Derived equivalences of twisted K3 surfaces

D. Huybrechts (Bonn)

(+ P. Stellari (Milano))

$C^{gr}(X):=$

K3 surface = smooth, projective $/\mathbb{C}$

$\omega_X \simeq \mathcal{O}_X,\ H^1(X, \mathcal{O}_X) = 0$

<u>Global Torelli Theorem:</u>

Let X, X' be K3 surfaces. Then

$X \simeq X' \iff \exists$ Hodge isometry

$\varphi : H^2(X,\mathbb{Z}) \simeq H^2(X',\mathbb{Z})$

$f : X \to X' \longmapsto \varphi := f_*$ (ample $\mapsto$ ample)

$\exists^1 f \longleftarrow$ If φ with – " –

<u>Example:</u> $\mathbb{P}^1 \simeq C \subset X$

$\rightsquigarrow [C] \in H^2(X,\mathbb{Z})$ with $[C]^2 = -2$

$\rightsquigarrow$ reflection in $[C]^\perp$:

$s_{[C]} : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$

$\gamma \longmapsto \gamma + (\gamma.[C])[C]$

Hodge isometry.

<u>Aim</u>: Survey of known generalizations (for K3 surfaces).

$\rightsquigarrow$ Derived Global Torelli

$X \rightsquigarrow \mathrm{Coh}(X)$

$\rightsquigarrow D^b(X)$

$H^2(X,\mathbb{Z}) \rightsquigarrow H^*(X,\mathbb{Z})$

$X \rightsquigarrow (X, \alpha \in \mathrm{Br}(X))$

$\rightsquigarrow \mathrm{Coh}(X,\alpha)$

$\rightsquigarrow D^b(X,\alpha)$

$X \rightsquigarrow (X$, B-field $B)$

$\rightsquigarrow H^*(X,B,\mathbb{Z})$

twisted Chern

$\mathrm{Coh}(X)$:= abelian category of coherent sheaves

$D^b(X) := D^b(\mathrm{Coh}(X))$:= its bounded derived category

- Study K3 surfaces X, X' with $\mathrm{Coh}(X) \simeq \mathrm{Coh}(X')$ or $D^b(X) \simeq D^b(X')$ (as triang. categories).
- Study twisted K3 surfaces $(X, \alpha), (X', \alpha')$ with $\mathrm{Coh}(X, \alpha) \simeq \mathrm{Coh}(X', \alpha')$ or $D^b(X, \alpha) \simeq D^b(X', \alpha')$.

<u>Gabriel:</u> $\mathrm{Coh}(X, \alpha) \simeq \mathrm{Coh}(X', \alpha')$
$\Leftrightarrow \exists f : X \simeq X' : f^* \alpha' = \alpha$

(X, X' algebraic and α, α' torsion)

Let $\alpha \in H^2(X, \mathcal{O}_X^\times)$ be represented by $\{\alpha_{ijk} \in \mathcal{O}^\times(U_i \cap U_j \cap U_k)\}$

$X = \bigcup U_i$ analytic open cover

$\{\alpha_{ijk}\}$ - twisted sheaf

$$\hat{=} \{ (E_i \in Coh(U_i), \varphi_{ij} : E_i|_{U_{ij}} \simeq E_j|_{U_{ij}})$$

s.t. $\cdot\ \varphi_{ij}\varphi_{ji} = id$

$\cdot\ \varphi_{ij}\varphi_{jk}\varphi_{ki} = \alpha_{ijk} \cdot id \}$

$Coh(X,\alpha) := Coh(X, \{\alpha_{ijk}\})$

:= abelian category of $\{\alpha_{ijk}\}$-twisted sheaves

$D^b(X,\alpha) := D^b(Coh(X, \{\alpha_{ijk}\}))$

<u>Remark</u>: If $[\{\alpha_{ijk}\}] = [\{\alpha'_{ijk}\}] \in H^2(X, \mathcal{O}^\times)$, then $Coh(X, \{\alpha_{ijk}\}) \simeq Coh(X, \{\alpha'_{ijk}\})$

(not canonical).

Examples of derived equivalences

Mukai: A = abelian variety

$\mathcal{P}$ = Poincaré bundle on $A \times \hat{A}$

$$\begin{tikzcd} & A \times \hat{A} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A & & \hat{A} \end{tikzcd}$$

$\rightsquigarrow \Phi_{\mathcal{P}} : D^b(A) \xrightarrow{\sim} D^b(\hat{A})$

$\mathcal{F} \longmapsto Rp_*(q^*\mathcal{F} \otimes^L \mathcal{P})$

Mukai: X = K3 surface

X' = fine, projective moduli space of stable sheaves on X

$\dim(X') = 2$

$\mathcal{E}$ = universal sheaf on $X \times X'$

$\rightsquigarrow \Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(X')$.

