Local geometric Langlands correspondence and representations of affine Kac-Moody algebras

(Joint work with Edward Frenkel)

Lecture 1

Aug. 10, 2005

1. Classical local Langlands correspondence.

Let G be a split reductive group over a ground field k. In this section k will be a finite field $\mathbf{F}_q$. We will consider the local field $k((t))$ and we will consider the group $G((t))_k$, which is by definition the group of $k((t))$-points of G.

It will be important for the sequel that $G((t))$ exists as an algebro-geometric object–it is a group ind-scheme, and $G((t))_k$ can be also interpreted as the group of k-points of $G((t))$.

The natural topology on $k((t))$ makes $G((t))_k$ into a topological group. One is interested in the category Rep$(G((t))_k, \mathbb{C})$ of continuous representations of $G((t))_k$ on $\mathbb{C}$-vector spaces.

Local Langlands correspondence is a way to describe the set of isomorphism classes of irreducible objects in $\mathrm{Rep}(G((t))_k, \mathbb{C})$.

One should remark that contrary to the case of the finite group G_k, it is interesting to study $\mathrm{Rep}(G((t))_k, \mathbb{C})$ as a category, and not just irreducibles in it. I.e., one can study projective modules, Exts, etc. Unfortunately, I don't know how to describe $\mathrm{Rep}(G((t))_k, \mathbb{C})$ in terms of Langlands parameters.

Let us return to the irreducibles, and consider first the commutative case $G = GL(1)$. Irreducible representations of $GL(1, k((t)))$ are the same as characters of $k((t))^\times$, and Class Field Theory tells us that those are in bijection with characters of the Galois (or rather Weil) group $\mathrm{Gal}(k((t)))$ of the local field $k((t))$.

Let $\check{G}$ be the Langlands dual group of G; this is a reductive group, defined combinatorially from the root data of G. Classical local Langlands correspondence is a generalization of Class Field Theory.

It asserts that there is a relation between the following the set Irr$\,(\mathrm{Rep}(G((t))_k, \mathbb{C}))$ of irreducibles in $\mathrm{Rep}(G((t))_k, \mathbb{C})$ and the set of conjugacy classes of homomorphisms

$$\sigma_{loc} : \mathrm{Gal}(k((t))) \to \check{G}(\overline{\mathbb{Q}_\ell}).$$

This relation is not a bijection, even on the conjectural level. However, certain more precise statements are known. E.g., for $G = GL(n)$, the above relation induces a bijection between the subset of cuspidal representations in Irr$\,(\mathrm{Rep}(G((t))_k, \mathbb{C}))$ and the subset of irreducible homomorphisms σ_{loc} as above, the latter being the same as irreducible n-dimensional ℓ-adic representations of $\mathrm{Gal}(k((t)))$.

2. Action of groups on categories.

We shall now take k to be an algebraically closed field of characteristic 0, e.g., $k = \mathbb{C}$, and consider the group ind-scheme $G((t))$ over k. It is possible to say what an algebrtaic representation of $G((t))$ is, but it is fairly easy to show that all such representations are essentially trivial.

Therefore, we have to look for an alternative notion of representation. Our guiding principle is that the passage "classical" $\Rightarrow$ "geometric" is associated with going one step up in the hierarchy

{Elements}, {Objects}, {Categories}, {2-Cat.}

Thus, instead of looking for ordinary representations of $G((t))$, which are, by definition, vector spaces with an action of $G((t))$, we will look for representations of $G((t))$ on categories.

Let H be an algebraic group over k (such as G), or more generally, a group ind-scheme (such as $G((t))$). If $\mathcal{C}$ is an abelian k-linear category, there is a notion of action of H on $\mathcal{C}$. In fact, there are two such notions– weak and strong (or Harish-Chandra) actions.

For example, if H acts on a scheme $\mathcal{Y}$, we have a weak action of H on the category QCoh($\mathcal{Y}$) of quasi-coherent sheaves on $\mathcal{Y}$. In addition, we have a strong action of H on the category D-mod($\mathcal{Y}$) of D-modules on $\mathcal{Y}$.

Naively, an action of H on $\mathcal{C}$ assigns to every point $h \in H$ a functor $\mathcal{C} \to \mathcal{C}$ with some natural associativity properties. Rigorously, to have a weak action of H on $\mathcal{C}$, we must have an algebraic family of such functors, parametrized by H. An action is strong if this family is infinitesimally trivialized.

In most examples $\mathcal{C}$ is a category of modules over an associative (topological) algebra $\mathbf{A}$. Then a weak action of H on $\mathcal{C}$ amounts to an action of H on $\mathbf{A}$ by automorphisms. A weak action is strong if the induced action of the Lie algebra $\mathfrak{h}$ of H is inner, i.e., comes from a homomorphism $\mathfrak{h} \to \mathbf{A}$.

