GENERALIZED BRUHAT DECOMPOSITIONS AND LANGLANDS DUALITY

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§1 B-N pairs algebraic groups and certain decompositions of groups and flag varieties.

First I wish to recall what a Coxeter presentation is. It is a pair (W, S) where W is a group and S is a set of generators (finite) called the "fundamental reflections" so that W is generated by $S = \{s_1 \cdots s_\ell\}$ and the only relations are of the form $s_i^2 = e$ and $(s_i s_j)^{m_{ij}} = e$. Then $\forall w \in W$ $w = s_{i_1} \cdots s_{i_g}$ is a reduced decomposition if it is of minimal length in which case $g = \text{length}(w) = \ell(w)$.

Let G be a group a B-N pair (Tits system) for G is a triple (B, N, S) so that:

i) B & N generate G
ii) $N \cap B$ is normal in N
iii) $S \to N/N\cap B$ is injective
and its element with $W = N/B \cap N$
constitute a Coxeter presentation.
iv) $s \in S$, $w \in W$ then

$$sBw \subset (BwB) \cup (BswB)$$

(NB: the cosets BwB and Bw are indep of the rep in N chosen for w)

In this case $G = \dot{\bigcup}_{w \in W} BwB$ (disj union)

Now suppose G is a linear algebraic group. Then G contains a maximal normal connected solvable group, its radical, $R \triangleleft G$, and R contains a maximal unipotent subgroup which is normal and normal in G as well. (Unipotent means that in any representation ~~e.g.~~ all representing transforms have eigenvalue only 1)

This normal unipotent subgroup is the <u>unipotent</u> <u>radical</u> of G and G is reductive if its unipotent radical is trivial.

A torus is a group which diagonalizes in every representation. If T is a k torus a character is an algebraic morphism $x : T \longrightarrow k^*$. For $t \in T$ write t^x for the value of x on t and write the group of characters $X(T) = \mathrm{Hom}(T\ k^*)$ additively. $X(T)$ is a finite free $\mathbb{Z}$ module (if $\mathbb{Z}$ is connected).

If G is a reductive group it contains a maximal torus, T and a maximal connected solvable subgroup B (a Borel subgroup). Since all the maximal k-split tori and the ~~[illegible]~~ Borel subgroups are conjugate ($k = \bar{k}$) we can take $T \subset B$. The normalizer $N = N_G(T)$ (a Cartan subg is a closed algebraic subgroup and, in a reductive group, N°, the connected component, is equal to T. Then $B \cap N = T$ and there is a set $S = \{s_1, \dots s_\ell\}$ so that $(B\ N\ S)$ is a B-N pair for G. The group $N/_{N\cap B} = N/_T$ is called the <u>Weyl group</u> of G. Since N operates on T by conjugation, $N/_T$ operates on $X(T) = X$ and so on $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$.

If $\mathfrak{g} = T_G(e)$ is the tangent space to G at e, G and hence T operates on it by conjugation. This is the adjoint representation. Since T is diagonable, we can write

$$\mathfrak{g} = \mathfrak{z} \oplus \coprod_{\alpha \in \Phi} \mathfrak{g}^\alpha \text{ where } \mathfrak{z} = T_e(\mathfrak{z}),$$

~~and~~ the $\alpha \in \Phi$ are a set of non trivial characters of T and $\mathfrak{g}^\alpha \neq (0)$ are

the distinct eigenspaces of T on $\mathfrak{g}$
Then the following summarizes what is classically known:

i) $\dim_k \mathfrak{g}^\alpha = 1$

(ii) W permutes the $\alpha \in \Phi$

iii) The $\alpha \in \Phi$ span a subgroup of finite index in X.

iv) For any $\alpha \in \Phi$ there is a unique $\alpha^\vee \in X^* = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ and an element $s_\alpha \in W$ so that for $\xi \in X$, $s_\alpha(\xi) = \xi - \alpha^\vee(\xi)\alpha$.

Moreover for $\alpha \in \Phi$, there are subgroups $U_\alpha, U_{-\alpha}$ each isomorphic to k^+ by isomorphisms $x_\alpha : k^+ \to G$ so that $t x_\alpha(a) t^{-1} = x_\alpha(t^\alpha a)$ and so that U_α and $U_{-\alpha}$ generate a subgroup isomorphic to $SL(2, k)$ or $PGL(2, k)$ and so that $\mathfrak{g}^\alpha$ is $T_{U_\alpha}(e)$.

