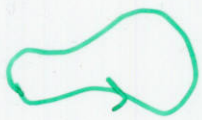


closed-oriented

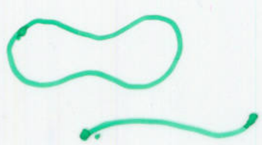
… so far

open-oriented

Homological Mirror Symmetry

$$D^b(\mathrm{Coh}\,M) \longleftrightarrow \mathrm{Fuk}(\tilde{M})$$

$\vdots \qquad\qquad \vdots$



unoriented

Spaces with involutions

real geometry

$\mathbb{C}$ with anti-involutions

Clifford algebra

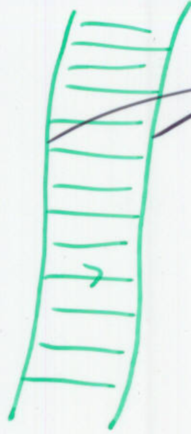
$\vdots$

heterotic

Spaces with G-bundles

"Flux"

Generalized Geometry

Open String  need

boundary conditions

$\parallel$

D-branes

NLσM $\quad \phi : \Sigma \to M$

- $N \subset M$ submanifold : b.c. $\phi : \partial\Sigma \to N$

- $(E, A) \to N$ hermitian vector bundle with connection :

path integral w.t. $\quad e^{-S}\, P\exp\left(-i\int_{\partial\Sigma} \phi^* A\right)$ holonomy factor "Chan-Paton factor"

Supersymmetry :  boundary relates $\left\langle \begin{matrix}\text{left} \\ \text{right}\end{matrix} \right.$ modes

$\to$ $Q_\pm, \bar{Q}_\pm$ cannot independently conserved.

best cases : $\frac{1}{2}$ is preserved.

A-brane : $Q_A = \bar{Q}_+ + Q_-$, $Q_A^\dagger$ conserved

mirror

B-brane : $Q_B = \bar{Q}_+ + \bar{Q}_-$, $Q_B^\dagger$ conserved

NLσM on **M** Kähler $\left\langle \begin{array}{l} \omega \text{ symplectic str.} \\ J \text{ complex str} \end{array} \right.$

(N, E) is an

A-brane if $N \subset (M, \omega)$ Lagrangian

E flat $(F_A = 0)$

B-brane if $N \subset (M, J)$ complex submfd

E holomorphic $(F_A \in \Omega^{1,1})$

LG model $W : M \to \mathbb{C}$ superpotential

$$A \Rightarrow \operatorname{Im} W|_N = \text{constant}$$

$$B \Rightarrow W|_N = \text{constant}$$

Open String States

Quantize on strip with boundary conditions **a** and **b** (both A or both B)

$$\rightsquigarrow \mathcal{H}_{a,b} = \text{space of states}$$

(∞ - dim)

On $\mathcal{H}_{a,b}$ $\exists$ operators Q, F $= \begin{cases} Q_A, F_A & \text{A-branes} \\ Q_B, F_V & \text{B-branes} \end{cases}$

optional

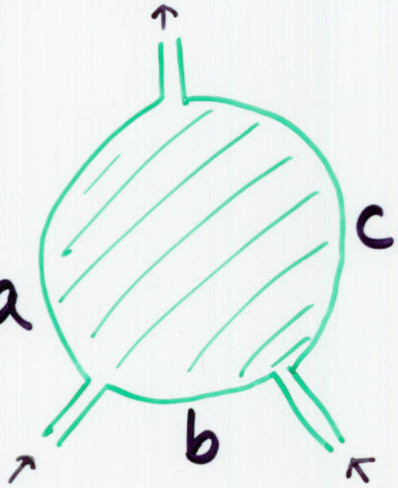
$$\{Q, Q^\dagger\} = 2H$$

$$Q^2 = 0$$ (in good cases : later)

$$[F, Q] = Q$$

F values $\subset \mathbb{Z}$: $\rightarrow \mathcal{H}^q_{a,b} \xrightarrow{Q} \mathcal{H}^{q+1}_{a,b} \xrightarrow{Q} \cdots$ <u>complex</u>

$$\mathcal{H}^q_{a,b}(\text{SUSY}) \cong H^q(\mathcal{H}^\bullet_{a,b} ; Q)$$

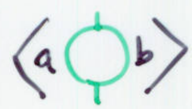


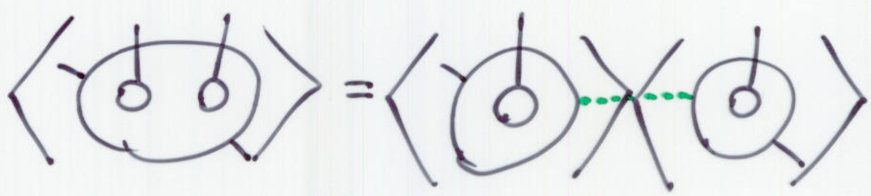
$$\mathcal{H}_{a,b} \times \mathcal{H}_{b,c} \longrightarrow \mathcal{H}_{a,c}$$

dg-Category $\mathcal{D}$ $\begin{cases} \text{Obj: D-branes} \\ \text{mor: } \mathcal{H}_{a,b} \end{cases}$

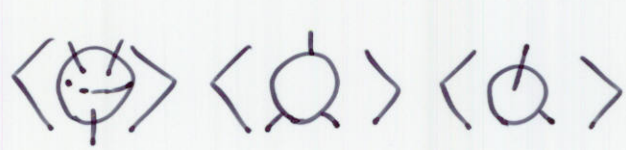
- Coupling to topological gravity ⇒  cyclic A_∞-category
- $H^{\cdot}(\mathcal{D})$ ⇝ open/closed TFT (no top. grav.)

