Density of rational points on K3 surfaces

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Guiding problem

F number field
(or field finitely generated over $\mathbb{Q}$ like $\mathbb{Q}(\sqrt{2}, t, u)$)

X/F variety defined over F

Does there exist a finite extension E/F for which

$$X(E) \subset X$$

is Zariski dense?

"Rational points of X are potentially dense"

Examples: where potential density holds

- rational varieties $\mathbb{P}^n$
- unirational varieties

$$X_3 \subset \mathbb{P}^n \quad n \geq 3$$

smooth cubics

- abelian varieties

e.g. simple abelian varieties of positive rank

And where potential density fails:

- curves of genus $g \geq 2$ (Faltings)
- Conjecturally (Lang Bombieri): Varieties of general type

$$X_d \subset \mathbb{P}^n \text{ smooth}$$
$$d \geq n+2$$

Potential density for K3 surfaces: classical examples

(Silverman)

$$X \subset \mathbb{P}^2 \times \mathbb{P}^2$$

$$= \{ f = g = 0 \} \qquad \begin{array}{l} \deg f = (1,1) \\ \deg g = (2,2) \end{array}$$

$$\pi_1, \pi_2 : X \longrightarrow \mathbb{P}^2$$

double covers branched over plane sextic

$$i_1, i_2 : X \longrightarrow X$$

corresponding involutions (non commuting)

$$\mathbb{Z}_2 * \mathbb{Z}_2 \subset \mathrm{Aut}(X)$$

$$\Rightarrow \quad |\mathrm{Aut}(X)| = \infty$$

Using height techniques Silverman shows that "most" points have infinite (hence Zariski dense) orbit $\Rightarrow$ potential density

(Lan Wang, Billard)

$$X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
$$\| $$
$$\{f = 0\} \qquad \deg f = (2,2,2)$$

$\pi_{12}, \pi_{13}, \pi_{23} : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ double covers with involutions i_{12}, i_{13}, i_{23}

Analysis of orbits $\Rightarrow$ potential density

K3 surfaces with infinite automorphisms

Basic facts:

1) $\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(H^2(X,\mathbb{Z}))$ has finite kernel

(Indeed, if h is ample then $\{\alpha \in \mathrm{Aut}(X) : \alpha^* h = h\}$ is finite)

2) $|\mathrm{Aut}(X)| = \infty \implies$ Rank $NS(X) \geq 2$

3) Conversely, if $NS(X) \geq 2$ and the intersection form on $NS(X)$ does not represent 0 or -2

(i.e. no smooth rational, elliptic curves)

Then $|\mathrm{Aut}(X)| = \infty$

<u>Theorem</u> (Bogomolov-Tschinkel)

X K3 surface with $\mathrm{Aut}(X)$ infinite

Then rational points are potentially dense

<u>Sketch</u>: Find a <u>singular</u> rational curve

$R \subset X$ $[R]$ ample

$\underset{(1)}{\Longrightarrow}$ $[R]$ and R have infinite orbit

Choose extension K/F so that

- generators of $\mathrm{Aut}(X)$
- R are defined

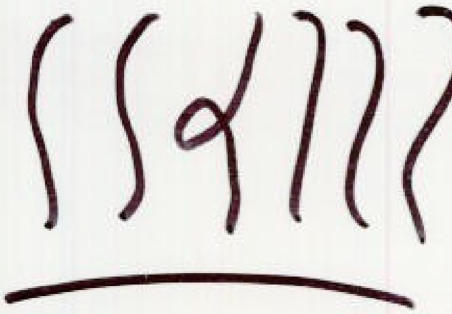
<u>and</u> $R^{\nu} \simeq \mathbb{P}^1_K$

$X(K) \subset X$ dense

$\square$

K3 surfaces with elliptic fibrations

$$X \xrightarrow{f} \mathbb{P}^1$$



Basic facts:

1) If X admits an elliptic fibration then

$$\text{Rank } NS(X) \geq 2$$

$$\left([f^{-1}(t)]^2 = 0 \right)$$

2) If the intersection form represents zero on $NS(X)$ then X admits an elliptic fibration

Theorem (Bogomolov - Harris - Tschinkel)