The fundamental reflections S are a particular subset of the s_α

corresponding to a linearly independent set of α's which span X over $\mathbb{R}$.

The set Φ with the α and $\alpha^\vee$ are the ROOT SYSTEM OF G.
This is a set

$$\Phi \subset V \text{ (real vector space)}$$

which is finite does not contain 0 and which spans V so that for any $\alpha \in \Phi$, $\exists \alpha^\vee \in V^*$ so that $s_\alpha(\Phi) = \Phi$ where $s_\alpha(\xi) = \xi - \alpha^\vee(\xi)\alpha$
$\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ is the DUAL ROOT SYSTEM.

The Φ classify semi simple groups, and a group exists for each Φ, up to isogeny. Φ with the character module $X \subset V$ classifies up to isomorphism. By existence $\Phi^\vee$ corresponds to a group $\widehat{G}$, the Langlands dual of G.

Aside from general existence theory there is no known functorial way to get $\hat{G}$ from G, a fact which is central to the difficulty of the Langlands program and the motivation of much of Geometric Langlands theory.

Now, for any closed subgroup $K \subseteq G$, the homogeneous space G/K (coset space) is a quasi projective variety. It is projective if and only if K contains a Borel subgroup, in which case K is called parabolic. Hence G/B is projective. The double coset BwB is of dimension $\dim B + l(w)$. Corresponding to the Bruhat decomposition, $G = \dot{\bigcup} BwB$, there is an orbit decomposition,
$G/B = \dot{\bigcup} B\bar{w}$, $\bar{w} = wB$ = coset of w.

Then $B\bar{w}$ is isomorphic to an affine k-space of dimension $\ell(w)$ and so the orbit stratification is a cell decomposition. The closures $\overline{B\bar{w}} = X_w$ are called generalized Schubert cells and they play a central role in the geometry of G/B, which is the generalized <u>flag</u> <u>variety</u> of G.

The most remarkable facts about these Schubert cells are that they are normal and Cohen-Macaulay. Moreover the inclusion relations amongst them are $X_{w'} \subset X_w$ if and only if $w' \leq w$ in the Chevalley Bruhat order, which I now explain:

$w' \leq w$ if and only if, $\exists$ a reduced decomposition

$w = s_{i_1} \cdots s_{i_q}$ so that w' is obtained from w by crossing out reflections

In any case, since $G/B = \dot{\bigcup}_{w \in W} B\bar{w}$ is a
<u>cell decomposition</u> (Recall it is $\cong \mathbb{A}_k^{\ell(w)}$)
whenever one wishes to compute invariants of G/B, these invariants are expressed in terms of these cells, or more properly their closures X_w. One finds that for example the Chow ring, the Grothendieck ring or the Borel-Moore homology where appropriate of G/B is generated by fundamental ~~cla~~ classes indexed by objects $X(w)$ associated to the root data of G.

2. <u>Valued fields</u> Suppose now that K is a valued field. Examples are $\mathbb{C}((t))$ or $K = \mathrm{Frac}(W(\bar{\mathbb{F}}_p))$, the fraction field of the Witt vectors of the alg. closure of $\mathbb{F}_p$. When we consider $G(K)$ something interesting happens. Write $v : K^* \longrightarrow \mathbb{Z}$ for the valuation and let π be the uniformizer.

I want to begin with an aside. One can construct the infinite Grassman in the following way. Consider $K^n \supset \mathcal{O}^n = F$. That is we consider a designated maximal $\mathcal{O}$-lattice in the vector space K^n. Then $SL(n,K)$ operates on the set of maximal lattices in K^n by the natural action and the stabilizer of F is $SL(n,\mathcal{O})$. The orbit of F under $SL(n,K)$ is the set of maximal lattices $L \subset K^n$ such that $\Lambda^n L = \Lambda^n F$ in $\Lambda^n K^n$. This orbit is canonically the infinite Grassman variety. How can we apply what we know to study the geometry of this thing, $G(K)/G(\mathcal{O})$ where $G = SL(n,K)$. Well in $G(K)$ there is yet another B·N pair structure and this is how Langlands duality enters.