Moore-Segal
Lazaroiu

  - $1 \in H(a,a)$
  - 

 : $H(a,b) \times H(b,a) \to \mathbb{C}$ non-degenerate
  - pinching axiom 

e.g.

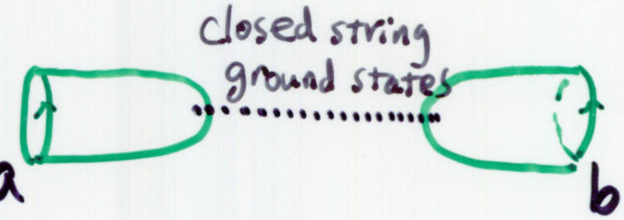
reconstructed by  obeying

associativity, Cardy, 

Witten index

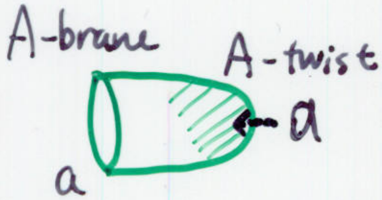
$$\mathrm{Tr}_{\mathcal{H}_{a,b}}(-1)^F = \sum_p (-1)^p \dim H^p(a,b) = \chi(Q)$$

$=$ 

$=$ 

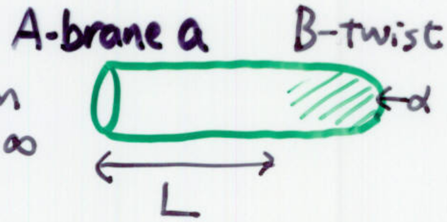
Riemann-Roch

Two possibilities

① 

… topological disc amplitudes

- depend on $\mathcal{M}_t$ (Kähler)
- independent on $\mathcal{M}_c$ (cplx str)

② $\Pi^a_\alpha = \lim_{L\to\infty}$ 

: Ramond-Ramond charge of the brane **a**

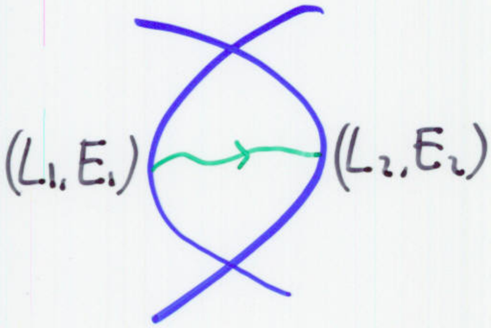
- independent on $\mathcal{M}_t$ (Kähler)
- dependence on $\mathcal{M}_c$ (complex)

$$\nabla_\alpha \Pi^a_\beta = \underbrace{C^{\gamma}_{\alpha\beta}}_{} \Pi^a_\gamma$$

R_B -structure constant.

A-branes in NLσM on M

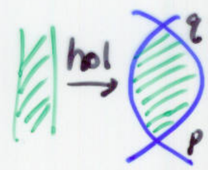
$L \subset M$ Lagrangian submfd $E \to L$ flat



finite model for $\mathcal{H}_{1,2}$, Q

$$\mathcal{H}^{i}_{1,2} = \bigoplus_{\substack{p\in L_1\cap L_2 \\ \mu(p)=i}} \mathbb{C}[p]\otimes \mathrm{Hom}(E_{1p},E_{2p})$$

$$Q[p] = \sum_{\mu(q)=\mu(p)+1} [q]\cdot \#\Big($$



$$\Big)\, e^{-A}\, \underbrace{\Phi(E_1)^{-T}\Phi(E_2)}_{\text{holonomy}}$$

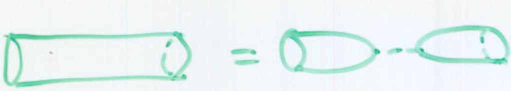
$Q^2=0$ may fail (disc bubble)

When $Q^2=0$, $H^{\bullet}(\mathcal{H}_{1,2}) = HF^{\bullet}((L_1,E_1),(L_2,E_2))$

Floer cohomology

$$= \overset{\mathrm{rank}E}{\vee}\int_L \Omega\cdot\mu \qquad \mu\in H^{0,\bullet}(M,\Lambda^{\bullet}T_M)$$

$$\mathrm{Tr}_{(L_1,E_1),(L_2,E_2)}(-1)^F = \mathrm{rank}E_1\cdot\mathrm{rank}E_2\ \#(L_1\cap L_2)$$



: Riemann's bilinear identity

B-branes in NLσM M

$$N \underset{\text{cplx}}{\subset} M, \quad E \xrightarrow[\text{hol}]{} N \rightsquigarrow \mathcal{E} \text{ coherent sheaf}$$

