On the tangent spaces to the space of algebraic cycles

Joint with Mark Green

Ref: Annals of Math Studies 157

In geometry there are various uses for linearization

- tangent spaces to a variety
- 1st order variations (normal spaces to a subvariety - deformation theory)
- iterative linear approximation to solving equations

In this talk we will discuss these when the geometric object is $Z^p(X)$

Outline

- Some background motivation
- Two heurestrics for $TZ^{\sim}(X)$
  - higher differentials
  - absolute differentials
- Definition of $TZ^{\sim}(X)$
- Tangent spaces to related groups
- Main result and some applications
- Issues raised

Some background motivation:

For a suitable functor

$$\text{Rings} \xrightarrow{F} \text{Abelian groups}$$

the formal tangent space is

$$T_f(R, F) = \ker\{F(R[\epsilon]) \to F(R)\}$$

Many years ago using the $p = 2$ case of

$$CH^p(X) \cong H^p(\mathcal{K}_p(\mathcal{O}_X))$$

for a smooth surface, Spencer Bloch arrived at

$$(*) \qquad T_f CH^2(X) \cong H^2(\Omega^1_{X/\mathbb{Q}})$$

We wanted to try to understand the geometric content of (*) by defining $TZ^p(X)$ leading to

$$T_g CH^p(X) = TZ^p(X)/TZ^p_{rat}(X)$$

and show that for X a surface

$$T_g(X) \cong T_f(X);$$

i.e., to "lift" (*) to the level of cycles. Will propose a definition of $TZ^p(X)$ for $p = n, 1$ and will (i) give some geometric consequences (ii) discuss issues raised

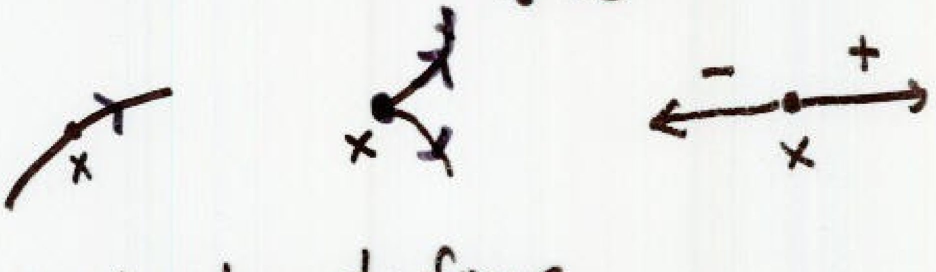
Will initially concentrate on the case $p = n$

- arc in $X^{(d)}$ is $(B, t) \to X^{(d)}$

  written $z(t) = \sum_i x_i(t)$

- arc in $Z^n(X)$ is $\mathbb{Z}$-linear combination of arcs in $X^{(d)}$'s

- $Z^n_{\{x\}}(X) = \left\{ \begin{array}{l} \text{arcs } z(t) \text{ with} \\ \lim\limits_{t \to 0} \text{supp } z(t) = x \end{array} \right\}$



Want to define

$$T_x Z^n(X) = Z^n_{\{x\}}(X) / \equiv_{1st}$$

(i) vector space

(ii) $\left(z(t) \pm \tilde{z}(t)\right)' = z(t)' \pm \tilde{z}(t)'$

Classical case $n = p = 1$

$$z(t) = \sum_i \pm x_i(t) \in Z^1_{\{x\}}(X)$$

$$\rightsquigarrow \frac{d}{dt}\left(\sum_i \pm \int_x^{x_i(t)} \omega \right)_{t=0}, \quad \omega \in \Omega^1_{X|\mathbb{C},x}$$

$$\rightsquigarrow Z^1_{\{x\}}(X) \to \mathrm{Hom}^0_{\mathbb{C}}(\Omega^1_{X|\mathbb{C},x}, \mathbb{C})$$

- Suggests duality, differential forms
- Creation/annihilation arcs present but not that important - changes for $n \geq 2$ when "cancellations" occur

Another view of classical case

$$0 \to \mathcal{O}_X^* \to \underline{\mathbb{C}}(X)^* \to \underline{\underline{\mathrm{Div}}}(X) \to 0$$

Considering arcs in $(\underline{\underline{\mathrm{Div}}}\, X)_x$ leads to

$$\underline{\underline{T}}\,\mathrm{Div}(X)_x \cong \underline{\mathbb{C}}(X)/\mathcal{O}_{X,x}$$
$$\cong PP(X)_x$$

$$z_f(t) = \mathrm{div}(f + tg)$$
$$z(t)' \leftrightarrow [\gamma_f] \in PP(X)_x$$

