Local geometric Langlands correspondence and representations of affine Kac-Moody algebras

(Joint work with Edward Frenkel)

Lecture 2

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1. Affine algebras and representations.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and G the corresponding adjoint group. We consider the loop algebra $\mathfrak{g}((t))$, and for any symmetric ad-invariant pairing $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ we have the Kac-Moody central extension

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0,$$

which splits as a vector space, and the bracket is given by the formula

$$[x \otimes f(t), y \otimes g(t)] = \kappa(x, y) \cdot \mathrm{Res}_{t=0}(df(t) \cdot g(t)) + [x, y] \otimes (f(t) \cdot g(t)).$$

We define the category $\widehat{\mathfrak{g}}_\kappa$-mod to have as objects (discrete) vector spaces M, endowed with an action of $\widehat{\mathfrak{g}}_\kappa$ as a Lie algebra, such that the following conditions are satisfied:

- The element $\mathbf{1} \in \mathbb{C} \subset \widehat{\mathfrak{g}}_\kappa$ acts on M as the identity operator.

- For every $m \in M$ there exists an integer n, such that for all $n' \geq n$, the elements $x \otimes t^{n'} \in \mathfrak{g}((t)) \subset \widehat{\mathfrak{g}}_\kappa$ annihilate m.

Let us give the most basic example of an object of $\widehat{\mathfrak{g}}_\kappa$-mod–the vacuum module, denoted $\mathbb{V}_\kappa$. It is generated by a single vector $\mathbf{1} \in \mathbb{V}_\kappa$, which is annihilated by the subalgebra $\mathfrak{g}[[t]] \subset \mathfrak{g}((t)) \subset \widehat{\mathfrak{g}}_\kappa$.

Functorially, the vacuum module is characterized by the property

$$Hom_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(\mathbb{V}_\kappa, M) = M^{\mathfrak{g}[[t]]}.$$

2. The center and the critical level.

Let us recall that the center of the category $\mathfrak{g}$-mod of modules over the finite-dimensional algebra $\mathfrak{g}$, which is isomorphic to the center of the universal enveloping algbera $U(\mathfrak{g})$, is isomorphic to the sub-algebra $U(\mathfrak{g})^{\mathfrak{g}}$ of ad-invariants in $U(\mathfrak{g})$.

Since the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ is locally finite, and since the category of finite-dimensional representations of $\mathfrak{g}$ is semi-simple, we obtain

$$gr(Z(U(\mathfrak{g}))) \simeq (gr(U(\mathfrak{g})))^{\mathfrak{g}} \simeq \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}},$$

where the associated graded is taken with respect to the PBW filtration on $U(\mathfrak{g})$.

Do we have a similar statement in the affine case? The answer is that *a priori* no, because we no longer have the semi-simplicity statement for the corresponding category.

Let $Z(\hat{\mathfrak{g}}_\kappa\text{-mod})$ denote the center of the category $\hat{\mathfrak{g}}_\kappa$-mod, or, which is the same, the center of the corresponding completed universal enveloping algebra. It carries a filtration induced by the PBW filtration. We always have an embedding

$$gr\,(Z(\hat{\mathfrak{g}}_\kappa\text{-mod})) \hookrightarrow (\mathrm{Sym}(\mathfrak{g}((t))))^{\mathfrak{g}((t))} \qquad (1)$$

Lemma 1. *If* $\kappa \neq \kappa_{crit} = -\frac{Killing}{2}$, *then*

$$Z(\hat{\mathfrak{g}}_\kappa\text{-mod}) \simeq \mathbb{C}.$$

Theorem 1. *[Feigin-Frenkel] For* $\kappa = \kappa_{crit}$, *the map* (1) *is an isomorphism.*

From now on we shall fix $\kappa = \kappa_{crit}$; we will use a short-hand notation $\widehat{\mathfrak{g}}_{crit}$-mod for $\widehat{\mathfrak{g}}_{\kappa_{crit}}$-mod. We will denote the algebra $Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$ by $\mathfrak{Z}$.

Consider the action of $\mathfrak{Z}$ on $\mathbb{V}_{crit}$. Along with Lemma 1 and Theorem 1 one shows the following:

Lemma 2. *The map*

$$\mathfrak{Z} \to End_{\widehat{\mathfrak{g}}_{crit}\text{-mod}}(\mathbb{V}_{crit}) \simeq (\mathbb{V}_{crit})^{\mathfrak{g}[[t]]}$$

is surjective.

