Outline

In view of time constraints and desire to go slower:

§ 84 Progress on Artin Conjecture and Serre's conjecture

§ 5 p-adic Hodge theory

We shall say nothing about p-divisible groups except:

By using p-adic Hodge theory methods, M. Kisin developed more powerful methods to use p-divisible groups and group schemes in deformation problems for Galois representations, providing a better approach to modularity.
§4. Progress on Artin and Serre conjectures.

Recall statements!

**Serre's conjecture**: Any odd, continuous irreducible \( p: G_\mathbb{Q} \to GL_2(\mathbb{F}_p) \) is modular (i.e., \( p \cong \tilde{\rho}_{f, \lambda} \) for some \( f \) prime \( \lambda \) of \( \mathbb{F}_p \)).

**Artin conjecture**: For any irreducible nontrivial \( p: G_F \to GL_n(\mathbb{C}) \) for \( F = \) number field, \( L(s, p) = \prod_p \det(-)^{-1} \) is entire.

**Ex**: For \( F = \mathbb{Q}, n=2 \), stronger to ask \( p \cong \tilde{\rho}_{f, \lambda} \) for some \( \lambda \) cuspidal eigenform \( f \in \text{H}^0(X, \omega(-\text{cusp})) \) of weight 1 \( \left( X = X_1(N), \text{some } N \geq 1 \right) \).
Some modest progress on Serre's cong.

Thm (Taylor, Manoharmayum, Ellenberg, Shepherd-Barron...)

Let $p : \mathbb{G}_a \to \text{GL}_2(\mathbb{F}_p)$ be odd, continuous, irreducible (over $\mathbb{F}_p$), $p = 3, 7, 13$.

Under some "local conditions" on $p\, \text{d}_3, p\, \text{d}_5$, it is modular.

Main point of proof: close study of geometry of certain "twisted" Hilbert modular varieties that classify suitable abelian varieties with $p$ identified in torsion subgroup. Show by "weak approximation" that these have global points corresponding to abelian variety s.t. Wiles method applies to another member of resulting compatible family.
By strengthening the geometric input and automorphic input, as well as work of Skinner-Wiles (adapting Wiles' methods to certain reducible \( \overline{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_p) \)), Taylor proved:

**Thm** Let \( \overline{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_p) \) be any continuous, odd, irreducible rep. s.t. \( \overline{\rho}|_{L_{E_{p}}} \) is "distinguished."

There exists an abelian variety \( A_{\mathbb{Q}} \) with action by \( \mathbb{Q} \)-isogenies on \( F \) of degree \( \dim A \) over \( \mathbb{Q} \), and prime \( p \) of \( \mathbb{O}_F \) s.t. \( A[\rho] \cong \overline{\rho} \) (over \( \mathbb{F}_p \)).

**Upshot:** EVERY such \( \overline{\rho} \) can be put in "compatible family" (or rather reduction of such). Can also control some local properties in family.
These advances on Serre's conjecture have provided key input ("\( \tilde{\rho} \) modular") to use Wiles' methods in weight 1 to attack Artin's conjecture! We gave one example:

**Theorem (Buzzard-Taylor).** Let \( \rho: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C}) \) be continuous, irreducible, odd. Subject to some "local conditions" at 3, 5, \( \rho \cong \rho_f \alpha \) for some \( f \) of weight 1 (so \( L(s, \rho) = L(s, f) = \text{entire} \)).

Geometric novelty in proof (beyond applying Wiles' deformation methods): the weight 1 form \( f \) is initially built over p-adic open in \( X_1(N)_{\mathbb{Q}_p} \), then analytically extended. Use rigid GAGA to relate to algebraic theory over \( \mathbb{Q}_p \supset \mathbb{C} \).
§5. \textit{p-adic Hodge theory}

\( x = \text{compact complex manifold} \)

\[ H_{\text{top}}^i(X, \mathbb{C}) \quad \Rightarrow \quad H^i(\Omega^k_X) = H^i_{\text{dR}}(X) \]

\[ H^i_c(X, \mathcal{O}_X) \xrightarrow{\cong} H^i(\Gamma(X, \mathcal{O}_X)) \]

\( \Gamma \) is a \( \mathbb{C} \)-valued \( \mathbb{C} \)-complex.