Above map is

$$\langle z(t), \omega \rangle = \mathrm{Res}_x(g\,\omega)$$

$\rightsquigarrow$ Two heuristics

$$Z^n(X) = \begin{cases} \text{group associated to} \\ \text{semi-group } X^{(\infty)} = \lim_d X^{(d)} \end{cases}$$

(inclusions via choice of base point)

- regular forms $\Omega^q_{X^{(d)}/\mathbb{C}}$ defined
- regular forms $\Omega^q_{X^{(\infty)}/\mathbb{C}}$ given

by $\varphi_d \in \Omega^q_{X^{(d)}/\mathbb{C}}$ with

$$\varphi_{d+1}\big|_{X^{(d)}} = \varphi_d \qquad \left(\begin{matrix}\text{hereditary} \\ \text{property}\end{matrix}\right)$$

- $\varphi \in \Omega^q_{X/\mathbb{C}} \rightsquigarrow \operatorname{Tr}\varphi \in \Omega^q_{X^{(\infty)}/\mathbb{C}}$

$$\operatorname{Tr}\varphi\Big(\sum_i x_i\Big) = \sum_i \varphi(x_i)$$

$X^{(d)}$ singular along diagonals

for $d \geqq 2$, $n \geqq 2$

<u>Differential geometric fact</u>: $\Omega^*_{X^{(\infty)}/\mathbb{C}}$

generated as an exterior algebra

over $\mathcal{O}_{X^{(\infty)}}$ by $\mathrm{Tr}\,\varphi$, $\varphi \in \Omega^q_{X/\mathbb{C}}$

where $1 \leqq q \leqq n$. Need <u>all</u>

degrees to generate

<u>Ex</u> $\sum_i dx_i \wedge dy_i + \Lambda^2$ (Traces 1-forms)

generates $\Omega^2_{X^{(\infty)}/\mathbb{C}}$

$\rightsquigarrow$ need higher degree forms to detect the infinitesimal geometry of 0-cycles for $n \geqq 2$

What is the information detected by higher degree forms?

$$dp = \sum_i \omega^i e_i$$
$$d\omega^i = \sum_j \omega^i_j \wedge \omega^j$$
} E. Cart

Simplest are in $Z^2(X)$ given by Puiseaux series

$$z(t) = z_+(t) + z_-(t)$$

$$z_\pm(t) = (x_\pm(t), y_\pm(t))$$

$$\begin{cases} x_\pm(t) = \pm a_1 t^{1/2} + a_2 t + \cdots \\ y_\pm(t) = \pm b_1 t^{1/2} + b_2 t + \cdots \end{cases}$$

Take "d" and the coefficient of dt

1-forms $\leadsto a_1^2, a_1 b_1, b_1^2, a_2, b_2$

eg

$\mathrm{Tr}\, dx \longrightarrow 2a_2$ $\qquad$ $\mathrm{Tr}\, x\,dy \to a_1 b_1$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \|$

$\mathrm{Tr}\, x\,dx \longrightarrow a_1^2$ $\qquad$ $\mathrm{Tr}\, y\,dx \to a_1 b_1$

Now go to 2-forms

$$\mathrm{Tr}\, dx \wedge dy \longrightarrow a_1 db_1 - b_1 da_1$$

$\|$

$dx_1 \wedge dy_1 + dx_2 \wedge dy_2$

not a consequence of 1-forms (only get $a_1 db_1 + b_1 da_1$)

What do we mean by $da_1, db_1 \dots$?

Hint given by the following

<u>Example</u>: Let $z_{\alpha\beta}(t)$ be the arc given by

$$\begin{cases} x^2-\alpha y^2=0 \\ xy-\beta t=0 \end{cases} \qquad \alpha\neq 0$$

(sums of Puiseaux series in $t^{1/2}$)

Let F be the free group generated and $\equiv$ the equivalence relation generated by