X K3 surface with elliptic fibration

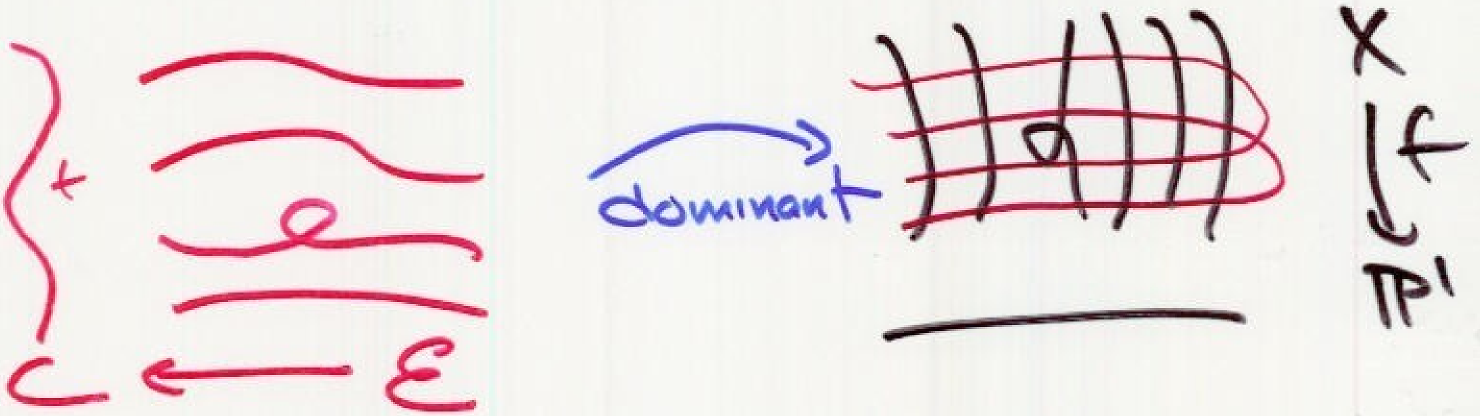
Then rational points are potentially dense

Sketch

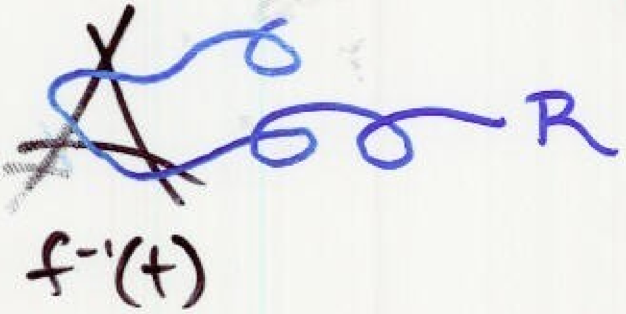
Lemma (Kollár)

(Existence of transversal elliptic family)

There exists a curve C and a fibration in genus one curves $\mathcal{E} \longrightarrow C$ dominating X so that $f|\mathcal{E}_t$ is non constant



Proof: deformation theory
Take singular fiber $f^{-1}(t)$ and rational curve $R \subset X$ not in some fiber of f with $\deg f|_R > 1$



A suitable union of R with components of $f^{-1}(t)$ deforms to an elliptic curve E_t

For a generic such deformation E

$$Y := X \times_{\mathbb{P}^1} E \longrightarrow E$$

$0 = \Delta_E$ diagonal

is an elliptic fibration with section of infinite order $s : E \longrightarrow Y$

Let K/F be an extension over which E is defined, s is defined, and $E(K) \subset E$ is dense

$$\Longrightarrow \bigcup_{N \geq 0} N\, s(E(K)) \subset Y \text{ dense}$$

$\square$

Problem:

Find an example of a K3 surface X/F with

$$\operatorname{rank} NS(\bar{X}) = 1$$

(geometric Néron Severi rank)

and $X(F) \subset X$ dense

van Luijk: Gives an example of $X/\mathbb{Q}$ with $\operatorname{rank} NS(\bar{X}) = 1$ and $X(\mathbb{Q})$ infinite but not necessarily dense

<u>Theorem</u>:

$X \subset \mathbb{P}^g$ K3 surface of degree $2g-2$

$X^{[g]} = \mathrm{Hilb}_g(X)$ Hilbert scheme of length-g zero dimensional subschemes

Then rational points on $X^{[g]}$ are potentially dense

N.B. Contrary to the case of curves, for any surface S

$$\kappa(S^{[n]}) = n\,\kappa(S)$$

i.e. Kodaira dimension is well behaved under symmetric products

K3 surfaces over complex function fields

B smooth proj curve $/\mathbb{C}$

$F = \mathbb{C}(B)$

X/F nonisotrivial K3 surface

Problem: Are rational points potentially dense on X?