We stipulate that a geometric replacement of the notion of representation of the topological group $G((t))_k$ will be the notion of abelian k-linear category endowed with an action of the group ind-scheme $G((t))$. We will call such a category a "1-representation of $G((t))$". An object of such a category (i.e., a D-module on a scheme on which $G((t))$ acts) is an analog of a vector in a vector space, underlying a representation of $G((t))_k$.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two categories acted on by G, there is a natural notion of a functor $\mathcal{C}_1 \to \mathcal{C}_2$ commuting with the G-action. Thus, we obtain the 2-category of 1-representations, which is a geometric replacement of Rep$(G((t))_k, \mathbb{C})$.

3. Categories over stacks

In order to be able to speak about geometric local Langlands correspondence, we need to introduce one more abstract notion: that of an abelian category over a stack.

Let $\mathcal{C}$ be an abelian category. The center of $\mathcal{C}$, denoted $Z(\mathcal{C})$, is by definition the algebra of endomorphisms of the identity functor on $\mathcal{C}$. I.e., an element of $Z(\mathcal{C})$ is a rule that assigns to every object $X \in \mathcal{C}$ its endomorphism, in a way functorial in X.

Such systems of endomorphisms can be naturally composed, making $Z(\mathcal{C})$ into an associative algebra. However, it is a simple exercise to show that $Z(\mathcal{C})$ is in fact commutative. For example, if $\mathcal{C}$ is the category of R-modules, where R is a ring, then $Z(\mathcal{C}) \simeq Z(R)$.

Let A be a commutative ring. We say that $\mathcal{C}$ is A-linear, or that $\mathcal{C}$ is a category over the affine scheme $Spec(A)$ if we are given a homomorphism $A \to Z(\mathcal{C})$. This is equivalent to endowing the group $Hom(X, Y)$ with a structure of A-module functorially in $X, Y \in \mathcal{C}$.

Let now $A \to A'$ be a homomorphism of commutative rings and let $\mathcal{C}$ be a category over $S := Spec(A)$. Then there exists a canonically constructed category $\mathcal{C}'$ over $S' := Spec(A')$, universal with respect to a certain natural property, which we will call "the base change" of $\mathcal{C}$ with respect to S', and denote by $\mathcal{C} \underset{S}{\times} S'$.

Let now $\mathcal{Y}$ be a stack (in the faithfully flat topology). A sheaf of categories $\mathcal{C}^{sh}$ over $\mathcal{Y}$ is a rule that attaches to every affine scheme S and an S-point $\mathcal{Y}$, a category $\mathcal{C}_S$ over S, and for a map $S' \to S$ of affine schemes, an equivalence

$$\mathcal{C}_{S'} \simeq \mathcal{C}_S \underset{S}{\times} S',$$

where $\mathcal{C}_{S'}$ is the category, corresponding to the induced S'-point of $\mathcal{Y}$. We need these data to satisfy some natural associativity properties.

If $\mathcal{C}^{sh}$ is a sheaf of categories over $\mathcal{Y}$, we can consider the category $\Gamma(\mathcal{Y}, \mathcal{C}^{sh})$ of its global sections. We shall say that an abelian category $\mathcal{C}$ *is a category over* $\mathcal{Y}$ if there exists (or, rather, we are given) a sheaf of categories $\mathcal{C}^{sh}$ as above and an equivalence $\mathcal{C} \simeq \Gamma(\mathcal{Y}, \mathcal{C}^{sh})$. For example, the category of quasi-coherent sheaves over $\mathcal{Y}$ is *a category* over $\mathcal{Y}$.

In particular, if y is a k-point of $\mathcal{Y}$, the corresponding category $\mathcal{C}_{Spec(k)}$ should be thought of as the fiber of $\mathcal{C}$ over y.

An instructive example of this situation is when $\mathcal{Y}$ is the classifying stack pt/H, where H is an affine algebraic group. It is a good exercise to show that a strcuture on $\mathcal{C}$ of category over pt/H is equivalent to an action on $\mathcal{C}$ of the tensor category of finite-dimensional representations of H.

4. Representations of local Galois groups.

In the case when k was the finite field $\mathbf{F}_q$, the appropriate object to consider was the set of conjugacy classes of homomorphisms from Gal($k((t))$) to $\check{G}(\overline{\mathbb{Q}_\ell})$.

This set can be also described as the set of isomorphism classes of ℓ-adic local systems on $Spec(k((t)))$ with respect to $\check{G}$. By the latter we mean a tensor functor from the category of finite-dimensional algebraic representations of $\check{G}$ to the category of ℓ-adic sheaves on $Spec(k((t)))$.