\([K : \mathbb{Q}_p] < \infty \), \( X = \text{smooth proper K-scheme} \)

\( H^i_{\text{et}}(X_K, \mathbb{Q}_p) = \text{finite-dimensional } \mathbb{Q}_p \text{-vector space with linear } G_K \text{-action} \)

\( H^i_{\text{dR}}(X/K) = H^i(\Omega^k_X/K) = \text{finite-dimensional } K \text{-vector space with Hodge filtration.} \)

- Can these be related?!?

- Are \( H^i_{\text{et}}(X_K, \mathbb{Q}_p) \)’s “special” as \( \mathbb{Q}_p[G_K] \)-modules?
\( \hat{Q}_p = \bar{Q}_p = \mathbb{C}_p \) algebraically closed!

By continuity, \( G_K \) acts on \( \mathbb{C}_p \) (by isometries).

**Thm (Tate)** \( \mathbb{C}_p^{G_K} = K \left[ \left\{ \text{no transcendental invariants} \right\} \right] \)

\[ Z_p(1) = T_p(G_m) = \lim \mu_{p^n}(\bar{Q}_p), \]

a finite free \( \mathbb{Z}_p \)-module of rank 1 with continuous \( G_K \)-action.

**Analogue for** \( K = \mathbb{C} \): for

\[ Z(1) = \ker (\exp : \mathbb{C} \to (\mathbb{C}^\times) = \pm 2\pi i \mathbb{Z}_p \]

\[ Z(1)/\mathbb{Z}_p, Z(1) \cong \mu_{p^n}(\mathbb{C}) \]

so \( \lim \mathbb{Z}_p \to Z_p \otimes \mathbb{Z}(1) \approx Z_p(1) \).

**Thm (Tate)** \( (\mathbb{C}_p \otimes \mathbb{Z}_p(1) \otimes i)^{G_K} = 0 \quad \forall i \neq 0 \).
Let $M = \mathbb{Z}_p[\mathbb{G}_k]$-module

$$M(i) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)^{\otimes 1}.$$ 

"Tate twist"

Then (Tate, Serre) Let $V$ be $\mathbb{Q}_p[\mathbb{G}_k]$-module with $\dim_{\mathbb{Q}_p} V < \infty$, continuous $\mathbb{G}_k$-action.

Then for $V\{i\} = ((\mathbb{C}_p \otimes V)(i))^G$ space have

$$\bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i) \otimes V\{i\} \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i) \otimes V$$

is injective (as $\mathbb{C}_p$-vector spaces with semilinear $\mathbb{G}_k$-action).

Pf: Chase minimal tensor sums, use previous theorem of Tate.

Cor: $\dim_k V\{i\} < \infty \text{ for all but finitely many } i$,

$$\sum \dim_k V\{i\} \leq \dim_{\mathbb{Q}_p} V$$

an equality $\iff$ $(\star)$ is isomorphism.
Def: $V$ is Hodge-Tate if

\[ \text{equality holds (preserved by } \otimes, \text{ duality)} \]

The $i$ such that $V \otimes \Omega^i \neq 0$ are Hodge-Tate w.r.t.

Ex: $X =$ abelian variety over $K$.

Using $p$-divisible groups over $\mathcal{O}_K$,

Tate showed if $X$ has good reduction

then $\exists$ canonical $\mathbb{C}_p$-linear

and $G_K$-linear isomorphism

\[
(C_p \otimes t_X^v) \oplus (C_p(H) \otimes t_X^*) = C_p \otimes H^2_{et}(X_K, \mathcal{O}_p) \]

Analogue of Hodge decomposition!

\[ \text{Thm (Faltings) } \exists \text{ canonical } \mathbb{C}_p \text{-linear and } G_K \text{-linear isomorphism } [\text{Hodge-Tate decomposition}] \]

\[ \oplus \mathbb{C}_p(-j) \otimes H^{n-d}(X, \mathcal{O}_K^j) \cong C_p \otimes H^d_{et}(X_K) \]

for $X = \text{smooth, proper over } K$. 