Căldăraru: X = K3 surface

X' = coarse, ...

$\rightsquigarrow \Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(X', \alpha')$ for certain $\alpha' \in H^2(X', \mathcal{O}^*_{X'})$.

($\mathcal{E} = p^*\alpha'$-twisted universal sheaf)

Hodge structures

$X =$ K3 surface $\rightsquigarrow$ $H^2(X,\mathbb{Z})$ with

- intersection pairing $(.)$ (sig $=(3,19)$)
- weight-two Hodgestructure:

$$H^2(X,\mathbb{C}) = \underbrace{H^{2,0}(X)}_{=\mathbb{C}\sigma} \oplus H^{1,1}(X) \oplus \underbrace{H^{0,2}(X)}_{=\mathbb{C}\bar{\sigma}}$$

(orthogonal wrt $(.)$)

$$\rightsquigarrow \widetilde{H}(X,\mathbb{Z}) = H^*(X,\mathbb{Z})$$
$$= H^2(X,\mathbb{Z}) \oplus (H^0 \oplus H^4)(X,\mathbb{Z})$$

- Mukai pairing $(.)$ (sig $=(4,20)$)
- weight-two Hodgestructure:

$$\widetilde{H}^{2,0}(X) = H^{2,0}(X),\ \widetilde{H}^{0,2}(X) = H^{0,2}(X)$$
$$\widetilde{H}^{1,1}(X) = (\widetilde{H}^{2,0} \oplus \widetilde{H}^{0,2})^{\perp}$$
$$= H^{1,1}(X) \oplus (H^0 \oplus H^4)(X,\mathbb{C})$$

Remark: $\mathrm{Re}\,\sigma, \mathrm{Im}\,\sigma, 1 - h^2/2, h$ with $h \in H^2$ ample span a positive four-plane. The order fixes an orientation of the positive directions.

Deformations of X yield deformations of $\widetilde{H}^{2,0} \subset \widetilde{H}$

(which stay in H^2).

<u>Q</u>: What is the geometric meaning of more general deformations of $\widetilde{H}^{2,0} \subset \widetilde{H}$.

<u>A</u>: Hitchin's generalized Calabi-Yau structures. (Ex: volume forms, symplectic structures, B-field twists.)

Fix B-field $B \in H^2(X, \mathbb{Q})$

$\rightsquigarrow$ $\widetilde{H}(X, B, \mathbb{Z}) = H^*(X, \mathbb{Z})$ with

- Mukai pairing
- weight-two Hodge structure:

$$\widetilde{H}^{2,0}(X, B) = \mathbb{C}\,(\sigma + \underbrace{\sigma \wedge B}_{= \sigma \wedge B^{0,2} \in H^4})$$

$$\widetilde{H}^{1,1}(X, B) = (\widetilde{H}^{2,0} \oplus \widetilde{H}^{0,2})(X, B)^{\perp}$$

B-field $\longrightarrow$ Brauer class

$B \in H^2(X,\mathbb{Q}) \rightsquigarrow B^{0,2} \in H^2(X,\mathcal{O}) \rightsquigarrow \alpha_B = \exp(B^{0,2})$
$\in H^2(X, \mathcal{O}_X^*)$

Derived Global Torelli Theorem:

Let $(X, \alpha = \alpha_B)$, $(X', \alpha' = \alpha_{B'})$ be twisted K3s

Then

$D^b(X,\alpha) \simeq D^b(X',\alpha') \iff \exists$ Hodge isometry

$\varphi : \widetilde{H}(X, B, \mathbb{Z}) \simeq \widetilde{H}(X', B', \mathbb{Z})$

(which preserves orientation)

- α, α' trivial: Mukai, Orlov
- special cases: Căldăraru, Donagi-Pantev
- general case: H. - Stellari

(use moduli spaces of twisted sheaves: Yoshioka, Lieblich, Simpson)

Roughly: $\Phi : D^b(X,\alpha) \simeq D^b(X',\alpha')$

$\longmapsto \varphi = \Phi_*$

$\exists\, \Phi \qquad \longleftarrow \qquad \varphi$

Untwisted

$\Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(X')$, $\mathcal{F} \longmapsto Rp_*(q^*\mathcal{F} \otimes^L \mathcal{E})$

$\rightsquigarrow \Phi_{\mathcal{E}*} : \widetilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', \mathbb{Z})$

$\gamma \longmapsto p_*(q^*\gamma . ch(\mathcal{E}) . \sqrt{td})$

<u>Mukai</u>: integral Hodge isometry.