In the geometric context, instead of the ℓ-adic sheaves on $Spec(k((t)))$ we will consider the category of holonomic D-modules. The latter are by definition finite-dimensional $k((t))$-vector spaces, endowed with an action of ∂_t, satisfying the Leibniz rule.

Thus, we obtain the notion of $\check{G}$-local system on the formal punctured disc $\mathcal{D}^\times := Spec(k((t)))$. However, by contrast with the classical situation, we will consider the collection of all $\check{G}$-local system on $\mathcal{D}^\times$ as an algebro-geometric object. This is a (non-algebraic) stack, which we will denote by $LocSys(\mathcal{D}^\times)_{\check{G}}$.

One can describe $LocSys(\mathcal{D}^\times)_{\check{G}}$ explictly as follows. Let $\check{\mathfrak{g}}$ denote the Lie algebra of $\check{G}$. One can describe $LocSys(\mathcal{D}^\times)_{\check{G}}$ as the quotient of the space $\Omega^1(\mathcal{D}) \otimes \check{\mathfrak{g}}$ of $\check{\mathfrak{g}}$-valued 1-forms on $\mathcal{D}^\times$ by the gauge action of $\check{G}((t))$.

For future use, let us remark that $LocSys(\mathcal{D}^\times)_{\check{G}}$ contains as a closed sub-stack the locus of tamely ramified (i.e., regular singular) local systems with unipotent monodromy, which we will denote by $LocSys(\mathcal{D}^\times)_{\check{G}}^{nilp}$.

The sub-stack $LocSys(\mathcal{D}^\times)^{nilp}_{\check{G}}$ is algebraic, and it is isomorphic to $\check{\mathcal{N}}/\check{G}$, where $\check{\mathcal{N}} \subset \check{G}$ is the cone of nilpotent elements.

In its turn, $LocSys(\mathcal{D}^\times)^{nilp}_{\check{G}}$ contains a closed sub-stack, corresponding to regular local systems (i.e., those without monodromy), which we will denote by $LocSys(\mathcal{D}^\times)^{reg}_{\check{G}}$, and which is isomorphic to the classifying stack $\mathrm{pt}/\check{G}$.

5. Local geometric Langlands correspondence.

We are now ready to formulate what we mean by local geometric Langlands correspondence.

We conjecture that there exists a (universal in a certain sense) abelian category $\mathcal{C}$, which is acted on by $G((t))$, and which is a category *over the stack* $LocSys(\mathcal{D}^\times)_{\check{G}}$, in a way, compatible with the $G((t))$-action.

For an individual $\check{G}$-local system σ_{loc}, which is the same as a k-point of $LocSys(\mathcal{D}^\times)_{\check{G}}$, the fiber of $\mathcal{C}$ over this point, denoted $\mathcal{C}_{\sigma_{loc}}$, should be thought of as a geometric analog of the irreducible representation of $G((t))_k$ if k were a finite field, corresponding to a given homomorphism of Gal$(k((t)))$ into $\check{G}(\overline{\mathbb{Q}_\ell})$.

At the moment we can neither construct the category $\mathcal{C}$, nor even characterize it uniquely.

However, we have predictions concerning some of its derivatives.

Namely, let $\mathcal{C}^{reg}$ and $\mathcal{C}^{nilp}$ be the categories, obtained by restricting $\mathcal{C}$ to the sub-stacks $LocSys(\mathcal{D}^\times)^{reg}_{\check{G}}$ and $LocSys(\mathcal{D}^\times)^{nilp}_{\check{G}}$, respectively.

Let $Gr_G = G((t))/G[[t]]$ and $Fl_G = G((t))/I$ be the affine Grassmannian of the group G and the affine flag variety, respectively, where $I \subset G[[t]]$ is the Iwahori subgroup. Let D-mod(Gr_G) and D-mod(Fl_G) be the corresponding categories of D-modules. Both these categories are naturally acted on by $G((t))$.

The Satake equivalence, which will be reviewed later, implies that D-mod(Gr_G) is naturally a category over the stack $\mathrm{pt}/\check{G}$.

We conjecture that $\mathcal{C}^{reg} \simeq$ D-mod(Gr_G), as categories with an action of $G((t))$.

The statement involving $\mathcal{C}^{nilp}$ is more complicated, since it involves passing to the realm of triangulated categories.

Let $\tilde{\check{\mathcal{N}}}$ be the Springer resolution of $\check{\mathcal{N}}$. According to the work of Arkhipov and Bezrukavnikov, the derived category $D(\text{D-mod}(Fl_G))$ is a category over the stack $\tilde{\check{\mathcal{N}}}/\check{G}$. We conjecture that

$$D(\mathcal{C}^{nilp}) \underset{\check{\mathcal{N}}/\check{G}}{\times} \tilde{\check{\mathcal{N}}}/\check{G} \simeq D(\text{D-mod}(Fl_G)).$$

6. Global geometric Langlands correspondence: the unramified case.

Let now X be a smooth projective curve over k. Let Bun_G denote the moduli stack of principal G-bundles on X. Let us recall the formulation of global geometric Langlands correspondence.

