \le i \le n$$

i.e. $D_i|_D = \frac{1}{t_i} \bar{D}_i$

but $j^* \mathcal{O}_{\mathfrak{X}}(D_i) = \mathcal{O}_{\mathfrak{D}}(\bar{D}_i)$

for $j : \mathfrak{D} \to \mathfrak{X}$

$t = \gcd(a_i t_i)$, $a_i t_i = t \bar{a}_i$

$\Rightarrow \quad \bar{a}_1 \bar{v}_1 + \cdots + \bar{a}_n \bar{v}_n = 0$

$$\bar{B} = \sum_{i=1}^{n} \Big(1 - \frac{1}{r_i t_i}\Big) \bar{D}_i \quad \text{on } D$$

$$D(X) = \langle j_* \bar{\varphi}^* D(F) \otimes \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i D_i),$$
$$f_* g^* D(Y) \mid 0 > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \geq - \sum_{i=1}^{n+1} \frac{a_i}{r_i} \rangle$$

SO decomposition

④ flip

$$\begin{tikzcd} X \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & Z & \end{tikzcd}$$

$$\bar{\varphi} : D = \bigcap_{i=p+1}^{n+1} D_i \longrightarrow F = \varphi(D) \quad \text{Mfs.}$$

$$\begin{tikzcd} & \mathcal{W} = (\mathcal{X} \times_Z \mathcal{X}^+)^\sim \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} & & \mathcal{X}^+ \end{tikzcd}$$

$$f_* g^* \mathcal{O}_{\mathcal{X}^+}(\sum_{i=1}^{n+1} k_i D_i^+) = \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i D_i)$$

if $0 \leq \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < \sum_{i=p+1}^{n+1} \frac{b_i}{r_i}$, $b_i = -a_i$

$$D(X) = \langle j_* \bar{\varphi}^* D(F) \otimes \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i D_i),$$
$$f_* g^* D(X^+) \mid 0 > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \geq - \sum_{i=1}^{n+1} \frac{a_i}{r_i} \rangle$$

SO decomp.

Cor complete exceptional collection

<u>dim 3 case</u>

(X, P) germ of a terminal singularity

$(X', P') \xrightarrow[m:1]{f} (X, P)$ canonical covering

$mK_X \sim 0$ X' Gorenst.

(universal covering in codim 1)

X normal projective, terminal

$\mathfrak{X}$ canonical covering stack

$D'(X) = D^b(\mathrm{Coh}\, \mathfrak{X})$

<u>Th</u> $K_X \equiv K_Y$

$\Rightarrow D'(X) \cong D'(Y)$

Bridgeland, Chen, K., VdBergh.

Consequences of D-equivalence

Th (X, B), (Y, C) satisfy assumptions

$\Phi : D(X, B) \to D(Y, C)$ f.f.

$D(Y, C) = \langle D(X, B)^{\perp}, D(Y, C) \rangle$

$\Rightarrow \exists_! E \in D^b(\mathcal{X} \times \mathcal{Y})$

s.t. $\Phi(-) = p_{2*}(p_1^* - \otimes E)$.

idea: L sufficiently ample on X

$$A = \bigoplus_{m \geq 0} A_m = \bigoplus H^0(X, mL)$$

$$B_m = \mathrm{Ker}(B_{m-1} \otimes_{\mathbb{C}} A_1 \to B_{m-2} \otimes_{\mathbb{C}} A_2)$$

$$R_m = \mathrm{Ker}(B_m \otimes_{\mathbb{C}} \mathcal{O}_X \to B_{m-1} \otimes_{\mathbb{C}} L)$$

$$\to L^{-m} \boxtimes R_m \to \cdots \to L^{-1} \boxtimes R_1 \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_{\Delta X} \to 0$$

$$E \leftrightarrow \{ \to L^{-m} \boxtimes \Phi(R_m) \to \cdots \}$$

Orlov, K.

Th $(X, B), (Y, C)$ satisfy cond.

$D(X, B) \simeq D(Y, C)$

$\Rightarrow$ (1) $\dim X = \dim Y$

(2) $\bigoplus_{m \in \mathbb{Z}} H^0(X, \llcorner m(K_X + B) \lrcorner)$

$\simeq \bigoplus_{m \in \mathbb{Z}} H^0(Y, \llcorner m(K_Y + C) \lrcorner)$

hence $\kappa(X, \pm(K+B)) = \kappa(Y, \pm(K+C))$

(3) $\pm(K_X + B)$ nef $\Leftrightarrow$ $\pm(K_Y + C)$ nef

$\nu(\pm(K_X + B)) = \nu(\pm(K_Y + C))$

(4) If $\kappa(X, \pm(K+B)) = \dim X$,

then $X \underset{bir}{\sim} Y$, $K_X + B = K_Y + C$.

idea: K is categorical, because Serre functor is unique.

related conjecture

(1) given X, $\#\{Y \mid \text{minimal}, Y \underset{\text{bir}}{\sim} X\}/_{\cong}$
$< \infty$.

(2) given X, $\#\{Y \mid D(X) \simeq D(Y)\}/_{\cong}$
$< \infty$.