The valuation v allows us to define a map $T(K)$ to $X(T)^* = \mathrm{Hom}(X, \mathbb{Z})$. We define $\eta : T(K) \longrightarrow X(T)^*$ by the equation

$$\langle \eta(t), \chi \rangle = v(t^\chi).$$

Take $N = N_G(T) = N_{G(K)}(T(K))$ as before but for B we change. The map $\mathcal{O} \to k$ (res. cl. field) induces a map $G(\mathcal{O}) \xrightarrow{\xi} G(k)$. If $B \subset G(k)$ is a Borel subgroup of $G(k)$, let $\tilde{B} = \xi^{-1}(B)$. Then $\tilde{B} \subset G(\mathcal{O})$. It is called an Iwahori subgroup of $G = G(K)$.

Then $\tilde{B}$, N and a particular set of generators for $N/N\cap\tilde{B}$ is another Tits system for G. First note that $N \cap \tilde{B} = T(\mathcal{O})$. Now note that $T(K) \simeq K^{*\ell}$ via a basis $\omega_1 \cdots \omega_\ell$ of $X(T)$. Then $\omega_i(t) \in \mathcal{O}^*$ for all

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i if and only if $v(\omega_i(t)) = 0$ for all i if and only if $v(\chi(t)) = 0$ for all characters $\chi \in X(T)$. That is $t \in \operatorname{Ker} \eta$ if and only if $t \in T(\mathcal{O}^*)$

Further one can check that as abstract groups $N_G(T(\mathcal{O})) = N_G(T(K)) = N$ and so if $\widetilde{W} = N/T(\mathcal{O})$ we have an exact sequence:

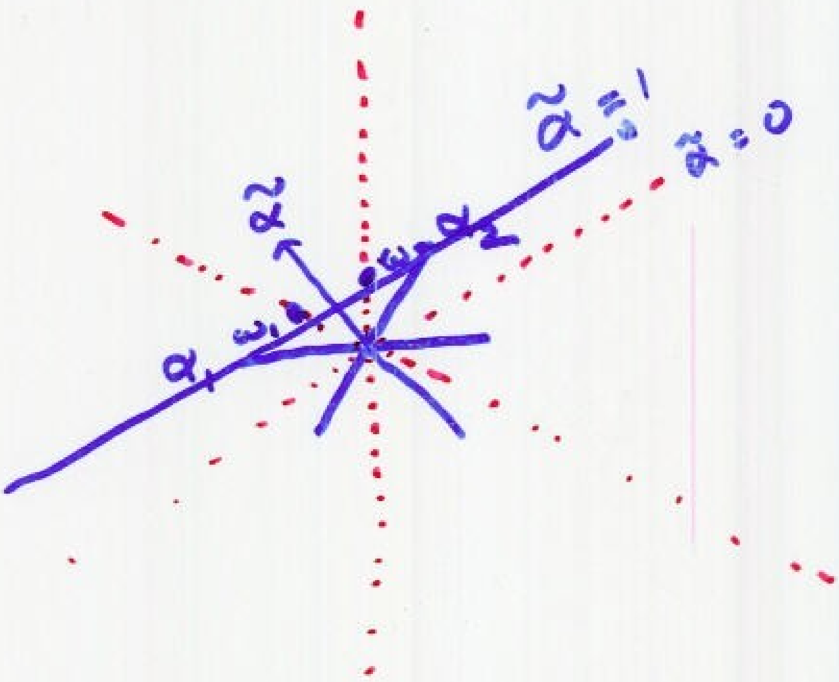
$$1 \to T(K)/T(\mathcal{O}) \longrightarrow N/T(\mathcal{O}) \longrightarrow W \to 1$$