Let σ_{glob} be a $\check{G}$-local system on X. We will think of it as a tensor functor from the category of finite-dimensional representations of $\check{G}$ to that of finite rank vector bundles on X with a connection.

For for a finite-dimensional representation V of $\check{G}$, we will denote by $V_{\sigma_{glob}}$ the corresponding D-module on X.

Recall that given a finite-dimensional representation V of $\check{G}$, there exists a naturally defined Hecke functor

$$H_V : D(Bun_G) \to D(Bun_G \times X),$$

where $D(\cdot)$ denotes the derived category of D-modules on a given stack.

An object $\mathcal{F} \in D(Bun_G)$ is called a Hecke eigen-sheaf with respect to σ_{glob} if for every V as above we are given an isomorphism

$$H_V(\mathcal{F}) \simeq \mathcal{F} \boxtimes V_{\sigma_{glob}},$$

compatible with tensor products of representations.

Hecke eigen-sheaves on Bun_G with respect to a fixed σ_{glob} form a triangulated category, which we shall denote by $Hecke(\sigma_{glob})$.

The global geometric Langlands correspondence predicts that (at least when σ_{glob} is sufficiently generic), the category $Hecke(\sigma_{glob})$ is equivalent to the derived category of the category of k-vector spaces. In particular, it contains a distinguished object $\mathcal{F}_{\sigma_{glob}}$, corresponding to k itself.

7. Global geometric Langlands correspondence with ramification, and the relation between local and global.

Let now σ_{glob} be a $\check{G}$-local system on the punctured curve $X - \{x_1, ..., x_n\}$.

(Note that by restricting σ_{glob} to the punctured disc $\mathcal{D}_i^\times$ around each x_i we obtain a point $\sigma_{loc,i}$ of the corresponding stack $LocSys(\mathcal{D}_i^\times)_{\check{G}}$.)

Let $Bun_G(x_1, ..., x_n)$ be the moduli stack of G-bundles on X with a full level structure at the points $x_1, ..., x_n$. We still have the Hecke functors H_V that map $D(Bun_G(x_1, ..., x_n))$ to

$$D(Bun_G(x_1, ..., x_n) \times (X - \{x_1, ..., x_n\})),$$

and therefore it makes sense to introduce the triangulated category $Hecke(\sigma_{glob}, x_1, ..., x_n)$ of D-modules on $Bun_G(x_1, ..., x_n)$, which satisfy the Hecke property

$$H_V(\mathcal{F}) \simeq \mathcal{F} \boxtimes V_{\sigma_{glob}}.$$

By construction, $Hecke(\sigma_{glob}, x_1, ..., x_n)$ is endowed with an action of $G((t))$.

Assume now that σ_{glob} as above is generic. In this case we conjecture that the category $Hecke(\sigma_{glob}, x_1, ..., x_n)$ introduced above is equivalent to

$$\bigotimes_i D(\mathcal{C}_{\sigma_{loc,i}}),$$

where $\sigma_{loc,i}$ are the corresponding *local* local systems, and each $\mathcal{C}_{\sigma_{loc,i}}$ is the fiber of $\mathcal{C}$ over it.

Such a relation is parallel to the relation between local and global Langlands correspondences in the classical setting.

8. Relation to representations at the critical level.

Although at the moment we cannot construct the category $\mathcal{C}$, we conjecture that it has a close relationship to the category of representations of affine Kac-Moody algebras at the critical level, which we will now explain.

We consider the loop algebra $\mathfrak{g}((t))$ and its Kac-Moody extension $\mathfrak{g}_{crit}$

$$0 \to k \to \widehat{\mathfrak{g}}_{crit} \to \mathfrak{g}((t)) \to 0,$$

corresponding to the critical value of the pairing $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k$, i.e., $\kappa = -\frac{Killing}{2}$.

We consider the category $\widehat{\mathfrak{g}}_{crit}$-mod. The center of this category, $Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$ can be described explicitly, and it was done in the works of Feigin and Frenkel.

As will be explained later there exists a natural map

$$Spec\,(Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})) \to LocSys(\mathcal{D}^{\times})_{\check{G}}.$$

We will study the category $\widehat{\mathfrak{g}}_{crit}$-mod, and our guiding principle will be the following conjecture:

$$\widehat{\mathfrak{g}}_{crit}\text{-mod} \simeq \mathcal{C} \underset{LocSys(\mathcal{D}^{\times})_{\check{G}}}{\times} Spec\,(Z(\widehat{\mathfrak{g}}_{crit}\text{-mod}))\,.$$