(i) $z_1(t)\equiv\tilde{z}_1(t)$ and $z_2(t)\equiv\tilde{z}_2(t)$

$$\Rightarrow z_1(t)\pm z_2(t)\equiv\tilde{z}_1(t)\pm\tilde{z}_2(t)$$

(ii) $z(\alpha t)\equiv\alpha z(t) \qquad \alpha\in\mathbb{Z}$

(iii) $z(t)$ and $\tilde{z}(t)$ two arcs in Hilb with same tangent

$$\Rightarrow z(t) \equiv \tilde{z}(t)$$

(iv) $\alpha z(t) \equiv \alpha \tilde{z}(t)$ for $\alpha \in \mathbb{Z}^*$

$$\Rightarrow z(t) \equiv \tilde{z}(t)$$

Theorem: $$F/\equiv \xrightarrow{\sim} \Omega^1_{\mathbb{C}/\mathbb{Q}}$$
$$z_{\alpha\beta}(t) \longrightarrow \beta \frac{d\alpha}{\alpha}$$

Corollary: $z_{\alpha\beta}(t) \equiv z_{1\beta}(t) \Leftrightarrow \alpha \in \overline{\mathbb{Q}}$

$\rightsquigarrow$ suggests we interpret $da_1, db_1, \ldots$ in $\Omega^1_{\mathbb{C}/\mathbb{Q}}$ - i.e. we have

$$T_{\{x\}} Z^2(X) \longrightarrow \mathrm{Hom}^0_{\mathbb{C}}(\Omega^2_{X/\mathbb{Q}}/\Omega^2_{\mathbb{C}/\mathbb{Q}}, \Omega^1_{\mathbb{C}/\mathbb{Q}})$$

Puiseaux series $\longrightarrow$ above construction

Step one: We will show that

$$(x^2 - m^2y^2, xy - t) \equiv (x^2 - y^2, xy - t), \quad m \in \mathbb{Z}. \tag{6.2}$$

By (i) and (iv),

$$\begin{aligned}(x^2 - m^2y^2, xy - t) &- (x^2 - y^2, xy - t)\\ &\equiv \left(x^2 - mt, y - \frac{x}{m}\right) + \left(x^2 + mt, y + \frac{x}{m}\right)\\ &\quad -(x^2 - t, y - x) - (x^2 + t, y + x)\end{aligned}$$

(when expanded as sums of Puiseaux series, both sides are the same), and by (iii)

which by (i) and (iv) again is

$$\equiv \left(x^2 - t, \left(y - \frac{x}{m}\right)^m (y + x)\right) + \left(x^2 + t, \left(y + \frac{x}{m}\right)^m (y + x)\right).$$

Now using (iii)

$$\begin{aligned}\left(x^2 - t, \left(y - \frac{x}{m}\right)^m (y + x)\right) \equiv \Bigg(x^2 - t, y^{m+1} &+ \left(\frac{\binom{m}{2}}{m^2} - 1\right) y^{m-2}x^2\\ &+ \left(\frac{\binom{m}{3}}{m^3} - \frac{\binom{m}{2}}{m^2}\right) y^{m-3}x^3\Bigg)\end{aligned}$$

because $x^4 \equiv t^2 \equiv 0$, and by the same idea the right-hand side is

$$\equiv \left(x^2 - t, y^{m+1} + \left(\left(\frac{\binom{m}{2}}{m^2} - 1\right) y^{m-1} + \left(\frac{\binom{m}{3}}{m^3} - \frac{\binom{m}{2}}{m^2}\right) y^{m-2}x\right) t\right)$$

which by (iii) and (iv) is

$$\begin{aligned}&\equiv (x^2 - t, y^{m-2}) + \left(x^2 - t, y^3 + t\left(\frac{\binom{m}{2}}{m^2} - 1\right) y\right)\\ &\quad + \left(x^2 - t, y^3 + t\left(\frac{\binom{m}{3}}{m^3} - \frac{\binom{m}{2}}{m^2}\right) x\right) - (x^2 - t, y^3).\end{aligned}$$

Similarly,

$$\begin{aligned}&\left(x^2 + t, \left(y + \frac{x}{m}\right)^m (y - x)\right)\\ &\quad\equiv (x^2 + t, y^{m-2}) + \left(x^2 + t, y^3 - t\left(\frac{\binom{m}{2}}{m^2} - 1\right) y\right)\\ &\quad\quad + \left(x^2 + t, y^3 + t\left(\frac{\binom{m}{3}}{m^3} - \frac{\binom{m}{2}}{m^2}\right) x\right) - (x^2 + t, y^3).\end{aligned}$$

Now by (ii)

$$(x^2 - t, y^{m-2}) \equiv -(x^2 + t, y^{m-2})$$
$$(x^2 - t, y^3) \equiv -(x^2 + t, y^3)$$

and

$$\left(x - t, y^3 + t\left(\frac{\binom{m}{2}}{m^2} - 1\right) y\right) \equiv -\left(x + t, y^3 - t\left(\frac{\binom{m}{2}}{m^2} - 1\right) y\right).$$

By (iii) and (iv)

$$\begin{aligned}\left(x^2 - t, y^3 + t\left(\frac{\binom{m}{3}}{m^3} - \frac{\binom{m}{2}}{m^2}\right) x\right) &+ \left(x^2 + t, y^3 + t\left(\frac{\binom{m}{3}}{m^3} - \frac{\binom{m}{2}}{m^2}\right) x\right)\\ &\equiv \left(x^4, y^3 + t\left(\frac{\binom{m}{3}}{m^3} - \frac{\binom{m}{2}}{m^2}\right) x\right)\end{aligned}$$

Now

$$0 \to \Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes \mathcal{O}_X \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{X/\mathbb{C}} \to 0$$

gives a filtration on $\Omega^2_{X/\mathbb{Q},x} / \Omega^2_{\mathbb{C}/\mathbb{Q}}$ with

$$\begin{cases} \Omega^2_{X/\mathbb{C},x} \longrightarrow \mathbb{C} \\ \Omega^2_{X/\mathbb{C},x} \dashrightarrow \Omega^1_{\mathbb{C}/\mathbb{Q}} \end{cases}$$

$\rightsquigarrow$ **Defn.** $\underline{\underline{T}} Z^2(X) = \lim\limits_{\substack{Z \text{ codim } 2 \\ \text{subscheme}}} \operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^1_{X/\mathbb{Q}})$

$$\underset{x \in X}{\oplus} H^2_x(\Omega^1_{X/\mathbb{Q}}) \overset{\text{SH}}{=}$$

<u>Puiseaux approach</u>

(i) additive

(ii) depends only on $z(t)$ as a cycle

(iii) depends on $z(t)$ to 1^{st} order

(iv) geometric

Algebraic approach

(i)´ depends on element of $\mathrm{THilb}^2($

determined by $z(t)$ (uses

work of Angéniol and

Lejeune-Jalabert)

(ii)´ computes well in examples

- not clear Puiseaux approach satisfies (i)´
- not clear algebraic approach satisfies (i), (ii)

Geometric interpretation

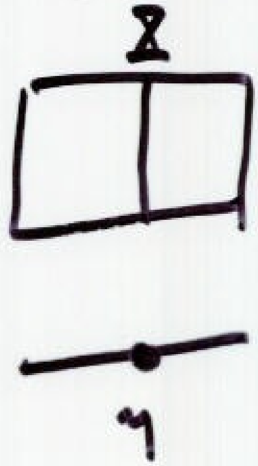
- Above does <u>not</u> work in analytic geometry ($\Omega^1_{X/\mathbb{C}} \neq \Omega^1_{X^{an}}$)
- In algebraic geometry one has

the notion of a **spread**.*

e.g. for X defined $/k$

the spread is a family

$$\begin{array}{c}\mathcal{X}\\ \downarrow\\ S\end{array}$$



defined over $\mathbb{Q}$ where

$\mathbb{Q}(S) = k$ and $X_\eta = X$. We

have $\Omega^1_{S/\mathbb{Q},\eta} \cong \Omega^1_{k/\mathbb{Q}}$

and the sequence

$$\mathcal{O}_{X(k)} \qquad 0 \to \Omega^1_{k/\mathbb{Q}} \otimes \mathcal{O}_{X(k)} \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{X/k} \to 0$$

($X = X(k)$) mentioned above for k=

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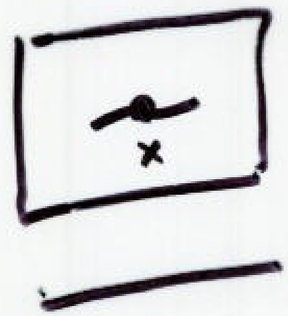
* Used by several people to study cycles, including M. Saito, S. Saito, Asakura, J. Lewis

The notion of a spread also works for cycles. An arc

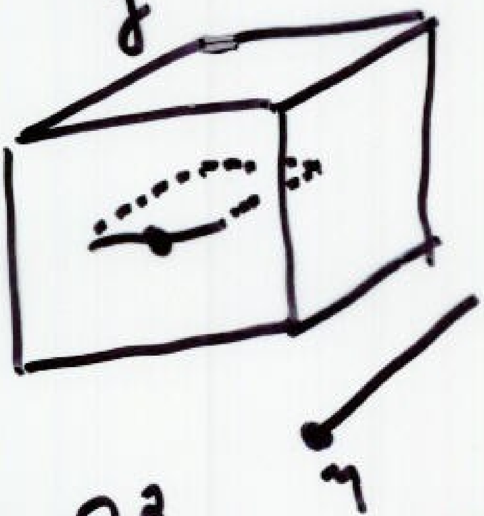
$$Z \subset X \times B \qquad /k$$



$(\dim T_x Z = 1)$

has a spread

$$\mathcal{Z} \subset \mathcal{X} \times B$$



Roughly $\Omega^2_{X(\mathbb{Q})/\mathbb{Q}} \cong \Omega^2_{\mathcal{X}(\mathbb{Q})/\mathbb{Q}}$ and $T_{(x,y)}\mathcal{Z}$ has (in this picture) dim 2

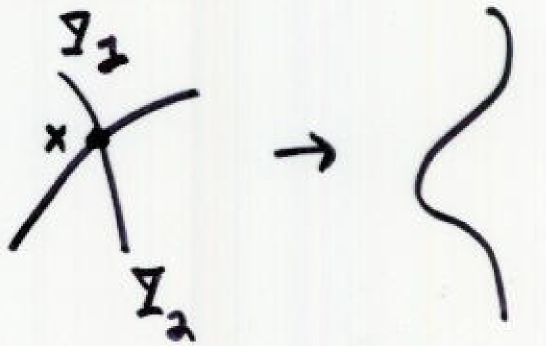
Tangent spaces to related groups

Defn: $$\underline{T}Z^1(X) = \lim_{\left\{\substack{Z \text{ codim } 1 \\ \text{subscheme}}\right.} \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X)$$

$$\cong$$

$$\ker\left\{ \bigoplus_{\left\{\substack{Y \text{ codim } 1 \\ \text{irreducible}}\right.} H^1_y(\mathcal{O}_X) \to \bigoplus_{x \in X} H^2_x(\mathcal{O}_X) \right\}$$

- Being in the kernel reflects geometrically



- **Thm**: $Z^1(X)$ is unobstructed

For X a smooth curve

$$\underline{\underline{Z}}^1(X) = \bigoplus_{y \in X} \underline{\underline{\mathbb{Z}}}_y$$

$$T\underline{\underline{Z}}^1(X) = \bigoplus_{y \in X} \underline{\underline{Hom}}^0_{\mathbb{C}}(\Omega^1_{X/\mathbb{Q},y}, \mathbb{C})$$

For

$$\underline{\underline{Z}}^1_1(X) = \bigoplus_{y \in X} \underline{\underline{\mathbb{C}}}^*_y$$

what is $T\underline{\underline{Z}}^1_1(X)$? Reasonable axioms as in the $z_{\alpha\beta}(t)$ example lead to the

<u>Defn</u>: $\underline{\underline{T}}Z^1_1(X) = \bigoplus_{y \in X} \underline{\underline{Hom}}^0_{\mathbb{C}}(\Omega^1_{X/\mathbb{Q},x}, \Omega^1_{\mathbb{C}/\mathbb{Q}})$

$$\cong \bigoplus_{y \in X} H^1_y(\Omega^1_{X/\mathbb{Q}})$$

For X a smooth surface

$$\underline{\underline{Z}}^1_1(X) = \bigoplus_{\substack{Y \text{ irred} \\ \text{codim } 1}} \underline{\underline{\mathbb{C}}}(Y)^*$$

Defn: $$\underline{\underline{T}} Z^1_1(X) = \bigoplus_{\substack{Y \text{ codim } 1 \\ \text{irred}}} H^1_y(\Omega^1_{X/\mathbb{Q}})$$

(no compatibility conditions when an irreducible curve becomes reducible - see below)

Defn $$T_y CH^2(X) = TZ^2(X) \Big/ \operatorname{image}\{TZ^1_1(X) \xrightarrow{\text{res}} TZ^2(X)\}$$