$\mathcal{E}_1, \mathcal{E}_2$ vector b'dles on M : 0-mode approx for $\mathcal{H}_{\mathcal{E}_1, \mathcal{E}_2}, Q$

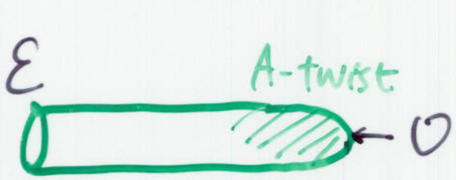
$$\mathcal{H}^i_{\mathcal{E}_1, \mathcal{E}_2} = \Omega^{0,i}(M, \mathcal{E}_1^* \otimes \mathcal{E}_2)$$

$$Q = \bar{\partial}_{\mathcal{E}_1^* \otimes \mathcal{E}_2}$$

$$H^\cdot(\mathcal{H}_{\mathcal{E}_1, \mathcal{E}_2}) = H^{0,\cdot}(M, \mathcal{E}_1^* \otimes \mathcal{E}_2)$$

$$= \mathrm{Ext}^\cdot(\mathcal{E}_1, \mathcal{E}_2)$$

$$\mathrm{Tr}_{\mathcal{E}_1, \mathcal{E}_2}(-1)^F = \sum_p (-1)^p \mathrm{Ext}^p(\mathcal{E}_1, \mathcal{E}_2) = \chi(\mathcal{E}_1, \mathcal{E}_2)$$



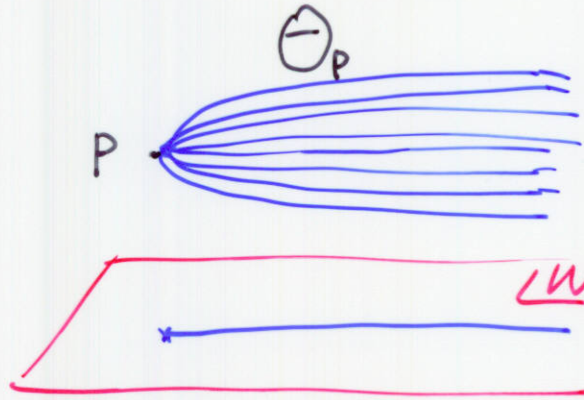
$\mathcal{O} \in H^\cdot_{DR}(M)$

$$= \int_M ch(\mathcal{E}^*) \sqrt{td(X)}\, e^{B + i\omega}\, \mathcal{O}$$

$$+ \underbrace{\cdots}_{O(e^{-t}) \text{ correction}}$$

A-branes in LG (M,W)

Assume $\mathrm{Crit}(W)$ all non-degenerate



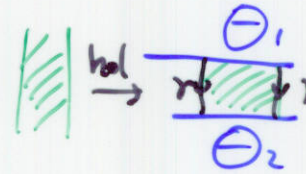
branes: Lefschetz thimbles

$p \in \mathrm{Crit}(W)$ $\quad \Theta_p = \bigcup \left(\begin{array}{l} \text{gradient flow lines} \\ \text{for Re}\,W \text{ starting } p \end{array} \right)$

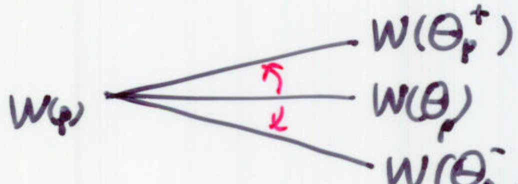
Finite model for $\mathcal{H}_{\Theta_1,\Theta_2}$, Q

$$\mathcal{H}_{\Theta_1,\Theta_2} = \bigoplus_{\substack{\gamma : ([0,1];0,1) \to (M,\Theta_1,\Theta_2) \\ \dot{\gamma} = -\mathrm{grad}\,\mathrm{Im}\,W}} \mathbb{C}[\gamma]$$

$$Q[\gamma] = \sum_{\substack{\gamma' \\ \mu(\gamma') = \mu(\gamma)+1}} [\gamma'] \cdot \#\left(\text{ } \right) e^{-A(\text{ })}$$



define $\Theta_p^{\pm}$ by

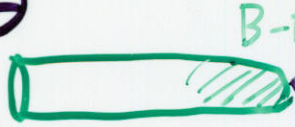


small rotation

$$H(\mathcal{H}_{\Theta_1,\Theta_2}) = \begin{cases} \mathbb{C}^{\#(\Theta_1^- \cap \Theta_2^+)} \\ \\ \mathbb{C} \end{cases}$$

$\leftarrow 0$ if $\mathrm{Im}\,W(\Theta_1) < \mathrm{Im}\,W(\Theta_2)$
"directed"