<u>Example</u>: $f : X \xrightarrow{\sim} X'$

$\Rightarrow f_* = [ch(\mathcal{O}_{\Gamma_f}) . \sqrt{td}]_*$

$\mathcal{E} = \mathcal{O}_{\Gamma_f}$

Twisted

$\Phi_{\mathcal{E}} : D^b(X, \alpha) \xrightarrow{\sim} D^b(X', \alpha')$

with $\mathcal{E} \in D^b(X \times X', \alpha^{-1} \boxtimes \alpha')$.

Need $ch(\mathcal{E})$ for twisted sheaves.

Twisted Chern character

$X =$ arbitrary

$B = [\{B_{ijk}\}] \in H^2(X, \mathbb{Q})$

$\rightsquigarrow \alpha_B = [\{\alpha_{ijk} = \exp(B_{ijk})\}] \in H^2(X, \mathcal{O}_X^*$

$K(X, \{\alpha_{ijk}\}) =$ K-group (!) of α_B-twis sheave

$$\begin{tikzcd}
K(X,\{\alpha_{ijk}\}) \arrow[d, "\color{red}{ch^B}"] & (E_i, \varphi_{ij}) \arrow[d, mapsto] \\
H^*(X, \mathbb{Q}) & ch(E_i, \varphi_{ij} \exp(-a_{ij}))
\end{tikzcd}$$

Here $a_{ij} \in \mathcal{C}^\infty(U_i \cap U_j)$ with
$$a_{ij} + a_{jk} + a_{ki} = B_{ijk}.$$

Then $(E_i, \varphi_{ij} \exp(-a_{ij}))$ is a $\mathcal{C}^\infty$-sheaf.

Note: ch^B is independent of $\{a_{ij}\}$.

General theory:

- ch^B is additive: $ch^B(E \oplus E') = ch^B(E) + ch^B(E')$

- ch^B is multiplicative:
$$ch^B(E) \cdot ch^{B'}(E') = ch^{B+B'}(E \otimes E')$$

- If $B = c_1(L) \in H^2(X, \mathbb{Z})$ $(\Rightarrow \alpha_B = 1)$, then $ch^B = \exp(B) \cdot ch$.

- If $B = [\{B_{ijk}\}] = [\{B'_{ijk}\}]$, then

$$\begin{tikzcd}
K(X, \{\exp(B_{ijk})\}) \arrow[rr, "\exp(c_{ij})"] \arrow[dr, "ch^B"'] & \arrow[d, phantom, "\circlearrowleft"] & K(X, \{\exp(B'_{ijk})\}) \arrow[dl, "ch^B"] \\
& H^*(X, \mathbb{Q}) &
\end{tikzcd}$$

$(B_{ijk} - B'_{ijk} = c_{ij} + c_{jk} + c_{ki})$

K3 surfaces

- $ch^B : K(X, \alpha_B) \longrightarrow \widetilde{H}^{1,1}(X, \overset{B}{\mathbb{Z}})$

- For $\mathcal{E} \in D^b(X \times X', \alpha_B^{-1} \boxtimes \alpha_{B'})$:

$$
\begin{tikzcd}
D^b(X, \alpha_B) \arrow[r, "\sim", "\Phi_{\mathcal{E}}"'] \arrow[d, "ch^B \cdot \sqrt{td}"'] \arrow[dr, phantom, "\circlearrowleft"] & D^b(X', \alpha_{B'}) \arrow[d, "ch^{B'} \cdot \sqrt{td}"] \\
\widetilde{H}(X, B, \mathbb{Z}) \arrow[r] & \widetilde{H}(X', B', \mathbb{Z})
\end{tikzcd}
$$

$\gamma \longmapsto p_*(q^*\gamma . ch^{-B \boxplus B'}(\mathcal{E}) . \sqrt{td})$

<u>Corollary</u>: Let $\alpha_0 \in Br(X) = H^2(X, \mathcal{O}_X^*)_{tor}$

Then

$$|\{\alpha \in Br(X) \mid D^b(X, \alpha) \simeq D^b(X, \alpha_0)\}| < \infty$$