Weak approximation
for rationally
connected varieties

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/ * although most
statements extend
to positive characteristic
§1 Geometric weak approximation

Background:
\[
F = \begin{cases} 
\text{number field} \\
C(B) \quad \text{B smooth proj. curve} 
\end{cases}
\]
\[
v = \text{place of } F \iff \exists p \in \text{Spec } \mathcal{O}_F, \alpha_1, \ldots, \alpha_n \in \mathcal{O}_F \\
b \in B
\]
\[
F_v \text{ completion} = \begin{cases} 
\mathbb{C}_p, R, \mathbb{C} \\
\mathbb{C}(\mathbb{C})
\end{cases}
\]

\[
X/F = \text{algebraic variety}
\]

\[
X(F) \subset X(F_v)
\]

rational points
points over completion
Defn \( X \) satisfies weak approximation (WA) if for any places \( v_1, \ldots, v_n \) and open \( \emptyset \neq U_i \subset X(F_{v_i}) \), there exists \( x \in X(F) \) with \( x \in U_i \).

Ex. \( X = \mathbb{P}^1 \), \( F = \mathbb{Q} \),

Chinese Remainder Theorem.

Geometric translation:

- \( F = \mathcal{O}(B) \)
- \( X \) proper / \( F \)

Choose a model \( \pi : X \to B \)

Flat, proper, \( \pi_* \mathcal{O}(B) = \mathcal{O}(X) \)
Rational points \( x \in X(C(B)) \) \( \leftrightarrow \) Sections \( \mathcal{S} \)

(valutative criterion)

WA holds iff for any \( b, \ldots, b \in B, N \geq 0 \), formal sections

\[ \hat{S}_c : \hat{B}_c = \text{Spec}(\delta_{b_1 \ldots b_c}) \rightarrow X \times B \hat{B}_c \]

There exists \( s : B \rightarrow X \) with

\[ s = \hat{S}_c (\text{mod } t^{N+1}) \]

(unionizer)

Assume

- \( X \) is regular model
- \( U \) (resolve singularities)

\( \text{Then } s(B) \subset X^\text{sm} \) smooth

locus for \( \pi \)
Hensel's Lemma $\Rightarrow$

WA holds iff for any collection of N jets of sections

There exists $s: B \to \mathbb{X}$ with $s = j_c (\text{mod } b^{n+1})$
General results

Birationality

\[ X_1 \dashrightarrow X_2 \Rightarrow \text{WA holds for } X_1 \text{ iff it holds for } X_2 \]

WA holds for fibers

\[ F = C(B) \]

\[ \mathbb{P}^n, \text{Gr}(k, m), Q_2, \mathbb{C}P^n \text{ n \geq 2} \]

(Raw holds)

Fibration property

\[ X \rightarrow Y \text{ fibration} \]

WA holds for Y and \( \Rightarrow \) for X

for fibers

\[ \checkmark \text{ Cone bundles} \]

\[ \checkmark \text{ Cubic hypersurfaces} \]

\[ \text{containing line } F \]

\[ l \in \mathbb{X}_3 \mathbb{C}P^n \text{ smooth } n \geq 3 \]

e.g. when \( n \geq 5 \)
Conjecture??

$X/F$ rationally connected

$F = C(B)$

WA holds for $X$

E.g. $X_d \subset \mathbb{P}^n$

den smooth

WA holds for $X_d$. 
Problem

$$X_3 \subset \mathbb{P}^3$$
smooth cubic surface /F

Does weak approximation hold?
§3 Work approximation at places of good reduction

\[ X / F = \text{Spec} (B) \] smooth proper

Define: \( b \in B \) is of good reduction if above exists a smooth proper model

\[ \pi : \hat{X} \to \hat{B}_b = \text{Spec} (\mathcal{O}_{B, b}) \]

(Easy) Theorem: There exists a proper flat algebraic space \( \pi : \hat{X} \to B \) with \( \hat{X}_{\mathcal{O}(B)} = X \) and \( \pi \)

smooth at places of good reduction

\# 3 places of bad reduction
Why algebraic spaces?