Θ



$\leftarrow f \in \mathrm{Fun}\, M/W \quad = \int_{\Theta^-} e^{-iW} f\, \Omega$

B-branes in LG (M, W)

$$Z \underset{\text{cplx}}{\subset} M \qquad W|_Z = \text{constant.}$$

Z_a, Z_b : $\mathcal{H}_{a,b}$ $\mathbb{Z}_2$ graded, $Q^2 = W(Z_a) - W(Z_b)$

If $W(Z_a) \neq W(Z_b)$, Q-fails to make a complex

If $W(Z_a) = W(Z_b)$, $Q^2 = 0$. 0-mode approx:

$$\mathcal{H}_{a,b} = \Omega^{0,\bullet}(Z_a \cap Z_b, \Lambda^\bullet N_{Z_a} \cap N_{Z_b})$$

$$Q = \bar{\partial} + \partial W \cdot$$

- $Z_a \cap Z_b = \phi \Rightarrow H^\bullet(Q) = 0$

- $Z_a = Z_b = \{p\}$ $\qquad N_{\{p\}} = \mathbb{C}^n$

$$\mathcal{H}_{p,p} = \Omega^{0,\bullet}(\{p\}, \Lambda^\bullet \mathbb{C}^n) = \Lambda^\bullet \mathbb{C}^n$$

$$0 \leftarrow \mathbb{C} \xleftarrow{\partial W(p)\cdot} \Lambda^1 \mathbb{C}^n \xleftarrow{\partial W(p)\cdot} \Lambda^2 \mathbb{C}^n \leftarrow \cdots \leftarrow \Lambda^n \mathbb{C}^n \leftarrow 0$$

$p \notin \mathrm{Crit}(W)$: $\partial W(p) \neq 0 \Rightarrow H^\bullet(Q) = 0$

$p \in \mathrm{Crit}(W)$: $\partial W(p) = 0 \Rightarrow H^\bullet(Q) = \Lambda^\bullet \mathbb{C}^n$

Alternative (more general) $M = \mathbb{C}^n$, W = Polynomial

..... Kontsevich

Objects : matrix factorizations of W

$$\mathcal{D} = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \quad 2N \times 2N \quad \text{entries in } \mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n]$$

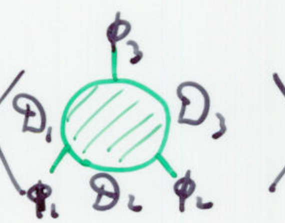
$$\mathcal{D}^2 = W \cdot \mathbb{1}_{2N} \qquad (\text{i.e. } fg = gf = W \mathbb{1}_N)$$

(or equivalently $M = \mathbb{C}[X]^N \oplus \mathbb{C}[X]^N$ $\mathbb{Z}_2$-graded $\mathbb{C}[X]$ module

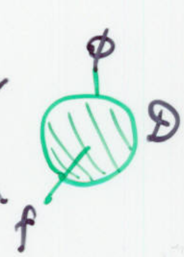
$\mathcal{D}$: odd endomorphism of M, $\mathcal{D}^2 = W \mathbb{1}_M$)

Morphisms : $\mathcal{H}_{\mathcal{D}_1, \mathcal{D}_2} = \mathrm{Hom}^{ev}(M_1, M_2) \oplus \mathrm{Hom}^{od}(M_1, M_2)$

$$Q\phi = \mathcal{D}_2 \phi - (-1)^{|\phi|} \phi \mathcal{D}_1$$

$\Big\langle$ 

$\Big\rangle = \mathrm{res}_W \left(\mathrm{Str}(\phi_3 \phi_2 \phi_1 (d\mathcal{D}_1)^n) \right)$

$\Big\langle$ 

$\Big\rangle = \mathrm{res}_W \left(f \, \mathrm{Str}(\phi (d\mathcal{D})^n) \right)$

Kapustin-Li

B-branes in LG orbifold $W/\mathbb{Z}_d$ $\quad x_i \to e^{\frac{2\pi i}{d}} x_i$

$$W(x_1, \dots, x_N) \text{ degree } d \quad \left(W(e^{2i\alpha/d} x) = e^{2i\alpha} W(x) \right)$$

require $\mathcal{D}(x_1, \dots, x_N)$ also "homogeneous" H-Walcher

$$e^{i\alpha R} \mathcal{D}(e^{2i\alpha/d} x) e^{-i\alpha R} = e^{i\alpha} \mathcal{D}(x)$$

$\rightarrow F_V$ conserved (necessary for conformal invariance)

$$\Rightarrow e^{i\pi R} \mathcal{D}(e^{2\pi i/d} x) e^{-i\pi R} = -\mathcal{D}(x)$$

$$\rho_\varphi(\omega) \mathcal{D}(\omega x) \rho_\varphi(\omega)^{-1} = \mathcal{D}(x) \quad \omega \in \mathbb{Z}_d\text{-invariant}$$

$\rho_\varphi(\omega) = e^{\pi i \varphi} (-1)^F e^{i\pi R}$ $\quad \varphi$ chosen so that $\rho_\varphi(\omega)^d = 1$

$\varphi \equiv \varphi + 2$ labels $\mathbb{Z}_d$ representations

$\mathbb{Z}_d$ action on $\mathrm{Hom}(M_1, M_2)$:

$$\phi(x) \to \rho_{\varphi_2}(\omega) \, \phi(\omega x) \, \rho_{\varphi_1}(\omega)^{-1}$$

$$\mathcal{H}^{orb}_{(\mathcal{D}_1, \varphi_1), (\mathcal{D}_2, \varphi_2)} = \mathrm{Hom}(M_1, M_2)^{\mathbb{Z}_d}$$