HT weights \[ \varepsilon = 0, 1. \]
Cor: \( (\mathbb{C}/(\mathbb{C}/G_K)) \otimes H^\infty_{\text{et}}(X_K, \mathbb{Q}) ) G_K \approx H^{n-d}(X, \Omega^d) \)

as \( K \)-vector spaces.

Q: i) Can we recover \( H^m_{\text{dR}}(X/K) \) as filtered vector space/\( K \) from \( H^m_{\text{et}}(X_K, \mathbb{Q}_p) = \mathbb{Q}_p[G_K]-\text{module} \)?

ii) Better: can we go in other direction, reconstructing Galois module from simple linear algebra data?

A: i) Yes, using better viewpoint (of period ring)

ii) Not quite: need more structure on \( H^\text{dR}(X/K) \) in general.
Let's restate Hodge-Tate decomposition (when it exists!) in a more convenient form.

\[ B_{HT} = \bigoplus_{i \in \mathbb{Z}} C_p(i) \quad (= C_p[t, t^{-1}], \quad [Z_p(i) = Z_p \cdot t]) \]

= graded \( C_p \)-algebra
with semi-linear \( G_K \)-action

Let \( V \) = finite-dim \( C_p \)-vector space
with continuous linear
\( G_K \)-action

\[ D_{HT}(V) = \left( B_{HT} \otimes_{C_p} V \right)^{G_K} \]

graded \( G_K \)-module
= graded \( K \)-vector space

Tate \( \Rightarrow \) \( \dim_K D_{HT}(V) \leq \dim_{C_p} V \),
with equality iff \( B_{HT} \otimes_K D_{HT}(V) \to B_{HT} \otimes_{C_p} V \)
as a graded \( B_{HT} \)-module with \( G_K \)-action is isom...
$B_{HT} \otimes_{K} D_{HT} (Y) \xrightarrow{\varphi} B_{HT} \otimes_{k} V$

$\otimes (b \otimes v) \rightarrow b \otimes v$

$\varphi$ in degree $r$ is $\mathbb{C}_{P}$-linear, $G_{K}$-linear

$\bigoplus C_{P}(j) \otimes (C_{P}(r-j) \otimes_{k} V) \xrightarrow{G_{K}} C_{P}(r) \otimes_{k} V$

This is $Q_{P}(r) \otimes_{k} (\cdot)$ applied to

$\bigoplus C_{P}(j) \otimes (C_{P}(j) \otimes_{k} V) \xrightarrow{G_{K}} C_{P} \otimes_{k} V$

$\left( j' = r-j \right)$

This has NOTHING to do with $r$!

Conclusion For study of $C_{P} \otimes_{k} V$

as semilinear $G_{K}$-module, when $V$ is

HT it is "equivalent" to work

with $D_{HT} (Y) = \text{f. dim } K$ vectorspa
Indeed, as $\mathbb{C}_p$-spaces with $G_K$-action,

$$gr^0(B_{HT} \otimes_{D_{HT}}(Y)) \simeq gr^0(B_{HT} \otimes_{Q_p} Y) = \mathbb{C}_p \otimes_{Q_p} V.$$

Though $\mathbb{C}_p = K$, no way we can extract $V$ from $D_{HT}(Y)$.