Let us denote by $\mathfrak{z}$ the quotient of $\mathfrak{Z}$ through which it acts on $\mathbb{V}_{crit}$. We have:

Theorem 2. *The map* (1) *induces an isomorphism*

$$gr(\mathfrak{z}) \to (gr(\mathbb{V}_{crit}))^{\mathfrak{g}[[t]]}.$$

Let us denote by $\widehat{\mathfrak{g}}_{crit}\text{-mod}^{reg}$ the subcategory of $\widehat{\mathfrak{g}}_{crit}$-mod, consisting of modules, on which the action of $\mathfrak{Z}$ factors through $\mathfrak{z}$. This is the category that we will be interested in today.

3. D-modules on the affine Grassmannian.

We consider the affine Grassmannian $Gr_G := G((t))/G[[t]]$. This is a strict ind-scheme of ind-finite type , i.e., it can represented as a union of finite-dimensional schemes, each being a closed subscheme in the next one.

Hence, it makes sense to speak about the category of D-modules on Gr_G; we will denote it by D-mod(Gr_G).

The group $G((t))$ acts naturally on Gr_G. Let us denote by $Sph_G :=$ D-mod$(Gr_G)^{G[[t]]}$ the category of D-modules on Gr_G, equivariant with respect to the subgroup $G[[t]] \subset G((t))$.

The convolution product makes Sph_G into a monoidal category, which acts on D-mod(Gr_G):

Consider the ind-scheme $G((t)) \underset{G[[t]]}{\times} Gr_G$, which maps to Gr_G using the action map. If $\mathcal{F}'$ is an object of D-mod(Gr_G) and $\mathcal{F}$ is a $G[[t]]$-equivariant D-module on Gr_G, we can form their twisted product

$$\mathcal{F}' \tilde{\boxtimes} \mathcal{F} \in \text{D-mod}(G((t)) \underset{G[[t]]}{\times} Gr_G),$$

taking $\mathcal{F}'$ along the first multiple and $\mathcal{F}$ along the second one.

We define $\mathcal{F}' \star \mathcal{F} \in$ D-mod(Gr_G) to be the direct image of $\mathcal{F}' \tilde{\boxtimes} \mathcal{F}$. *A priori*, this would be an object of the derived category, but one can show that it is acyclic off degree 0, i.e., it is a single D-module.

If $\mathcal{F}'$ was also $G[[t]]$-equivariant, then so will be the result of the convolution.

The following theorem (due to Drinfeld, Ginzburg, Lusztig, Mirković and Vilonen) is the basis for geometric Langlands duality:

Theorem 3. *The monoidal category* Sph_G *has a natural commutativity constraint, and the resulting tensor category is equivalent to that of algebraic representations of* $\check{G}$*.*

We shall denote the functor Rep$(\check{G}) \to Sph_G$ by $V \mapsto \mathcal{F}_V$.

Thus, we obtain that the category Rep$(\check{G})$ acts as a monoidal category on D-mod(Gr_G). According to the previous lecture, this means, by definition, that D-mod(Gr_G) is *a category over the stack* pt $/\check{G}$.

4. Critically twisted global sections.

Recall that to every integral (and in particular, critical) value of κ, there corresponds a line bundle $\mathcal{L}_\kappa$ on Gr_G, such that the action of $\mathfrak{g}((t))$ on Gr_G lifts to an action of $\widehat{\mathfrak{g}}_\kappa$ on $\mathcal{L}_\kappa$.

Given a D-module $\mathcal{F}$ on Gr_G, we can consider the vector space

$$\Gamma_\kappa(Gr_G, \mathcal{F}) := \Gamma(Gr_G, \mathcal{F} \otimes \mathcal{L}_\kappa),$$

and it will be an object of $\widehat{\mathfrak{g}}_\kappa$-mod.

For example, if we take $\mathcal{F}$ to be the δ-function δ_{1,Gr_G} at the point $1 \in Gr_G$, we obtain

$$\Gamma_\kappa(Gr_G, \delta_{1,Gr_G}) \simeq \mathbb{V}_\kappa.$$

From this, we obtain:

Lemma 3. *For any* $\mathcal{F} \in$ D-mod(Gr_G),

$$\Gamma_{crit}(Gr_G, \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{-mod}^{reg}.$$

We shall now state a theorem, which is at the origin of the relation, discovered originally by Feigin and Frenkel, of the algebras $\mathfrak{Z}$ and $\mathfrak{z}$ and the stack $LocSys(\mathcal{D}^\times)_{\check{G}}$.

Theorem 4. *For* $V \in \mathrm{Rep}(\check{G})$, *the object*

$$\Gamma_{crit}(Gr_G, \mathcal{F}_V) \in \hat{\mathfrak{g}}_{crit}\text{-mod}^{reg}$$

is isomorphic to $\mathbb{V}_{crit} \underset{\mathfrak{z}}{\otimes} \mathcal{V}$ *for some locally free* $\mathfrak{z}$*-module* $\mathcal{V}$.

