On the tangent spaces
to the space of
algebraic cycles

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In geometry, there are various uses for linearization:

- Tangent spaces to a variety
- 1st order variations (normal spaces to a subvariety - deformation theory)
- Iterative linear approximation to solving equations

In this talk, we will discuss those when the geometric object is $\mathbb{Z}_p(\mathbb{X})$. 
Outline

1a

- Some background motivation

- Two heuristics for TZ^w(\mathcal{F})
  - higher differentials
  - absolute differentials

- Definition of TZ^w(\mathcal{F})

- Tangent spaces to related groups

- Main result and some applications

- Issues raised
Some background motivation:

For a functor $F : \text{Rings} \to \text{Abelian groups}$

the formal tangent space is

$$T_f (R, F) = \ker \{ F(R \{ e \}) \to F(R) \}$$

Many years ago using the $p = 2$ case of

$$CH^p (\mathcal{X}) \cong H^p (\mathcal{X}, \mathcal{O}^\wedge_p)$$

for a smooth surface, Spencer Bloch arrived at

$$T_f CH^2 (\mathcal{X}) \cong H^2 (\Omega^1_{\mathcal{X} / \mathcal{A}})$$
We wanted to try to understand the geometric content of (x) by defining $TZ^p(X)$ leading to

$$T_g CH^p(X) = TZ^p(X)/TZ^p_{rat}(X)$$

and show that for $X$ a surface

$$T_g(X) \cong T_f(X)$$

i.e., to “lift” (x) to the level of cycles. We will propose a definition of $TZ^p(X)$ for $p = n, 1$ and will (i) give some geometric consequences (ii) discuss issues raised.
Will initially concentrate on the case $p=n$.

- Arc in $\mathbb{R}^{(d)}$ is $(B,t) \rightarrow \mathbb{R}^{(d)}$.

- Arc in $Z^m(\mathbb{R})$ is $\mathbb{Z}$-linear combination of arcs in $\mathbb{R}^{(d)}$'s.

- $Z^m_{\{x\}}(\mathbb{R}) = \left\{ \text{arcs } z(t) \text{ with } \lim_{t \to 0} \text{ supp } z(t) = x \right\}$

Want to define

$$T_x Z^m(\mathbb{R}) = Z^m_{\{x\}}(\mathbb{R}) / \sim_{Z^m}$$

(i) Vector space

(ii) $(z(t) \pm \bar{z}(t))' = z(t)' \pm \bar{z}(t)'$
Classical case \( n = p = 1 \)

\[ z(t) = \sum_{i} \pm x_{i}(t) \mapsto \mathbb{Z}^{2}(X) \]

\[ m \mapsto \frac{d}{dt} \left( \sum_{i} \pm \int_{X} x_{i}(t) \right) \]

\( t = 0 \)

\[ \mapsto \mathbb{Z}^{2}(X) \mapsto \operatorname{Hom}_{\mathbb{C}}(\Omega^{2}_{X}, \mathbb{C}) \]

- Suggests duality, differential forms
- Creation/annihilation arcs present but not that important - changes for \( n \leq 2 \) when "cancellations" occur
Another view of classical case

\[ 0 \to \mathcal{O}_X^* \to \mathcal{C}(X) \to \text{Div}(X) \to 0 \]

Considering arcs in \((\text{Div} X)_x\), leads to

\[ \mathcal{T}_{\text{Div}(X)} \cong \mathcal{C}(X)/\mathcal{O}_X^* \]

\[ \cong \mathcal{PP}(X)_x \]

\[ z(t) = \text{div} \left( f + t g \right) \]

\[ z(t) \leftrightarrow \left( z_f \right) \in \mathcal{PP}(X)_x \]

Above map is

\[ \langle z(t), w \rangle = \text{Res}_x \left( g \cdot w \right) \]

\[ w \to \text{Two heuristics} \]
\[ Z^n(X) = \left\{ \text{group associated to semi-group } X^{(\infty)} = \lim_{d \to \infty} X^{(d)} \right\} \]

(inclusions via choice of base point)