$U(1)_V$ R-sym: $\phi(x) \longrightarrow e^{i\alpha R_2} \phi(e^{2i\alpha/d} x) e^{-i\alpha R_1}$

Charge can be made $\subset \mathbb{Z}$ by dressing $e^{i\alpha(\varphi_2 - \varphi_1)}$

$\rightsquigarrow$ $\mathbb{Z}$ grading $\left(\varphi_i \not\equiv \varphi_i + 2 \text{ for } \mathbb{Z}\text{-graded B-branes} \right)$ Walcher matrix factorizations of W $\overset{\text{Eisenbud}}{\longleftrightarrow}$ Maximal Cohen Macaulay modules over $R = \mathbb{C}[X]/W$

1980's Auslander, Reiten
Knorrer, Buchweitz, ...

Buchweitz (1980's)

$$\underline{MF}(W) \cong \underbrace{\underline{MCM}}_{\text{projectives}=0}(R) \cong \underbrace{\underline{D}^b}_{\text{modulo perfect cplxs}}(R)$$

Orlov (2005) $W = G(\phi_1, \cdots, \phi_N)$ degree d

$M = \{ W = 0 \} \subset \mathbb{C}P^{N-1}$ hypersurface

$d = N$: $\underline{grMF}(W) \xrightarrow[\sim]{\Psi_i} D^b(\mathrm{Coh}\, M)$

$\mathbb{Z}$ graded B-branes in LG orbifold $W = G/\mathbb{Z}_d$

$d > N$: $\underline{grMF}(W) \xrightarrow{\sim} \langle \underbrace{\bullet, \cdots, \bullet}_{(d-N)\text{ objects}}, D^b(\mathrm{Coh}M) \rangle$

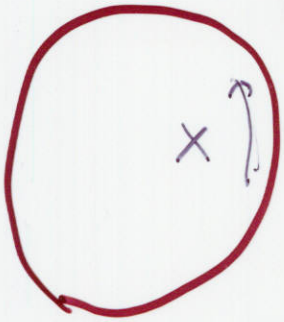
semiorthogonal decompo

$d < N$: $D^b(\mathrm{Coh}M) \xrightarrow{\sim} \langle \underbrace{\mathcal{O}_M(1-(N-d)), \cdots, \mathcal{O}_M}_{(N-d)\text{ objects}}, \underline{grMF}(W) \rangle$

Mirror Symmetry for D-branes

T-duality

$$(S^1, R) \cong (\tilde{S}^1, \alpha'/R)$$



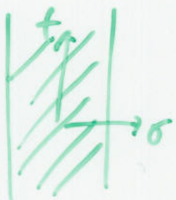
momentum $\longleftrightarrow$ winding #

winding # $\longleftrightarrow$ momentum

$$\Rightarrow \quad \partial_t X = \partial_\sigma \tilde{X}$$

$$\partial_\sigma X = \partial_t \tilde{X}$$

Open string:



Neumann b.c. $\longleftrightarrow$ Dirichlet b.c.

$\partial_\sigma X = 0$ on $\partial\Sigma$ $\qquad$ $\partial_t \tilde{X} = 0$ on $\partial\Sigma$

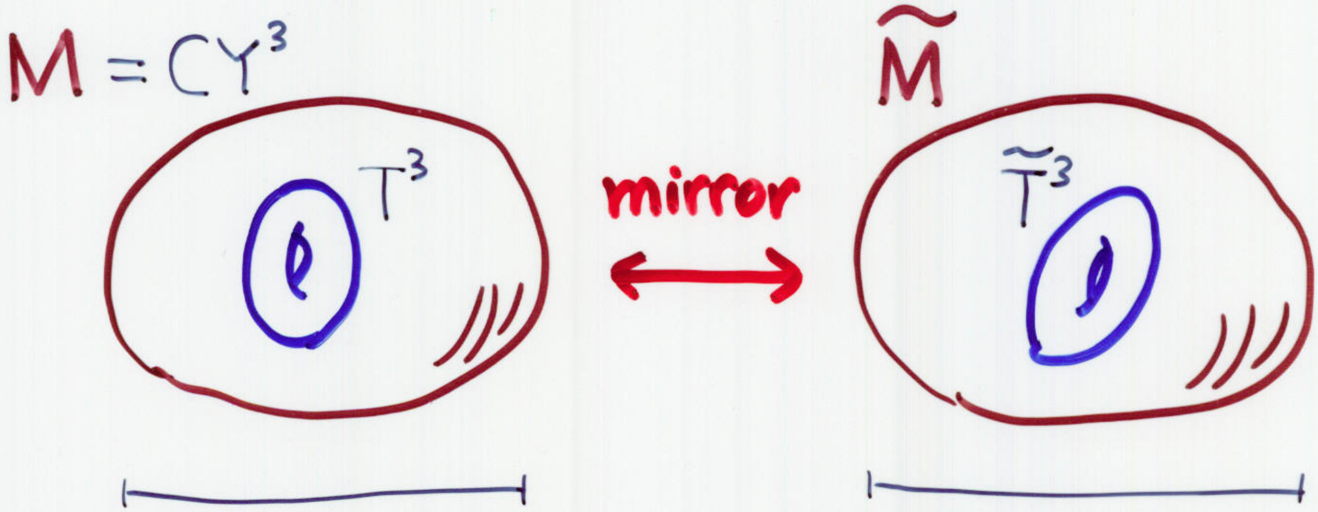
$$\boxed{\text{D1-brane wrapped on } S^1 \longleftrightarrow \text{D0-brane at a point of } \tilde{S}^1}$$