**Ex:** $\dim Y = 1$, $G_K$ acts by finite-order non-trivial character.

Then $(\mathbb{C}_p \otimes_{Q_p} V)^{G_K} \neq 0$, so $\mathbb{C}_p \otimes_{Q_p} V \cong \mathbb{C}_p$ as $\mathbb{C}_p$-vector space with $G_K$-action. \[ D_{HT}(Y) = (B_{HT} \otimes_{Q_p} V)^{G_K} = (\oplus \mathbb{C}_p^{(i)} \otimes_{Q_p} V)^{G_K} = K \mathcal{L} = gr^0. \]

**Ex:** If $\lambda \in HT$ with $wts 0, k-1$ for $f$ of weight $k$ (up to coefficients...)

**Ex:** Weird "non-geometric" $p$-adic operations ($\exp ...$) make some non-HT rep...
Ex: Let $\pi : G_K \rightarrow \mathbb{Z}_p^\times$ be an action of $G_K$ on $\mathbb{Z}_p$ such that $\pi (g) = \lim_{\rightarrow} \pi (R) = \mathfrak{m}^n$ for $p$-power roots of unity $\mathfrak{m}$.

Using $p$-adic log/exp between neighborhood of $1, 0$ in $\mathbb{Z}_p^\times$, $\mathbb{Z}_p$ resp, can define $x^r$ for any $r \in \mathbb{Z}_p$ if $K \cong \mathcal{O}_p(\mathfrak{m})$. This is Hodge-Tate iff $r \in \mathbb{Z}$.

Rem: For $K/K'$ finite, $V = \text{repn of } G_K$ is HT $\iff$ $V$ as repn of $G_{K'}$ is HT (same with $G_{K'}$)

Fontaine defined a remarkable field $B_{dR} = \text{Frae}(B_{dR}^+)$ (field of $p$-adic periods) $\hookrightarrow$ dvr, complete where

- $B_{dR}^+$ is topological $\mathcal{O}_p[G_K]$-algebra
- residue field $\mathcal{O}_p$ as topological ring with $G_K$-action
- 3 canonical $\mathbb{Z}_p(l) \hookrightarrow B_{dR}^+$ giving basis of $\mathfrak{m}/\mathfrak{m}^2$ over $\mathcal{O}_p$. 

For example, what really happens in construction of $B^{+}_{dR}$ is that one makes some very non-trivial extensions of $C_{p}(r)$ by $C_{p}(s)$ for $r \neq s$.

\[
0 \to \mathfrak{m}/\mathfrak{m}^2 \to B^{+}_{dR}/\mathfrak{m}^2 \to B^{+}_{dR}/\mathfrak{m} \to 0
\]

\[
\begin{align*}
(B^{+}_{dR}/\mathfrak{m}) & \otimes_{p(1)} \\
C_{p}(1) & \\
C_{p}(1)
\end{align*}
\]

is an exact sequence of topological $C_{p}[G_{K}]$-modules, NOT split as such.

By commutative algebra, $B^{+}_{dR} \simeq C_{p}[I_{T}]$. However, there is no structure on topological ring $B^{+}_{dR}$ of $C_{p}$-algebra respecting $G_{K}$-action and residue field identification with $C_{p}$.
The field $B_{\text{dr}}$ is filtered by powers of maximal ideal of $B_{\text{dr}}^+$,

$$\text{gr}^i(B_{\text{dr}}) = \bigoplus_{i \in \mathbb{Z}} (B_{\text{dr}}^+)_{m^i/m^{i+1}}$$

as graded $C_p$-algebras with $G_K$-action. By Tate, $((B_{\text{dr}})^{G_k})_{G_K} = K$.

$$D_{\text{dr}}(V) = (B_{\text{dr}} \otimes_{C_p} V)^{G_K}$$

filtered $K$-vector space with compatible $G_K$-action.

$$\dim_K D_{\text{dr}}(V) \leq \dim_{C_p} V$$

Def V is de Rham if equality holds (preserved by $\otimes$, duality)
Another viewpoint:

\[ D_{dR}(V) = (B_{dR} \boxtimes V)^{\mathbb{G}_k} \]

\[ = \text{Hom}_{\mathbb{A}_p}[\mathbb{G}_k](V^*, B_{dR}) \]

is finite-dimensional $K$-vector space, so using basis \{\( \varphi_i \)\} get finite-dimensional $K$-subspace

\[ \sum \varphi_i(V^*) \leq B_{dR}. \]