\[ \mathcal{E} = 3 \leq 5s_0 + 5s_1 = 0 \exists \mathcal{C} \times \mathcal{P}^3 \times \mathcal{P}' \]

\[ \pi : \mathcal{P}' \]

generic pencil of cubic surfaces

3 \[ \mathcal{P}' \] as \[ \mathcal{P}_3 \times \mathcal{C} \times \mathcal{P}' \]

places of bad reduction

\[ B \to \mathcal{P}' \] branched double cover

Blowdown gives resolution

\[ Y \to \mathcal{X} \times \mathcal{P}' \]

\[ \to B \]

\[ Y \]

Minimal resolution

\[ Y \text{ is not a scheme - no divisors meeting exceptional } \mathcal{P}' \text{'s} \]
Main Theorem

\( X \) proper rationally connected \( / F = C(B) \)

Then \( X \) satisfies weak approximation at places of good reduction

Proof \( \xrightarrow{\text{good}} \) regular model

\( b, b' \in B \)

\( J \subseteq J' \) \( N \)-jets

Graber-Harris-Starr \( \Rightarrow \)

\( \exists \) section \( s: B \rightarrow X \)

Induction on \( N \)

\( (N=0) \) \( \) (Kollar-Miyaoka-Mori)

Goal: Given \( x \in X \), smooth hypersurface \( s: B \rightarrow X \), \( s(b') = x \).
\[ Y_c = s'(b\xi) \]
\[ C = 1 - \nu \text{ very} \]
Choose free curves \( s'(b) \)
\[ f_c : \mathbb{P}^1 \to \mathbb{X} ; f_c(x) = y_c \]
\[ T_c = f_c(\mathbb{P}^1) \]

**Claim** There exist additional very free curves
\[ f_k : \mathbb{P}^1 \to \mathbb{X} ; f_k(x) = s(\xi_k) \]
for \( k = \nu + 1, -\nu + 5 \)

so that the "comb"
\[ T_1, T_2, T_{\nu+5} \]
deforms to a smooth curve in \( \mathbb{X} \) containing \( x_1, \ldots, x_5 \)

\[ \Rightarrow \text{desired section} \]
Problem

Generalize to case where $x_c \in X_{bc}$ are smooth points of singular fibers.

$\Rightarrow$ Full weak approximation
Given $x_i \in X_{hi}$

tangent directions $U \in T_{x_i}X \rightarrow T_{x_i}X$

Find a section

$s : B \rightarrow X$

$s(b_c) = x_i$

$s'(b_c) = u_c$

Have $s' : B \rightarrow X$

with $s'(b_c) = x_i$

$\tilde{X} = B \varepsilon \ldots \times \varepsilon \chi$
\[ T_{c,1} = \text{line joining } \tilde{x}_c \text{ and } \tilde{y}_c \]

\[ T_{c,0} = \text{very free curve in } BL_{x_c x_b} \text{ with } T_{c,0} \cap E_c = T_{c,1} \cap BL_{x_c x_b} \]

Claim: There exists "broken comb"

defining to a section \( \tilde{s} : B \to X \) containing \( \tilde{x}_1, \ldots, \tilde{x}_n \)
Theorem (with same proof)
\[ X \rightarrow B \text{ regular proper model} \]
Assume: For each \( b \in B \)
\( X^s_B \) is strongly rationally connected

(For each \( x \in X^s_B \) there exists \( f : T^1 \rightarrow X^s_B \)
\( f(0) = x \quad f(\infty) = \text{generic} \))

Then weak approximation holds

NB. Excludes reducible curves completely!!
Corollary

1. $X \rightarrow B$ regular proper with fibers cubic surfaces with at worst rational double points

Then weak approximation holds

2. (Subcase of 1)

Weak approximation holds for cubic surfaces with square-free discriminant "generic"