- regular forms \( \Omega^g \bigwedge X^{(d)} / C \)
- regular forms \( \Omega^g \bigwedge X^{(\infty)} / C \) given

by \( \varphi_d \in \Omega^g \bigwedge X^{(d)} / C \) with

\( \varphi_{d+1} / X^{(d)} = \varphi_d \) (hereditary property)

\( \varphi \in \Omega^g \bigwedge X / C \) \( \to \) \( \text{Tr} \varphi \in \Omega^g \bigwedge X^{(\infty)} / C \)

\( \text{Tr} \varphi \left( \sum_i x_i \right) = \sum_i \varphi(x_i) \)
\( X(d) \) singular along diagonals
for \( d \geq 2, n \leq 2 \)

**Differential geometric fact:** \( \Omega^* \mathcal{X}^{(\infty)} / \mathcal{C} \)

generated as an exterior algebra
over \( \Omega^\infty \mathcal{X} \) by \( Tr\phi, \phi \in \Omega^\infty \mathcal{X} / \mathcal{C} \)

where \( 1 \leq g \leq n \). Need all
degrees to generate

\[ \exists \sum \text{d}x_i \wedge \text{d}y_i + \Lambda^2 (\text{Traces 1-forms}) \]

generates \( \Omega^\infty \mathcal{X} / \mathcal{C} \)

\[ \rightarrow \text{need higher degree forms to detect} \]
the infinitesimal geometry of \( 0 \)-cycles for \( n \leq 2 \)
What is the information detected by higher degree forms?

\[ dp = \sum \omega^w e^w \]
\[ dw^w = \sum \omega^w v^w \]

Simplest are in \( \mathbb{Z}^2(\mathbb{R}) \) given by Puiseaux series

\[ z(t) = z_+(t) + z_-(t) \]

\[ z_{\pm}(t) = (x_{\pm}(t), y_{\pm}(t)) \]

\[ \begin{cases} x_{\pm}(t) = \pm a_2 t^{\frac{1}{2}} + a_3 t + \ldots \\
y_{\pm}(t) = \pm b_2 t^{\frac{1}{2}} + b_3 t + \ldots \end{cases} \]

Take "d" and the coefficient of dt

1-forms \( \rightarrow a_2, a_2 b_2, b_2^2, c_2, b_2 \)
\[ \text{Tr } d\mathbf{x} \rightarrow 2a_2 \]

\[ \text{Tr } x dx \rightarrow a_2 \]

\[ \text{Tr } x dx \rightarrow a_1 \]

\[ \text{Tr } x dy \rightarrow a_1 b_1 \]

\[ \text{Try } dy \rightarrow a_2 b_1 \]

\[ \text{Now go to } 2\text{-forms} \]

\[ \text{Tr } d\mathbf{x} \wedge d\mathbf{y} \rightarrow a_1 d b_1 - b_1 d a_2 \]

\[ \text{not a consequence of 1\text{-forms}} \]

\[ \text{only get } a_2 d\mathbf{x}_2 + b_2 d\mathbf{a}_2 \]
What do we mean by $d\varepsilon_2, d\varepsilon_3, \ldots$?

Hint given by the following:

**Example:** Let $Z_{ab}(t)$ be the arc given by

\[\begin{align*}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 0 \\
xy - \beta t &= 0
\end{align*}\]

(sums of Puiseaux series in $t^{1/2}$)

Let $F$ be the free group generated and $\equiv$ the equivalence relation generated by:

(i) $Z_3(t) \equiv Z_1(t)$ and $Z_2(t) \equiv Z_2(t)$

\[\Rightarrow Z_2(t) \pm Z_2(t) \equiv Z_2(t) \pm Z_2(t)\]

(ii) $Z(dt) \equiv Z(t)$ \quad $d \in \mathbb{Z}$
(iii) \( \mathcal{Z}(t) \) and \( \mathcal{Z}'(t) \) two arcs in Hilb with same tangent
\[ \Rightarrow \mathcal{Z}(t) \equiv \mathcal{Z}'(t) \]

(iv) \( \alpha \mathcal{Z}(t) \equiv \alpha \mathcal{Z}'(t) \) for \( \alpha \in \mathbb{Z}^* \)
\[ \Rightarrow \mathcal{Z}(t) \equiv \mathcal{Z}'(t) \]

