Outline

In view of time constraints and desire to go slower:

§4 Progress on Artin conjecture and Serre's conjecture

§5 p-adic Hodge theory.

We shall say nothing about p-divisible groups except:

By using p-adic Hodge theory methods, M. Kisin developed more powerful methods to use p-divisible groups and group schemes in deformation problems for Galois representations, providing a better approach to modularity

§4. Progress on Artin and Serre conjectures.

Recall statements!

Serre's conjecture: Any odd, continuous irreducible $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is modular (ie, $\rho \simeq \bar{\rho}_{f,\lambda}$ some f, prime λ of K_f)

Artin conjecture For any continuous irreducible nontrivial $\rho : G_F \to GL_n(\mathbb{C})$ for $F =$ number field, $L(s,\rho) = \prod_{\mathfrak{p}} \det(-)^{-1}$ ($\mathrm{Re}\, s > 1$) is entire.

Ex: For $F = \mathbb{Q}$, $n=2$, stronger to ask $\rho \simeq \rho_{f,\lambda}$ for some cuspidal eigenform $f \in H^0(X, \bar{\omega}(-\text{cusps}))$ of weight 1 ($X = X_1(N)$, some $N \geq 1$).
($\Rightarrow L(s,\rho) = L(s,f)$)

Some modest progress on Serre's conj:

<u>Thm</u> (Taylor, Manoharmayum, Ellenberg, