Hodge theory and algebraic cycles

Joint with Mark Green

Ref: IMRN, no 9

Outline

1. Algebraic cycles and their basic Hodge-theoretic invariants
2. Two main conjectures (status)
   (i) Generalized Hodge conj. (GHC)
   (ii) Bloch-Beilinson conj. (B^2)
3. Complete set of Hodge-theoretic invariants of $CH^p(X)_{\mathbb{Q}}$ (assuming GHC and B^2(iv))
4. Some examples and applications

X = smooth projective variety

$$F_\lambda(x) = \sum_I a_{\lambda I} x^I = 0 , \quad a_{\lambda I} \in k$$

$$Z^p(X) = \left\{ \begin{array}{l} \text{group of codimension-}p \\ \text{alg. cycles } Z = \sum_i n_i Z_i \end{array} \right.$$

$Z^p_{rat}(X)$ = subgroup generated by

$$\left\{ \begin{array}{l} Z_1 \equiv_{rat} Z_2 \text{ if have } \mathcal{Z} \subset X \times \mathbb{P}^1 \\ \text{with } \mathcal{Z} \cdot X \times \{0\} = Z_1,\ \mathcal{Z} \cdot X \times \{0\} = Z_2 \end{array} \right.$$

<u>Ex</u>

X

(X_v, f_v), $f_v \in \mathbb{C}(X_v)^*$

$$\sum_v \operatorname{div} f_v \equiv_{rat} 0$$

<u>Defn:</u> $CH^p(X) = Z^p(X) / Z^p_{rat}(X)$

The basic classical Hodge-theoretic invariants of $[Z]$ are encapsulated in its Deligne class

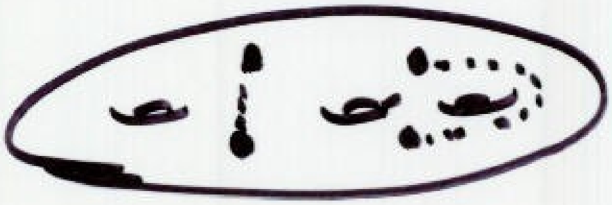
$$[Z]_{\mathcal{D}} \in H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

Roughly this consists of

$\psi_0(Z) \in H^{2p}(X, \mathbb{Z})$ (fund. class)

$\psi_1(Z) \in J^p(X)$ (if $\psi_0(Z) = 0$)

For algebraic curves



$$Z = \sum_i n_i p_i$$

- $\varphi_0(Z) = \deg Z = \int_Z 1$

where "1" $\in H^0(\Omega^0_X)$

- $\varphi_1(Z)(\omega) \equiv \int_\gamma \omega$ where

$\partial\gamma = Z$ and $\omega \in H^0(\Omega^1_X)$

In the "classical" case $p=1$

$[Z]_D$ captures rational equivalence

$$\varphi_0(Z) = \varphi_1(Z) = 0 \Leftrightarrow Z \equiv_{rat} 0$$

(moreover - φ_0 is onto the group

$Hg^p(X)$ and ψ_1 is onto $J^1(X)$)

When $p \geq 2$ we have, e.g.

(Mumford): $H^0(\Omega^2_X) \neq 0 \Rightarrow \dim(\ker \psi_1) = \infty$

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X is a compact Kähler manifold and we have

$$\left\{ \begin{array}{l} H^r(X, \mathbb{C}) = \bigoplus\limits_{p+q=r} H^{p,q}(X) \\ H^{p,q}(X) = \overline{H^{q,p}(X)} \end{array} \right.$$

and

$$H^{p,q}(X) \cong H^q(\Omega^p_X)$$

Setting

$$Hg^p(X) = H^{p,p}(X) \cap H^{2p}(X, \mathbb{Z})$$

we have

$$\begin{array}{ccc} CH^p(X) & \xrightarrow{\psi_0} & Hg^p(X) \\ \cup & & \\ CH^p(X)_0 & \xrightarrow{\psi_1} & J^p(X) \end{array}$$

where*

$$J^p(X) = F^p \backslash H^{2p-1}(X, \mathbb{C}) / H^{2p-1}(X, \mathbb{Z})$$

$$\cong F^{n-p+1\,*} / H_{2n-2p+1}(X, \mathbb{Z})$$

(Known that ψ_1 can seldom be onto for $p \geq 2$)

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* $F^m = \bigoplus_{p \geq m,\, q} H^{p,q}$, $F^{n-p+1} = $ "1^{st}" half of H $\underbrace{\cdot\ \cdot\ \cdot\ \cdot}\ \cdot\ \cdot\ \cdot\ \cdot$

Main conjectures

(HC) : Ψ_0 is onto

(GHC) : $\left\{ \begin{array}{l} \text{sub Hodge structure of} \\ \text{coweight } q \text{ is supported on} \\ \text{a subvariety of codimension } q \end{array} \right.$

(HC is case $r = 2p$, $q = p$)

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(B^2) : There exists $F^q CH^p(X)$

$\left\{ \begin{array}{l} \text{(i) } F^0 = \ker \Psi_0 \text{ and } F^1 = \ker \Psi_1 \\ \text{(ii) } Gr^q = F^q / F^{q+1} \text{ is described} \\ \qquad \text{Hodge theoretically} \\ \text{(iii) } F^{p+1} = 0 \end{array} \right.$

(iv) for X defined over $\bar{\mathbb{Q}}$

$$F^2 CH^p(X(\bar{\mathbb{Q}})) = 0$$

Status

- HC known for $p=1$ (Lefschetz)

two proofs - Poincaré-Lefschetz (normal functions)

Kodaira-Spencer - two steps

(i) $\zeta \in Hg^1(X) \Rightarrow \zeta = c_1(L)$ (Kähler fa

(ii) L is algebraic $\Rightarrow c_1(L) = [Z]$ (GAGA)

- for $p \geq 2$ false for torsion (Atiyah-Hirzebruch, Kollár)

→ For $p \geq 2$, everything is modulo torsion

- (i) in Kodaira-Spencer proof is false for $p \geq 2$ (Voisin)

- a few special examples and consequences of the $p=1$ case

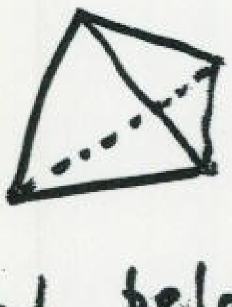
- GHC - few special examples

- first unknown cases

  HC : $n=4$, $p=2$

  GHC : $n=3$, $p=2$

- B^2 - one significant example



(Bloch-Suslin)

discussed below if time permits

- known "to 1^{st} order" for the first unknown case $n=p=2$ of points/0-cycles on a surface

New Hodge-theoretic invariants

Will first discuss a special case that gives some of the flavor of the story. For 0-cycles on a surface the classical invariants are

$$\psi_0(Z) = \int_Z 1, \qquad 1 \in H^0(\Omega^0)$$

$$\psi_1(Z) \equiv \int_\gamma \omega, \qquad \omega \in H^0(\Omega^1)$$

where $\partial\gamma = Z$ - those give F^0, F^1 on $CH^2(X)$ and we remarked

$$H^0(\Omega^2_X) \neq 0 \Rightarrow CH^2(X)_2 = \infty\text{-dim}$$

Bloch conjectured

$$CH^2(X)_{\mathbb{Z}} = 0 \Longleftrightarrow H^0(\Omega^2_X) = 0$$

Geometrically the issue is:

If $\Psi_0(Z) = \Psi_1(Z) = 0$, then

$$\Psi_2(X) \overset{?}{=} \int_\Gamma \varphi \quad , \; \varphi \in H^0(\Omega^2_X)$$

This question has a very beautiful "answer" (in general).

$\left\{\begin{array}{l} \text{GHC} \Rightarrow \text{construction well-defined} \\ B^2(\text{iv}) \Rightarrow \text{construction captures } \Xi_{rat} \end{array}\right.$

For simplicity assume $X/\mathbb{Q}$

and $Z \in Z^2(X(k))$

$$X: \quad F_\lambda(x) = \sum_I a_{\lambda I} x^I = 0$$

$$(*) \quad Z: \quad G_\alpha(x) = \sum_I b_{\alpha I} x^I = 0$$

where

$$k = \mathbb{Q}(\ldots, b_{\alpha I}, \ldots) \cong \mathbb{Q}(S)$$

$$\boxed{\rightsquigarrow \ \underline{\text{spread}} \ \mathcal{Z} \in Z^2(X \times S(\mathbb{Q}))}$$

Think of $\mathcal{Z} = \{Z_s\}_{s \in S}$ where

Z_s is given by $(*)$ where

the coefficients satisfy the same

relations $/\mathbb{Q}$ as the $b_{\alpha I}$. (Use of spreads has been "in the air")

<u>Issues</u>: (i) Ambiguities (choice of S, choice of k, later on in Ξ_{rat})

(ii) Why must spreads be considered only for $p \geq 2$?

(iii) Using spreads and factoring out ambiguities to define $F^q CH^p(X)$, why does the construction terminate at $q = p+1$?

(i) requires GHC

(iii) requires GHC + B^2(iv)

(ii) comes out just right

Back to 0-cycles on a surface

So what is $\int_\Gamma \varphi$, $\varphi \in H^0(\Omega^2_{\mathfrak{X}})$

There are two steps

(a) $$\int_\Sigma \mathrm{Tr}_{\mathfrak{z}}(\varphi), \quad \Sigma \in H_2(S, \mathbb{Z})$$

where

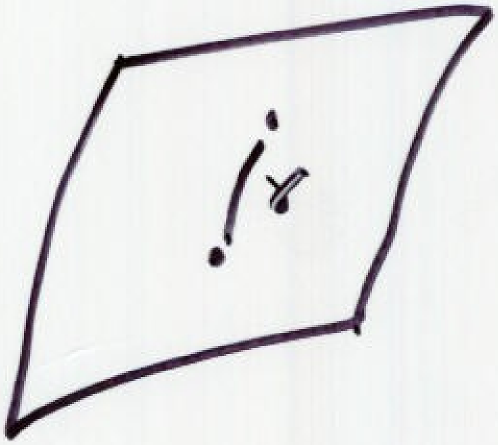
$$\mathrm{Tr}_{\mathfrak{z}}(\varphi)(s) = \sum_i n_i\, \varphi(p_i(s))$$

Assume $(a) = 0$. For $\lambda \in H_2(S, \mathbb{Z})$

$$\begin{cases} \{Z_s\}_{s \in \lambda} = \mathfrak{z}_\lambda \\ \{\gamma_s\}_{s \in \lambda} = \Gamma_\lambda \end{cases}, \quad \partial\gamma_s = Z_s$$

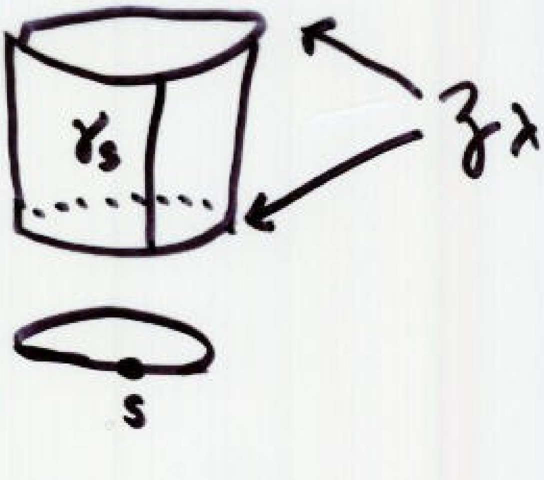
$\Rightarrow \partial \Gamma_\lambda = \mathfrak{z}_\lambda$

Σ



$S =$

$\Gamma_\lambda = \{$



We then consider

$$\text{(b)} \qquad \int_{\Gamma_\lambda} \varphi$$

By (a) $= 0$ this depends only the homology class of λ. Assuming the GHC and B^2 (iv) we have

<u>(a) and (b) are well-defined</u>

<u>and (a) $=$ (b) $= 0 \Rightarrow Z \equiv_{\text{rat}} 0$</u>

(mod torsion)

General case uses Künneth

$$H^*(X \times S) \cong H^*(X) \otimes H^*(S)$$

in total degrees $2p, 2p-1$

*

$$\boxed{\begin{array}{llllll} 2p & \psi_0(Z)_0 & \psi_0(Z)_1 & \psi_0(Z)_2 & \cdots & \psi_0(Z)_p \\ 2p-1 & 0 & \psi_1(Z)_0 & \psi_1(Z)_1 & \cdots & \psi_1(Z)_{p-1} \end{array}}$$

<u>Fact</u>: $\psi_0(Z)_0 = \cdots = \psi_0(Z)_i = 0$

$\Rightarrow \psi_1(Z)_0, \ldots, \psi_1(Z)_{i-1}$ defined

<u>Fact</u>: GHC $\Rightarrow$ everything to the right is in "ambiguities"

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* This is modulo sub-Hodge structures arising from ambiguities

Write above as

$$
\begin{array}{c|c|c|c|c}
\varphi_0 & \varphi_1 & \varphi_3 & \cdots & \varphi_{2p-1} \\
 & \varphi_2 & \varphi_4 & \cdots & \varphi_{2p}
\end{array}
$$

$$F^1 \longleftrightarrow \varphi_0 = 0$$

$$F^2 \longleftrightarrow \varphi_1 = \varphi_2 = 0$$

$$F^3 \longleftrightarrow \varphi_3 = \varphi_4 = 0$$

$$\vdots$$

<u>Note</u> $\varphi_1 = 0 \Rightarrow \mathrm{Alb}\, S \to J^p(X)$ constant for $\{Z_s\}_{s \in S}$, and

$\varphi_2 = 0 \Rightarrow AJ_X(Z_{s_0}) = 0$ $\varphi_{2m-1} = 0$

(also $\Leftarrow$), but for $m \geq 2$, φ_m defined only if

$$\text{GHC} \Rightarrow \left\{ \begin{array}{l} \text{if } \varphi_0 = \cdots = \varphi_{2p} = 0 \\ \text{then may choose } \mathfrak{Z} \\ \text{with } \psi_0(\mathfrak{Z}) = \psi_1(\mathfrak{Z}) = 0 \end{array} \right\}$$

on $\mathfrak{X} \times S$ - may also assume

$\mathfrak{Z}$ defined over $\bar{\mathbb{Q}}$ - then

$$B^2(\text{iv}) \Rightarrow \mathfrak{Z} \equiv_{\text{rat}} 0 \text{ on } \mathfrak{X} \times S$$

- arguments make essential use of fact that $\mathfrak{Z}$ is a spread, Lefscetz theorems, etc.
- for $p = 1$, everything beyond $\varphi_0, \varphi_1, \varphi_2 \longleftrightarrow \psi_0, \psi_1$ is in ambiguities

For $\mathfrak{X}$ defined over a general field k, then we must take the spread of both $\mathfrak{X}$ and Z to have

$$z \in Z^p(\mathcal{X}), \quad \mathcal{X} \xrightarrow{\pi} S$$

Modulo ambiguities we may assume that π is smooth and use the degeneration of the Leray s.s. to define F^m

Test for

$$Z \underset{\text{rat}}{\equiv} 0 \quad \text{(mod torsion)}$$

is "algorithmic"

Simplest example: Y_1, Y_2 are algebraic curves defined $/k$

$$Z = [p_1 - q_1] \times [p_2 - q_2]$$

where $q_i \in Y_i(k)$, $p_i \in Y_i(\mathbb{C})$.

Then:

$$\Psi_0(Z) = \Psi_1(Z) = 0$$

$$p_1, p_2 \text{ alg. ind}/k \Rightarrow \Psi_2(Z) \neq 0$$

$\mathrm{Tr\,deg}_k\,[p_1, p_2] = 1$, $\varphi_3 = 0$ but $\varphi_4 \neq 0$ in gen

Corollary (i) $[Z] \in F^{g+2}CH^p(X)$ and trdeg $k \leq g \Rightarrow Z \equiv_{rat} 0$

(ii) $[Z] \in F^g CH^p(X)$ is a sum of cycles defined over fields of tr deg g (here $X/\bar{\mathbb{Q}}$)