**Theorem:**
\[ F/\mathcal{Z} \overset{\sim}{\longrightarrow} \Omega^1_{\Sigma}/\mathcal{O} \]

\[ z_{\beta}(t) \rightarrow \beta \frac{d\phi}{dt} \]

**Corollary:**
\[ z_{\beta}(t) \equiv z_{\beta}(t) \Leftrightarrow \beta \in \mathcal{O} \]

\[ \text{suggests we interpret } dz, d\beta \text{ as } \in \Omega^1_{\Sigma}/\mathcal{O} \]

\( \Rightarrow \text{ in } \Omega^2_{\Sigma}/\mathcal{O} \text{ - i.e. we have } \]

\[ T_{d \times \mathbb{Z}} Z^2(G) \rightarrow \text{Hom}^0(\Omega^2_{\Sigma}/\mathcal{O}, \Omega^1_{\Sigma}/\mathcal{O}) \]

Puiseaux series \( \rightarrow \) above construction
Step one: We will show that

\[(x^2 - m^2 y^2, xy - t) \equiv (x^2 - y^2, xy - t), \quad m \in \mathbb{Z}. \]

By (i) and (iv),

\[
(x^2 - m^2 y^2, xy - t) - (x^2 - y^2, xy - t)
\]

\[
= (x^2 - mt, y - \frac{x}{m}) + (x^2 + mt, y + \frac{x}{m})
\]

\[
= (x^2 - t, y - x) - (x^2 + t, y + x)
\]

(when expanded as sums of Puiseaux series, both sides are the same), and by (iii)

which by (i) and (iv) again is

\[
= \left( x^2 - t, \left( y - \frac{x}{m} \right)^m (y + x) \right) + \left( x^2 + t, \left( y + \frac{x}{m} \right)^m (y + x) \right).
\]

Now using (iii)

\[
(x^2 - t, \left( y - \frac{x}{m} \right)^m (y + x)) \equiv (x^2 - t, y^{m+1} + \left( \frac{m}{m^2} - 1 \right) y^{m-1} x^2
\]

\[
+ \left( \frac{m}{m^2} - \frac{m}{m^2} \right) y^{m-2} x^3
\]

because \(x^4 \equiv t^2 = 0\), and by the same idea the right-hand side is

\[
= \left( x^2 - t, y^{m+1} + \left( \frac{m}{m^2} - 1 \right) y^{m-1} + \left( \frac{m}{m^2} - \frac{m}{m^2} \right) y^{m-2} x \right) t
\]

which by (iii) and (iv) is

\[
\equiv (x^2 - t, y^{m-2}) + \left( x^2 - t, y^3 + t \left( \frac{m}{m^2} - 1 \right) y \right)
\]

\[
+ \left( x^2 - t, y^3 + t \left( \frac{m}{m^2} - \frac{m}{m^2} \right) x \right) - (x^2 - t, y^3).
\]

Similarly,

\[
(x^2 + t, \left( y + \frac{x}{m} \right)^m (y - x))
\]

\[
= (x^2 + t, y^{m-2}) + \left( x^2 + t, y^3 - t \left( \frac{m}{m^2} - 1 \right) y \right)
\]

\[
+ \left( x^2 + t, y^3 + t \left( \frac{m}{m^2} - \frac{m}{m^2} \right) x \right) - (x^2 + t, y^3).
\]

Now by (ii)

\[
(x^2 - t, y^{m-2}) = -(x^2 + t, y^{m-2})
\]

\[
(x^2 - t, y^3) = -(x^2 + t, y^3)
\]

and

\[
(x - t, y^3 + t \left( \frac{m}{m^2} - 1 \right) y) \equiv -(x + t, y^3 - t \left( \frac{m}{m^2} - 1 \right) y).
\]