$U(1)$ holonomy $\oint_{S^1} A \longleftrightarrow$ position in $\tilde{S}^1$

$$\tilde{S}^1 = \{\text{D0-brane in T-dual}\} = \{\text{D1-brane in } S^1\} = H^1(S^1, U(1))$$

$$\tilde{S}^1 = \text{dual space } H^1(S^1, U(1)) \qquad \text{space of } U(1) \text{ holonomy}$$

Strominger-Yau-Zaslow



$$\left\{ \begin{matrix} \text{D3 brane wrapped on} \\ \text{Special Lagrangian } T^3 \subset M \end{matrix} \right\} = \widetilde{M}$$

$$M = \left\{ \begin{matrix} \text{D3 brane wrapped on} \\ \text{SLAG } \widetilde{T}^3 \subset \widetilde{M} \end{matrix} \right\}$$

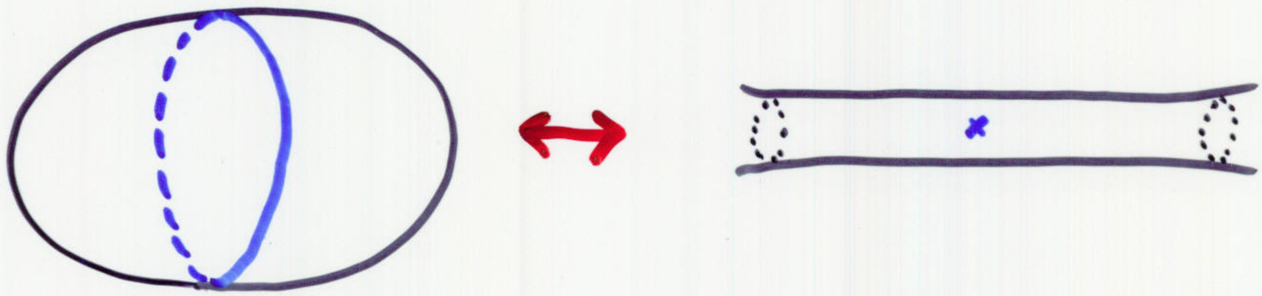
Mirror Symmetry

$$= \text{T-duality} + \underbrace{\text{Quantum Correction}}_{\sim\, T^3 \text{ degeneration}}$$

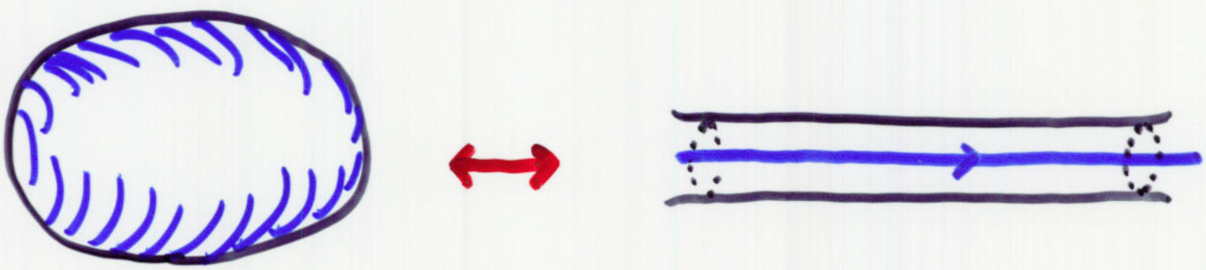
$\left.\begin{matrix}\text{Gross} \\ \text{W.-D. Ruan} \\ \text{D. Morrison} \\ \vdots\end{matrix}\right\}$ topological construction of the mirror

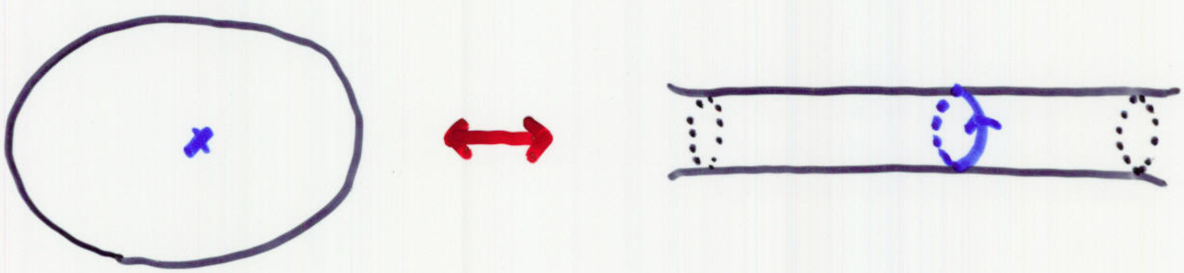
$$X = \mathbb{CP}^1 \quad \longleftrightarrow \quad W = e^{-Y} + e^{-t+Y}$$