More canonically, this is image of canonical map

\[ V^{\text{dR}}(V) = V^* \boxtimes (B_{dR} \boxtimes V)^{\mathbb{G}_k} \to B_{dR}. \]

Elements of image are the $p$-adic periods of $V$. The $dR$ reps are those with the "most" such periods, given their $\mathbb{A}_p$-dimension.
There is always a canonical map
\[ \mathcal{B}_{\text{dR}} \otimes \mathcal{D}_{\text{dR}}(V) \to \mathcal{B}_{\text{dR}} \otimes_{\mathbb{Q}_p} V \]
This is isomorphism iff \( V \) is deRham (always injective)

Thus (Faltings) \( X = \) smooth, proper /\( K \).
Then \( V = H^{m}_{\text{et}}(X, \mathbb{Q}_p) \) is deRham,
and have canonically \( \mathcal{D}_{\text{dR}}(V) \cong H^{m}_{\text{dR}}(X/K) \)
as filtered \( K \)-vector spaces. Thus,
have \( \mathcal{B}_{\text{dR}} \)-linear isomorphism
\[ \mathcal{B}_{\text{dR}} \otimes H^{m}_{\text{et}}(X/K) \cong \mathcal{B}_{\text{dR}} \otimes H^{m}_{\text{dR}}(X, \mathbb{Q}_p) \]
as filtered \( K \)-vector spaces with \( \mathcal{B}_{\text{dR}} \)-action (a) respects cycle maps,
Poincare duality, cup products

Still insufficient to extract \( H^{m}_{\text{et}}(X, \mathbb{Q}_p) \) from "linear algebra"
\( \text{via } \mathcal{B}_{\text{dR}}, \text{etc.} \)
Fontaine defined more intricate subrings

$$B_{cris} \subset B_{st} \subset B_{dR}$$

with more "linear algebra" structure
(Frobenius operator, monodromy operator...)
Defined functors $\text{Dens}$, $D_{st}$ into
suitable "linear algebra" categories,
again "dim $D_{st} \leq \dim \omega_0 V$, say V
is crystalline, semistable (resp.)
when equality holds.

Thus $\text{Dens}$, $D_{st}$ are fully faithful
on crystalline, ss-t reps, respectively
Thus (Faltings, Tsuji...) $H^m_{et}(X, \mathcal{O}_p)$ as
above is crystalline (resp. ss-t) if
$X$ admits proper smooth (resp. proper ss-t
model over $\mathcal{O}_K$)
Then (Fontaine-Colmez) the essential unipage of $D_{st}$ can be described in terms of "linear algebra" structures alone! (in terms of Newton and Hodge polygons) (Kisin uses Hodge polygons)

Consequence: to "deform" a semistable Galois repn, can try to deform the associated linear algebra datum! (This viewpoint)

There is a (reasonable) notion of "potentially semistable", $D_{pst}$-functor

Then (Berger, Kedlaya, ...) $dR = pst$ !

By deJong's alterations, $H^m_{et}(X,\mathcal{O}_X)$ is pst for ANY proper $K$-scheme $X$. 
Significance: when formulating global deformation problems for $p$-adic representations, for local condition "at $p" need to impose some property relate to $p$-adic period rings.

Fontaine-Mazur Conjecture: If $p: G_F \to GL_n(K), [K:Q_p] < \infty$, is irreducible, a continuous representation and
  
  • $p$ is unramified at all but finitely many places

  • $A \sim p, p|p$, is pst then $p$ "arises from algebraic geometry" (= subquotient of some $H_{et}^n(X_{\overline{F}_p}, \Omega_p(r))$.

For $F = Q, n=2$, Taylor proved many cases.