Now W is $N/T(K)$ the classical Weyl group. On the other hand $N/T(\mathcal{O}) = \widetilde{W}$ is an extension of W by $T(K)/T(\mathcal{O})$ and $T(K)/T(\mathcal{O}) = \operatorname{Im} \eta = X(T)^*$ the dual of the character group. Write $X^\vee$ for this "group of cocharacters". This extension, $\widetilde{W}$:

$$0 \to X^\vee \to \widetilde{W} \to W \to 1$$

is the "Affine Weyl group" and it is also a Coxeter group. For example if G is simply connected X is the weights of Φ and $X^\vee$ is the co-root lattice. One takes the longest co root $\tilde{\alpha}^\vee$ and one considers the hyperplane $\tilde{\alpha}^\vee(x) = 1$. Add this reflection to ~~and~~ the reflections generating W and one obtains $\widetilde{W}$. This is not always true but it is true for the simply connected group and then one can work out the appropriate "Coxeter geometry" from this case. The picture for $SL(3\ K)$ is below

A_3:



To pass to smaller character grps & more complicated root systems one needs to work a bit but one *thinks of this picture as "true in principle". So we think that we have a new ~~#~~ decomposition:

$$G = \dot{\bigcup_{w \in \tilde{W}}} \tilde{B} w \tilde{B}.$$

Write $\tilde{W} = X^*_{\cdot} W$ and remember that there is a "positive Weyl chamber" T. (~~so~~ a connected component of the compl of the hyper planes $\alpha^\perp = H_\alpha$, $\alpha \in \Phi$.)

The classical Weyl group operates on these components simply transitively so that any $\gamma \in X^*$ can be written $\gamma = w\lambda w^{-1}$, $\lambda \in T$.

Thus one gets various decompositions of G and $G/G(O)$ the inf grassm

$$\text{First } G = \dot{\bigcup_{\substack{\gamma \in X^* \\ w \in W}}} \tilde{B}\gamma w \tilde{B} = \dot{\bigcup_{\gamma \in X^*}} \tilde{B}\gamma \underset{w}{\cup} \tilde{B} w \ldots$$

$$= \dot{\bigcup_{\gamma \in X^*}} \tilde{B}\gamma G(O) = \dot{\bigcup_{\substack{\gamma \in T \\ w \in W}}} \tilde{B} w\gamma w^{-1} G(O)$$

$$= \dot{\bigcup_{\gamma \in X^* \cap T}} G(O)\cdot\gamma\, G(O).$$

Now this is a cell decomposition of G which descends to the orbit decomposition

$$G/G(O) = \dot{\bigcup_{\gamma \in X^* \cap T}} G(O)\cdot\bar{\gamma}.$$

Now here the cells are indexed

by __cocharacters__ and these are things associated to the Langlands dual group! Hence when one computes the invariants of the space $G(K)/G(\mathcal{O})$ one expects to obtain algebraic objects associated to the Langlands dual group $\hat{G}$. Developing and expounding upon this observation is what has become known as "Geometric Langlands". One considers such things as Grothendieck groups and Chow groups but also, in honor of the theory emerging from the proofs of the Kazhdan - Lusztig conjecture, the bounded derived category of $\mathcal{D}$-modules on ~~k~~ $G(K)/G(\mathcal{O})$ which are $G(\mathcal{O})$ equivariant, ~~etc.~~ Perverse sheaves with respect to the orbit stratification on certain [illegible]
etc.

It is a large program. I think of it as having started in the work of Lusztig & later Victor Ginzburg. After that one thinks of Mirkovic - Vilonen, Arkipov, and most particularly Bezrukavnikov & Gaitsgory. All figured out important parts of this. Of course Drinfeld & Beilinson also contributed. The influence of the latter comes through his role in establishing the Kazhdan Lusztig conjecture.

Some references:

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A survey article:

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Also things like

Astérisque 100

Bruhat-Tits Groupes réductifs sur un corps local ~~IHES~~
I IHÉS #41 II IHÉS 60

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For basics:

LINEAR ALGEBRAIC GROUPS, T. A Springer
Birkhauser:
Bourbaki for B-N pairs root systems
affine Weyl gps etc.

Kac-Moody groups, their flag
varieties and representation theory
Birkhauser Boston (Progress in Math
204)

2 papers by me (trying to rapport
this back to the Langlands program)
Infinite dimensional alg. geometry;
algebraic structures on p-adic
groups and their homogeneous spaces
Tohoku math J 57 2005

Projective embeddings of varieties
of special lattices Contemp Math
325 2003

Talk Thurs by A SANO
my student will talk on this then.