A-branes in $\mathbb{CP}^1$ B-branes in LG

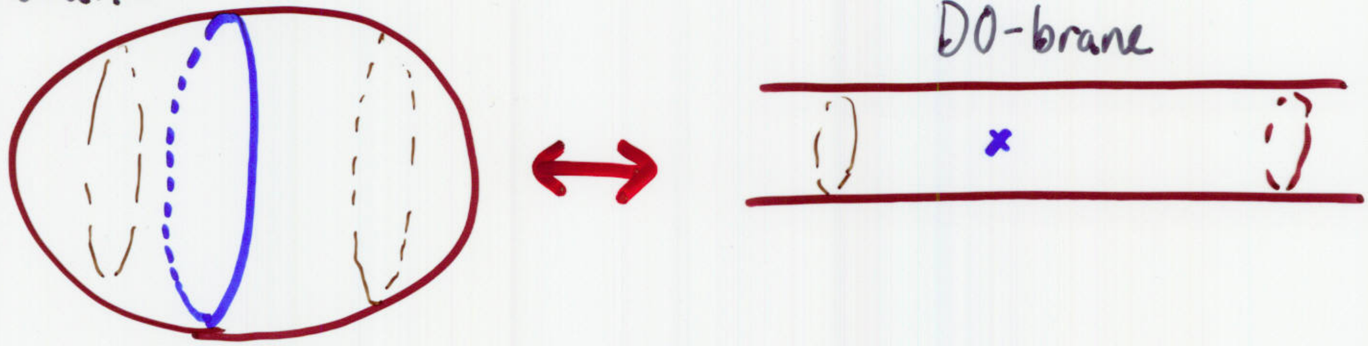


B-branes in $\mathbb{CP}^1$ A-branes in LG





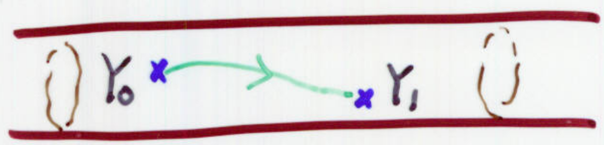
D1-brane



$$\mathrm{Area}(\quad) = \mathrm{Re}\, Y_{\times}$$

$$\int A = \mathrm{Im}\, Y_{\times}$$

For two D0-branes in LG



$$Q_B^2 = W(Y_0) - W(Y_1) = e^{-Y_0} + e^{-t+Y_0} - e^{-Y_1} - e^{-t+Y_1}$$

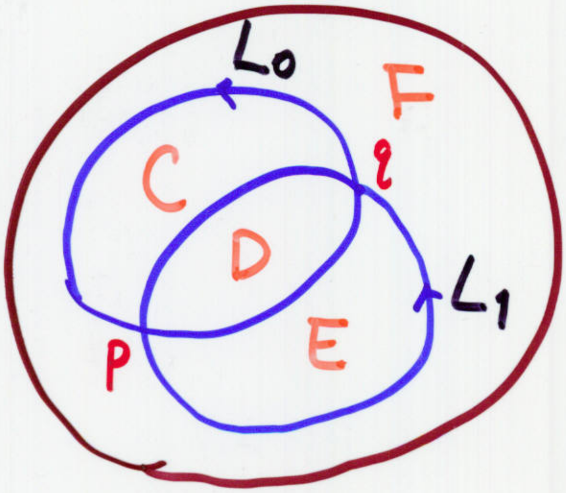
$$Q_B^2 = 0 \quad \text{iff} \quad Y_1 = Y_0 \quad \text{or} \quad Y_1 = t - Y_0$$

$$\mathcal{H}_{Y_0,Y_1}(S^1\times\mathbb{R}) = \begin{cases} \Lambda^\bullet \mathbb{C} = \mathbb{C}\oplus\mathbb{C} & \text{if } Y_0 = Y_1 \in \mathrm{Crit}(W) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Crit}(W) = \{ e^{-Y} = \pm e^{-t/2} \} \rightarrow \quad \mathrm{Area}(\quad) = \frac{r}{2}$$

$$\int A = 0 \text{ or } \pi$$

Computation in NLσM FOOO



$$C \cup D =: D_0$$

$$D \cup E =: D_1$$

$$Q_A[q] = e^{-A(D)}[p] - e^{-A(F)}[p]$$

$$Q_A[p] = e^{-A(C)}[q] - e^{-A(E)}[q]$$

$$Q_A^2[q] = \left[e^{-A(D_0)} - e^{-A(D_1)} - e^{-r+A(D_1)} + e^{-r+A(D_0)} \right][q]$$

$$Q_A^2 = 0 \quad \text{iff} \quad A(D_1) = A(D_0) \quad \text{or} \quad A(D_1) = r - A(D_0)$$

If $A(D_1) = A(D_0)$ (i.e. $L_1 \sim L_0$)

$$Q_A[q] = e^{-A(D)}\left(1 - e^{-r+2A(D_0)}\right)[p]$$

$$Q_A[p] = 0$$

$$\therefore HF^{\cdot}(L_0, L_1) = \begin{cases} \mathbb{C}[q] \oplus \mathbb{C}[p] & A(D_0) = A(D_1) = \frac{r}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The prediction of mirror symmetry was indeed correct.