By (ii) and (iv)

\[
(x^2 - t, y^3 + t \left( \frac{m}{m^2} - \frac{m}{m^2} \right) x) + \left( x^2 + t, y^3 + t \left( \frac{m}{m^2} - \frac{m}{m^2} \right) x \right)
\]

\[
\equiv \left( x^4, y^3 + t \left( \frac{m}{m^2} - \frac{m}{m^2} \right) x \right).
Now

\[ 0 \to \Omega^1_{X/Q} \to \Omega^1_{X/Q} \to \Omega^1_{X/Q} \to 0 \]

gives a filtration on \( \Omega^2_{X/Q} \to \Omega^2_{X/Q} \)

with

\[ \begin{align*}
\Omega^2_{X/Q} & \to C \\
\Omega^2_{X/Q} & \to \Omega^2_{C/Q}
\end{align*} \]

\[ \text{Defn. } \mathcal{I} \mathcal{Z}^a(X) = \lim \text{ Ext}^2_{X/Q} (\Omega^2_{X/Q}, \Omega^2_{X/Q}) \]

\[ \{ \text{Z codim 2 subscheme} \} \]

\[ \oplus \mathcal{H}^2_x \left( \mathcal{Z}^a(X) \right) \]

\[ x \in X \]

Pfaffian approach

(i) additive

(ii) depends only on \( Z(\theta) \) as a cycle

(iii) depends on \( Z(\theta) \) to 1st order

(iv) geometric
Algebraic approach

(i) depends on element of $\mathcal{H}ilb^2$ determined by $z(t)$ (uses work of Angéniol and Lejeune-Jalabert)

(ii) computes well in examples

- not clear Puiseaux approach satisfies (i)
- not clear algebraic approach satisfies (i), (ii)

Geometric interpretation

- Above does not work in analytic geometry ($\Omega^1 / \Omega^1 = \Omega^1_{\text{an}}$

- In algebraic geometry one has
the notion of a spread -
eq for $X$ defined $/ \mathbb{A}$
the spread is a family

\[ X \rightarrow S \]

defined over $\mathbb{A}$ where
$Q(S)=\mathbb{A}$ and $X_{\eta}=X$. We
have $\Omega^1_{S/\mathbb{A}_{\eta}} \cong \Omega^1_{X/\mathbb{A}}$

and the sequence

\[ 0 \rightarrow \Omega^1_{S/\mathbb{A}} \rightarrow \Omega^1_{X/\mathbb{A}} \rightarrow \Omega^1_{X/\mathbb{A}} \rightarrow 0 \]

($X=\overline{X}(\mathbb{A})$ mentioned above for $\mathbb{A}$)

is used by several people to study cyclot,
The notion of a spread also works for cycles. An arc $Z \subset X \times B$ has

\[
\dim T_x Z = 1
\]

has a spread $Z \subset X \times B$.

Roughly, $\Omega^2_{X(\mathbb{Q})/\mathbb{Q}} \cong \Omega^2_{X(\mathbb{Q})/\mathbb{Q}}$ and $T_{(x, y)} Z$ has (in this picture) $\dim 2$.
Tangent spaces to related groups

**Defn:** \( T^2(X) = \lim_{\text{codim} z \to \infty} \text{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_z, \mathcal{O}_X) \)

**Thm:** \( Z^2(X) \) is unobstructed

- Being in the kernel reflects geometrically
- \( \text{ker} \) reflects \( \mathcal{O}_z \to \bigoplus_{x \in X} \mathcal{H}^2(\mathcal{O}_{x}) \)
For $\mathbf{Y}$ a smooth curve

$$Z^0_Z(\mathbf{Y}) = \bigoplus_{y \in \mathbf{Y}} \mathbb{Z}_y$$

$$T Z^0_Z(\mathbf{Y}) = \bigoplus_{y \in \mathbf{Y}} \text{Hom}^0_\mathbb{C} \left( \Omega^2 \mathbb{Z}_{1/\mathbf{X}}, \mathbb{C} \right)$$

For

$$Z^1_Z(\mathbf{Y}) = \bigoplus_{y \in \mathbf{Y}} \mathbb{C}$$

what is $T Z^1_Z(\mathbf{Y})$? Reasonable axioms as in the $\mathbb{Z}_\mathbf{p}(\mathbf{t})$ example lead to the