In general, this is completely open

Theorem

There exists a K3 surface $X/\mathbb{C}(t)$, nonisotrivial with

$$NS(X)=1$$

with $X(\mathbb{C}(t)) \subset X$

dense

Precisely: Consider the fibration

$$\pi : \mathfrak{X} \longrightarrow \mathbb{P}^1$$

$$\mathfrak{X} = \{ s_0 f_0 + s_1 f_1 = 0 \}$$

f_0, f_1 suitably general quartics in $\mathbb{C}[x,y,z,w]$

Then sections of π are dense in $\mathfrak{X}$

proof:

$$B = \{ f_0 = f_1 = 0 \} \subset \mathbb{P}^3$$

degree 16 $\qquad g(B) = 33$

Assume B is smooth

$$\begin{tikzcd} E \subset \mathrm{Bl}_B \mathbb{P}^3 = \mathcal{X} \arrow[r] \arrow[d] & \mathbb{P}^1 \\ \mathbb{P}^3 & \end{tikzcd}$$

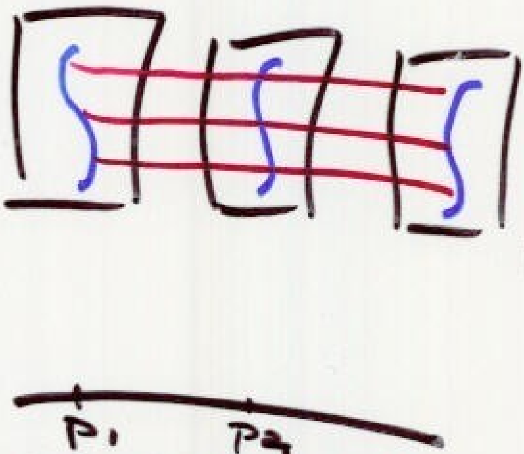
exceptional

$$\begin{tikzcd} b \times \mathbb{P}^1 \subset B \times \mathbb{P}^1 \simeq E \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

$\| $

E_b

$B \times p_1 \quad B \times p_2$



$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \\ \mathbb{P}^1 \end{tikzcd}$$

Each point $b \in B \longleftrightarrow$ section of π

$\rightsquigarrow$ infinite number of sections of π

(This can't work over number fields!)

Choose $t_d \in \mathbb{P}^1$ (15)

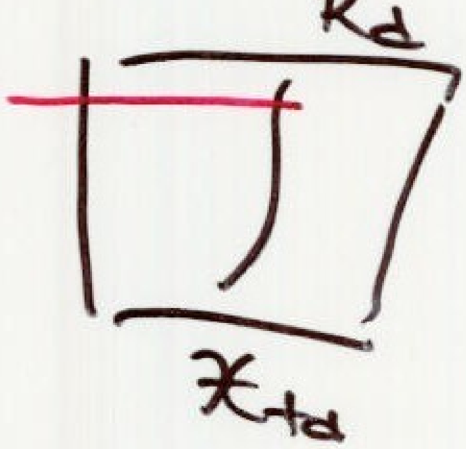
so that there exists a smooth degree d rational curve

$$R_d \subset \mathcal{X}_{t_d} \subset \mathcal{X}$$

For generic pencil

$$N_{R_d/\mathcal{X}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$$

Pick $b \in B \cap R_d$ $\quad b \times \mathbb{P}^1 = E_b$



<u>Claim</u> $C = R_d \cup_b E_b$ deforms in a one parameter family $\mathcal{C} \to W$

The generic member is a smooth section of π

$\Longrightarrow$ $R_d \subset$ Zariski closure of sections of $\pi : \mathcal{X} \to \mathbb{P}^1$

$\underset{d \to \infty}{\Longrightarrow}$ These sections are dense $\square$

We can visualize these sections in terms of the projective geometry of

$$B = \{ f_0 = f_1 = 0 \} \subset \mathbb{P}^3$$

- 3-secant lines to B meet each $\mathcal{X}_t$ in one point outside B

$\rightsquigarrow$ section expected dimension = 1

- 7-secant conics to B meet each $\mathcal{X}_t$ in one point outside B

$\rightsquigarrow$ section expected dimension = 1

- $(4d-1)$-secant rational curves of degree d to B

$\rightsquigarrow$ sections varying in family with expected dimension = 1

<u>Problem</u> Will this work over $\overline{\mathbb{Q}}(t)$?