As a corollary we obtain:

Lemma 4. *For any* $\mathcal{F} \in \text{D-mod}(Gr_G)$ *and* V *as above, we have*

$$\Gamma_{crit}(Gr_G, \mathcal{F} \star \mathcal{F}_V) \simeq \Gamma_{crit}(Gr_G, \mathcal{F}) \underset{\mathfrak{z}}{\otimes} \mathcal{V}.$$

The above lemma implies that the assignment $V \mapsto \mathcal{V}$ is a monoidal functor from the category $\mathrm{Rep}(\check{G})$ to that of locally free modules over $\mathfrak{z}$. It is fairly easy to show that this functor is compatible with the commutativity constraints.

Hence, the data of such functor is equivalent to that of a torsor $\mathcal{P}_{\check{G}}$ (i.e., a principal bundle) over $Spec(\mathfrak{z})$ with respect to $\check{G}$, such that for $V \in \mathrm{Rep}(\check{G})$, what we denoted by $\mathcal{V}$ is the corresponding associated vector bundle.

5. A localization conjecture at the critical level.

Let us recall the theorem of Beilinson and Bernstein that says that the category of $\mathfrak{g}$-modules with a given central character is equivalent to the category of twisted D-modules on the flag variety G/B.

What we are going to state now is a conjecture, that is supposed to give a similar description to the category $\hat{\mathfrak{g}}_{crit}$-mod^{reg} in terms of D-modules on Gr_G.

In terms of the previous lecture, this conjecture is the combination of two statements:

$$\mathcal{C} \underset{LocSys(\mathcal{D}^\times)_{\check{G}}}{\times} Spec(\mathfrak{Z}) \simeq \hat{\mathfrak{g}}_{crit}\text{-mod}$$

and

$$\mathcal{C} \underset{LocSys(\mathcal{D}^\times)_{\check{G}}}{\times} LocSys(\mathcal{D}^\times)^{reg}_{\check{G}} \simeq \text{D-mod}(Gr_G).$$

By the construction of the map $Spec(\mathfrak{Z}) \to LocSys(\mathcal{D}^\times)_{\check{G}}$ (that we have not yet explained), the composition

$$Spec(\mathfrak{z}) \to Spec(\mathfrak{Z}) \to LocSys(\mathcal{D}^\times)_{\check{G}}$$

equals

$$Spec(\mathfrak{z}) \to \mathrm{pt}/\check{G} \simeq LocSys(\mathcal{D}^\times)^{reg}_{\check{G}} \hookrightarrow LocSys(\mathcal{D}^\times)_{\check{G}},$$

where the first arrow corresponds to the torsor $\mathcal{P}_{\check{G}}$, introduced above.

We obtain that

$$\widehat{\mathfrak{g}}_{crit}\text{-mod}^{reg} \simeq \widehat{\mathfrak{g}}_{crit}\text{-mod} \underset{Spec(\mathfrak{Z})}{\times} Spec(\mathfrak{z})$$

should be equivalent to

$$\text{D-mod}(Gr_G) \underset{\mathrm{pt}/\check{G}}{\times} Spec(\mathfrak{z}).$$

Here is a way to reformulate this conjecture:

Let $Hecke(Gr_G)$ denote the category, whose objects are D-modules $\mathcal{F}$ on Gr_G, endowed with an action of the algebra $\mathfrak{z}$ by endomorphisms, and a system of isomorphisms

$$\mathcal{F} \star \mathcal{F}_V \simeq \mathcal{F} \underset{\mathfrak{z}}{\otimes} \mathcal{V},$$

for $V \in \text{Rep}(\check{G})$, comptible with tensor products of representations. Morphisms in this category are D-module morphisms that respect the other pieces of structure.

In fact $Hecke(Gr_G)$ is, by definition, equivalent to D-mod$(Gr_G) \underset{\text{pt}/\check{G}}{\times} Spec(\mathfrak{z})$.

Conjecture 1. *The category $Hecke(Gr_G)$ is equivalent to $\widehat{\mathfrak{g}}_{crit}$-mod^{reg}.*

Corollary 1. *For every pair of central characters $\chi_1, \chi_2 \in Spec(\mathfrak{z})$, there exists a canonical equivalence of the categories*

$$\widehat{\mathfrak{g}}_{crit}\text{-mod}_{\chi_1} \simeq \widehat{\mathfrak{g}}_{crit}\text{-mod}_{\chi_2}$$

for every choice of an isomorphisms of $\check{G}$-torsors $(\mathcal{P}_{\check{G}})_{\chi_1} \simeq (\mathcal{P}_{\check{G}})_{\chi_2}$.