**Def.** $T Z^1_Z(\mathbf{Y}) = \bigoplus_{y \in \mathbf{Y}} \text{Hom}^0_\mathbb{C} \left( \Omega^2 \mathbb{Z}_{1/\mathbf{X}}, \Omega^2 \mathbb{D}_\mathbf{D} \right) \approx \bigoplus_{y \in \mathbf{Y}} H^1_Z(\mathbb{Z}_t(\mathbf{Q}))$
For $X$ a smooth surface

$$Z^2_1(X) = \bigoplus C^*(Y)$$

$$= \text{Irred} \oplus \text{Cohom}_z$$

**Defn:** $T^2 Z^2_1(X) = \bigoplus H^2_2(\Omega^2_{\mathcal{O}/X})$

$$= \text{Irred} \oplus \text{Cohom}_z$$

(no compatibility conditions when an irreducible curve becomes reducible - see below)

**Defn:** $T Y^2_1(X) = T Z^2_1(X) \setminus \text{image} \{ T Z^2_1(X) \rightarrow T Z^2_0 \}$

where we have

$$Z^2_1(X) \rightarrow Z^2_0(X) \subset Z^2_0(X)$$

$$\text{div} \rightarrow \text{not}$$
Ex Consider are in $\mathbb{Z}_2^2(X)$
given by

$$xy = t$$

$$f_t = \frac{x^2 - y^2}{x^2 + y^2}$$

Over $t = 0$, $f_0$ has different values ±1 at origin on the two components.
Remark

\[ T_{\mathbb{Z}^2}(X) = \lim_{\mathbb{Z} \to \mathbb{R}} \text{Ext}^2_{\mathcal{A}}(\mathcal{O}_Z, \mathcal{O}_X) \]

\[ \text{S} \ni \ker \{ \bigoplus H^2_y(\cdot) \to \bigoplus H^2_x(\cdot) \} \]

\[ T_{\mathbb{Z}^2}(X) = \bigoplus H^2_y(\cdot) \]

\[ \lim_{\mathbb{Z} \to \mathbb{R}} \text{Ext}^2_{\mathcal{A}}(\mathcal{O}_Z, \mathcal{O}_X) / \mathcal{A} \]

(no compatibility conditions)
Theorem: $H^a_\mathcal{F}(X) \cong \hat{T}_\phi \hat{C}H^a(X)$

Corollary: Suppose $X/\overline{Q}$ and $(x_i, \tilde{w}_i) \in T_{x_i}^\nu(X(\overline{Q}))$ with

$$\sum_i \langle \omega(x_i), \tilde{w}_i \rangle = 0$$

for all $\omega \in H^0(\Omega^2/\overline{Q})$. Then there exists $(x^*, \phi^*) \in TZ^\nu_\mathcal{F}(X(\overline{Q}))$ with

$$\sum_{\omega \in \Omega^2} \text{res}(x^*, \phi^*) = \sum_i \langle x_i, \tilde{w}_i \rangle$$

(infinite dimensional version of a special case of Bloch-Beilinson conjecture)
$T^2(X)$ can be used for non-existence ($f' \neq 0 \Rightarrow f$ non-constant).

Mumford: $H^0(\Omega^2_X) \neq 0 \Rightarrow \dim \mathcal{C}H^0(X) = \infty$.

(assume) $w \rightarrow$ generic $z, z' \in X^{(d)}$ are not rationally equivalent.

where "generic" means outside a countable union of proper subvarieties.

Suppose $X/\Theta$ and $x, y \in \Theta(X)$.

$z = \sum z_i, z_i = (x_i, y_i)$ and $\pi_i: T_{x_i}X \rightarrow X^{(d)}$ with

- $dx_i, dy_i$ linearly independent,
- $\omega(\pi_i) \neq 0$.

$\Rightarrow$ no rational equivalence with tangent $T = \sum (z_i, T_i)$.