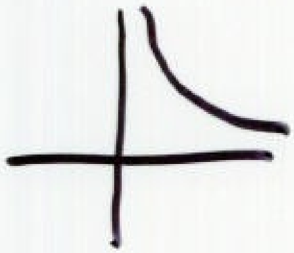
where we have

$$Z^1_1(X) \xrightarrow{\text{div}} Z^2_{\text{rat}}(X) \subset Z^2(X)$$

__Ex__ Consider arc in $Z_1^1(\mathfrak{X})$

given by

$$xy = t$$



$$f_t = \left.\frac{x^2 - y^2}{x^2 + y^2}\right|_{\mathfrak{X}_t}$$

Over $t = 0$, f_0 has

different values ± 1 at origin

on the two components

<u>Remark</u>

$$T\underline{\underline{Z}}^1(X) = \lim_{Z} \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X)$$

$$\begin{array}{c} \mathrel{\rotatebox{90}{$\subseteq$}}\mathrel{\|} \\ \ker\{\oplus H^2_y(\cdot\cdot) \to \oplus H^2_x(\cdot\cdot)\} \end{array}$$

$$T\underline{\underline{Z}}^1_1(X) = \oplus H^2_y(\cdot\cdot)$$

$$\nparallel$$

$$\lim_{Z} \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^1_{X/\mathbb{Q}})$$

(no compatibility conditions)

$$\boxed{\underline{\underline{\text{Theorem}}} : T_g CH^2(X) \cong T_f CH^2(X) = H^2(\Omega^1_{X/\mathbb{Q}})}$$

Corollary: Suppose $X/\bar{\mathbb{Q}}$ and $(x_i, \tau_i) \in T_{x_i}(X(\bar{\mathbb{Q}}))$ with

$$\sum_i \langle \omega(x_i), \tau_i \rangle = 0$$

for all $\omega \in H^0(\Omega^1_{X/\bar{\mathbb{Q}}})$. Then there exists $(Y_\nu, \varphi_\nu) \in TZ^1_1(X(\bar{\mathbb{Q}}))$ with

$$\sum_\nu \operatorname{res}(Y_\nu, \varphi_\nu) = \sum_i (x_i, \tau_i)$$

(infinitesimal version of a special case of Bloch-Beilinson conjecture)

$TZ^2(X)$ can be used for non-existence ($f' \neq 0 \Rightarrow f$ non constant)

<u>Mumford</u>: $H^0(\Omega^2_X) \neq 0 \Rightarrow \dim CH^2(X) = \infty$

(assume $g = 0$) $\rightsquigarrow$ generic $z, z' \in X^{(d)}$ are not rationally equivalent

where "generic" means outside a countable union of proper subvarieties

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Suppose $X/\bar{\mathbb{Q}}$ and $x, y \in \bar{\mathbb{Q}}(X)$

$z = \sum_i z_i$, $z_i = (x_i, y_i)$ and

$\tau_i \in T_{z_i}(X(\mathbb{C}))$ with

- dx_i, dy_i linearly independent ($\mathbb{C}$
- $\omega(\tau_i) \neq 0$

$\Rightarrow$ no rational equivalence with tangent $\tau = \sum_i (z_i, \tau_i)$

$\rightsquigarrow$ "generic" has arith meaning generic surface of degree 5 in $\mathbb{P}^3$ contains no rational curves (Clemens)

$\rightsquigarrow$ for each k there is $d(k)$ such that a generic surface of degree $d \geq d(k)$ has no g^1_k

$k = 1 \quad d(k) = 5$

$k = 2 \quad d(k) = 6$

$\vdots$

<u>Voisin</u> On a generic surface of degree ≥ 7 no two points are rationally equivalent -

"generic" means transcendental independence of coefficients of $F(x) = 0$

Issues

(i) Defn of $TZ^p(X)$

- for general p
- axiomatically
- unobstructedness

(ii) **Null curve**

$$Y \subset B \times X$$

(**) $\begin{cases} J(Y) \to CH^2(X) \text{ non-constant} \\ \text{but differential} \equiv 0 \end{cases}$

- (**) gives null curve if X regular and everything defined $/ \bar{\mathbb{Q}}$
- $CH^2(P^2, T)_2 \cong K_2(\mathbb{C})$ (Bloch-Suslin)