The analysis is extended to general toric Fano.

Cheol-Hyn Cho & YG Oh

Also, compare

$$Q_A^2 = e^{-A(D_0)} + e^{-r+A(D_0)} - e^{-A(D_1)} - e^{-r+A(D_1)}$$

$$Q_B^2 = W|_{Y_0} - W|_{Y_1} = e^{-Y_0} + e^{-t+Y_0} - e^{-Y_1} - e^{-t+Y_1}$$

— identical! (as the should be)

$Q_B^2 = W|_{Z_0} - W|_{Z_1}$ has enough info to determine W (up to a shift)

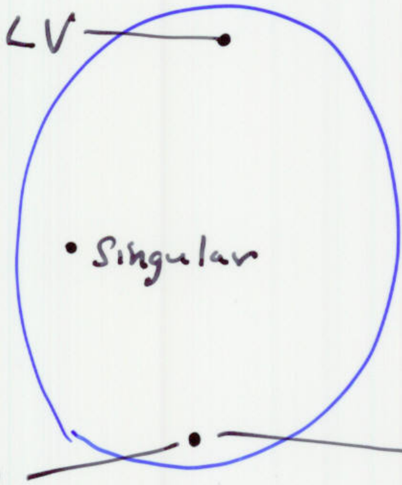
$\Rightarrow$ Computation of Q_A^2 in NLσM can reproduce the LG superpotential of the mirror LG.

... This is one way to take into account the quantum correction at degenerating torus fibres.

MF mirror to real quintic

Kähler moduli for quintic $\subset \mathbb{CP}^4$

Complex structure moduli space for mirror quintic



Fermat: $\sum_{i=1}^{5} X_i^5 = 0$

$$\frac{\sum_{i=1}^{5} \tilde{X}_i^5 - 5\psi \tilde{X}_1 \cdots \tilde{X}_5 = 0}{\mathbb{Z}_5^3}$$

$$\frac{W = \sum_{i=1}^{5} X_i^5}{\mathbb{Z}_5}$$

LGO

$\widetilde{\text{LGO}}$

$$\frac{\widetilde{W}_0 = \sum_{i=1}^{5} \tilde{X}_i^5}{\mathbb{Z}_5^4}$$

$\mathbb{R}M \subset M$ real quintic $\left(\overset{\text{homeo}}{\approx} \mathbb{RP}^3 \right)$

in $\widetilde{\text{LGO}}$ $\quad \mathcal{D}_0 = \sum_{i=1}^{5} \left(\tilde{X}_i^2 \eta_i + \tilde{X}_i^3 \bar{\eta}_i \right)$ $\qquad \eta_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \bar{\eta}_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

on $\underbrace{\mathbb{C}[\tilde{X}]^2 \hat{\otimes} \cdots \hat{\otimes} \mathbb{C}[\tilde{X}]^2}_{5}$

$$H^i(\mathcal{H}^{orb}_{\mathcal{D}_0,\varphi_0,\mathcal{D}_0,\varphi_0}) = \begin{cases} \mathbb{C} & 0 \\ \mathbb{C} & 1 \\ \mathbb{C} & 2 \\ \mathbb{C} & 3 \end{cases}$$

H^1 : $\Phi = (\eta_1 - \tilde{X}_1 \bar{\eta}_1) \cdots (\eta_5 - \tilde{X}_5 \bar{\eta}_5)$

H^2 : $\tilde{X}_1 \cdots \tilde{X}_5 \cdot \mathbb{1} = \Phi^\dagger$

They obey $\Phi^2 = -\Phi^\dagger = -\tilde{X}_1 \cdots \tilde{X}_5$

Deformation

$$W_0 \to W_\psi = W_0 - 5\psi\, \tilde{X}_1 \cdots \tilde{X}_5$$

$$\mathcal{D}_0 \to \mathcal{D}_\varphi = \mathcal{D}_0 + \varphi\, \Phi$$

require $\mathcal{D}_\varphi^{\,2} = W_\psi$

$$\Longrightarrow \quad \boxed{\varphi^2 = 5\psi}$$

$\forall \psi \neq 0$, $\exists$ two solutions $\varphi = \pm\sqrt{5\psi}$

$\longleftrightarrow \quad \pi_1(\mathbb{RM}) = \mathbb{Z}_2 \Rightarrow$ Two flat $U(1)$ b'dles on $\mathbb{RM}$

$$H^i(\mathcal{H}_{\mathcal{D}_\varphi, \mathcal{D}_\varphi}) = \begin{cases} \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \end{cases} \text{at } \varphi = 0, \quad \begin{cases} \mathbb{C} \\ 0 \\ 0 \\ \mathbb{C} \end{cases} \text{at } \varphi \neq 0$$

$$\longleftrightarrow \quad HF^i(\mathbb{RM}, \mathbb{RM}) = \begin{cases} \mathbb{C} \\ 0 \\ 0 \\ \mathbb{C} \end{cases} \text{(near Large volume)}$$

$$\xrightarrow{\text{analytic continuation}} \begin{cases} \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \end{cases} \text{at } e^t = 0$$