Application: X regular surface$/\bar{\mathbb{Q}}$ and $Z \in Z^2(k)$. Then

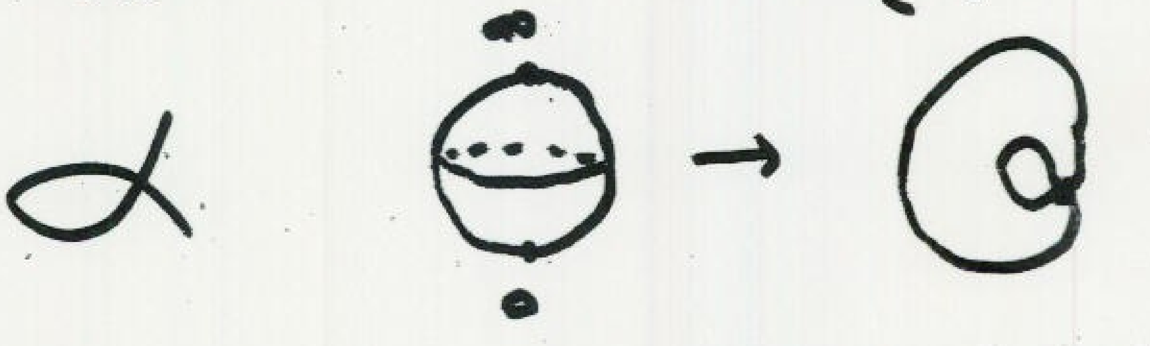
$$\left.\begin{array}{r}\psi_0(Z)=0\\ h^{2,0}(k)=0\end{array}\right\} \Rightarrow Z \equiv_{rat} 0$$

Same result for general X if assume $h^{q,0}(k)=0$

Conclusion: Hodge theory of fields of definition enter into Abel's theorem and its converse for $p \geq 2$

<u>Example</u>: Simplest curve with

$h^{1,0}(X) \neq 0$ is $(\mathbb{P}^1; \{0,\infty\})$



Functions are $f(z)$ with $f(0)=f(\infty)$

Differential is $\omega = dz/z$

$$Z = \sum_i n_i z_i \,, \quad z_i \in \mathbb{C}^*$$

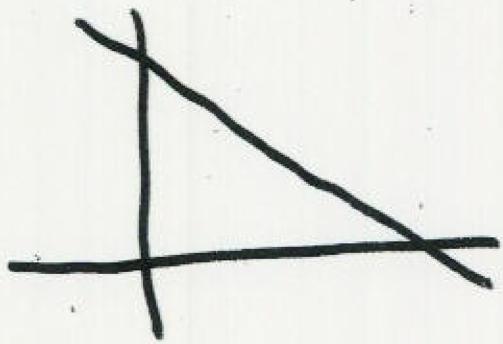
$$Z \overset{?}{=} (f)$$

$$\psi_0(Z) = \sum_i n_i = 0 \Rightarrow Z = \partial\gamma$$

$$\psi_1(Z) = \int_\gamma \omega \equiv 0$$

$$\Leftrightarrow \prod z_i^{n_i} = 1 \in K_1(\mathbb{C})$$

Surface analogue is $(\mathbb{P}^2, T)$



1-forms are dx/x, dy/y and

$$\varphi = \frac{dx}{x} \wedge \frac{dy}{y}$$

For $Z = \sum_i n_i (x_i, y_i)$ the

Hodge-theoretic conditions are

$\psi_0(Z) = \sum n_i$, $\psi_1(Z) = \left(\prod x_i^{n_i}, \prod y_i^{n_i}\right) =$

For ψ_2, if Z has spread

$$Z_s = \sum_i n_i (x_i(s), y_i(s))$$

$$(a) = \mathrm{Tr}_{Z} \cdot \varphi = 0$$

For (b) we let λ be a closed curve in S and Λ_i the curve $(x_i(s), y_i(s))_{s \in \lambda}$. then

$$(b) = \sum n_i \left(\int_{\Lambda_i} \log x \frac{dy}{y} - \log y(s_0) \frac{dx}{x} \right)$$

= regulator

($(a) = 0 \Rightarrow (b)$ well-defined on $H_1(S)$

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<u>Application</u>: $\dim X = 4$ and $h^{q,0}(X) = 0$ for $1 \leq q \leq 3$ (e.g., $X \subset \mathbb{P}^5$). Then for

$$Z \in Z^2(X)$$

with

$$\psi_0(Z) = 0$$

we have (mod torsion)

$$Z \equiv_{alg} 0$$

Relation between $F^m CH^p(X)$ above and definitions proposed by Murre, Saito, Jannsen. Heuristic argument that

$$\begin{cases} F^m_M = F^m \\ F^m_{S-J} \subseteq F^m \end{cases}$$

Heuristic depends on GHC and global properties of the decomposition theorem of B-B-D-G (uses intersection homology).