$\Rightarrow$ "generic" has arithmetic meaning.
generic surface of degree 5 in $\mathbb{P}^3$ contains no rational curves (Clemens)

for each $k$ there is $d(k)$ such that a generic surface of degree $d \geq d(k)$ has no $k$ points

$k = 2 \quad d(k) = 5$
$k = 3 \quad d(k) = 6$

Voisin: On a generic surface of degree $\geq 7$ no two points are rationally equivalent —
"generic" means transcendental independence of coefficients of $F(x) = 0$
Issues

(i) Definition of $TZ^p(X)$
   - for general $p$
   - axiomatically
   - unobstructedness

(ii) **Null curve**

$\mathcal{I} \subset B \times \Xi$

\[
\mathcal{I} \xrightarrow{\mathcal{J}} \mathcal{CH}^q(\Xi) \quad \text{non-constant}
\]

but differential $\equiv 0$

- $(**)$ gives null if $\mathcal{I}$ regular
  and everything defined

- $\mathcal{CH}^q(P^2, T) \simeq \mathcal{K}_2(C)$ (Block-Suslin)

\{1 + td, $\beta$\} gives null curve if $\beta \in \mathcal{K}^*$

- Block-Beilinson $\Rightarrow$ only way can happen
\[ TK_2(C) \cong \Omega^2 C/\Theta \]

\[ \{z + td, \beta\} \rightarrow d \frac{df}{\beta} \]

\[ \rightarrow z(t) \text{ non-constant does not imply } z(t) \neq 0 \]
- Formal vs. actual

Assume $Z^p(X)$ has been defined (say algebraically) such that

\[ \text{gives } c \in V^3 \]

(i) can $c$ be lifted to

\[ \text{spec}(D[I^p]/I^p) \to Z^2(V) \]

for all $\epsilon$?

(ii) is $c$ tangent to geometric

are in $Z^2(V)$?
\[ \overline{V} \text{ defined over } \overline{\mathbb{Q}} \text{ and } H^{2,2}(V) \oplus H^{4,2}(V) \]

not \text{ a sub-Hodge structure.}

We have (assuming \( h^{4,0}(V) = 0 \))

\[ H^2(\Omega^2_{\overline{X}/\overline{\mathbb{Q}}}) \rightarrow H^2(\Omega^2_{\overline{X}/\overline{\mathbb{Q}}}) \]

\[ \Rightarrow \quad TCH^2(V) \rightarrow T\mathcal{J}^2(V) \]

where top arrow is onto since \( V/\overline{\mathbb{Q}} \) and bottom arrow is \((AJ_V)_*\).
Estimates: Assume $X$ is regular and defined over $\mathbb{Q}$ and assume $BB$

for $p, q \in X(\overline{\mathbb{Q}})$ there is

\[
\begin{align*}
&\exists f_{p,q} : \mathbb{P}^2 \rightarrow X \\
&\text{such that} \\
&f_{p,q}(0) = p + z \\
&f_{p,q}(\infty) = q + z
\end{align*}
\]

Set $\delta_{p,q} = \deg f_{p,q}$

\[
\begin{cases}
\delta_{p,q} & \text{cannot bound } d_{p,q} \text{ and } \\
\delta_{p,q} & \text{for all } p, q \in X(\overline{\mathbb{Q}})
\end{cases}
\]

Let $H(p, q) = \text{height}$.
Setting $D(p,q) = d_{p,q} + \delta p, q$ may be

$$H(p,q) \to \infty \Rightarrow D(p,q) \to \infty$$

$$\exists \quad \mathcal{C}^n(P^{q,T}) = \mathcal{C}^{x,y}$$

$$\Rightarrow \quad p \otimes q = \prod_{r \in \mathbb{R}}$$

?? $T\mathbb{F}^0(\mathbb{X})$ needs additional arithmetic information in order for any sort of iterative process to be convergent
Specific question: For $X$ regular, we have

$$\mathcal{Z} = \sum (x_i, \tau_i) \in TZ^a(X(\bar{a}))$$

we have

$$\mathcal{Z} = \text{Ran} \sum \mathcal{Z}''(y_i, f_i)$$

can we bound

$$D(\mathcal{Z}(y_i, f_i)) + H(\mathcal{Z}(y_i, f_i))$$

in terms of

$$D(\mathcal{Z} + H(\mathcal{Z}))$$

needs nullstellensatz with bounds on $D + H$
Conclusions: Proposed definition of $\mathcal{H}^2(\mathcal{X})$ "explains" geometrically why in codimension $p \geq 2$

- higher differentials (up to degree $p$) appear
- absolute differentials (specifically $\Omega^{p-2}_{\mathcal{X}/\mathcal{O}}$ appear)
- But: definition needs some "arithmetic supplement" to go further