$\{1 + t\alpha, \beta\}$ gives null curve if $\beta \in \bar{\mathbb{Q}}^*$

- Bloch-Beilinson $\Rightarrow$ only way can happen

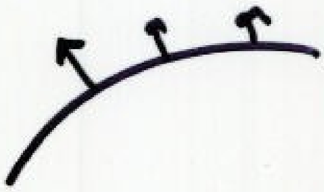
$$TK_2(\mathbb{C}) \cong \Omega^1_{\mathbb{C}/\mathbb{Q}}$$ (van der Kallen)

$$\begin{array}{ccc} \cup & & \cup \\ \{1+t\alpha, \beta\}' & \longrightarrow & \alpha \dfrac{d\beta}{\beta} \end{array}$$

$\rightarrow$ $z(t)$ non-constant does <u>not</u> imply $z(t)' \neq 0$

- Formal vs. actual

Assume $Z^p(X)$ has been defined (say algebraically) such that

 $C \subset V^3$

gives $\tau \in TZ^2(V)$

(i) can τ be lifted to

$$\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^{k+1}) \to Z^2(V)$$

for all k?

(ii) is τ tangent to geometric arc in $Z^2(V)$?

__No__: V defined over $\bar{\mathbb{Q}}$ and

$$H^{2,1}(V)\oplus H^{1,2}(V)$$

__not__ a sub-Hodge-structure -

We have (assuming $h^{3,0}(V)=0$)

$$\begin{tikzcd}
H^2(\Omega^1_{X/\mathbb{Q}}) \arrow[r, two heads] \arrow[d, phantom, sloped, "="] & H^2(\Omega^1_{X/\mathbb{C}}) \arrow[d, phantom, sloped, "\subset"] \\
TCH^2(V) \arrow[r] & TJ^2(V)
\end{tikzcd}$$

where top arrow is onto since $V/\bar{\mathbb{Q}}$ and bottom arrow is $(AJ_V)_*$

<u>Estimates</u>: Assume X is regular and defined over $\bar{\mathbb{Q}}$ and assume B-B

$$\Rightarrow \begin{cases} \text{for } p,q \in X(\bar{\mathbb{Q}}) \text{ there is} \\ \quad f_{p,q} : \mathbb{P}^1 \longrightarrow X^{(d_{p,q})} \\ \text{such that} \\ \quad f_{p,q}(0) = p + z \\ \quad f_{p,q}(\infty) = q + z \end{cases}$$

Set $\delta_{p,q} = \deg f_{p,q}$

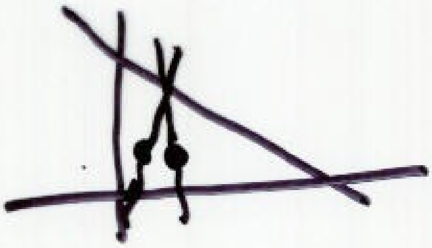
$$\rightsquigarrow \left\{ \begin{array}{l} \text{cannot bound } d_{p,q} \text{ and} \\ \delta_{p,q} \text{ for all } p,q \in X(\bar{\mathbb{Q}}) \end{array} \right\}$$

Let $H(p,q)$ = height .

Setting $D(p,q) = d_{p,q} + \delta_{p,q}$ maybe

$$H(p,q) \to \infty \Rightarrow D(p,q) \to \infty \; ?$$

<u>Ex</u> $\mathbb{C}H^2(\mathbb{P}^2, T)_2$, $\mathbb{P}^2 \setminus T \cong \mathbb{C}^* \times \mathbb{C}^*$



$$\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

$$\rightsquigarrow \quad p \otimes q = \prod_r r \otimes 1-r$$

?? $TZ^p(X)$ needs additional arithmetic information in order for any sort of iterative process to be convergent

<u>Specific question</u>: For X regular $/\bar{\mathbb{Q}}$ and

$$\tau = \sum_i (x_i, \tau_i) \in TZ^2(X(\bar{\mathbb{Q}}))$$

we have

$$\tau = \mathrm{Res} \sum_\nu (Z_\nu, f_\nu)$$

Can we bound

$$D\left(\sum_\nu (Z_\nu, f_\nu)\right) + H\left(\sum_\nu (Z_\nu, f_\nu)\right)$$

in terms of

$$D(\tau) + H(\tau) \quad ?$$

—(o)—

Needs nullstellensatz with bounds on $D + H$

<u>Conclusions</u> Proposed definition of $TZ^n(X)$ "explains" geometrically why in codimension $p \geq 2$

- higher differentials (up to degree p) appear
- absolute differentials (specifically $\Omega^{p-1}_{X/\mathbb{Q}}$ appear)
- <u>But</u> - definition needs some "arithmetic supplement" to go further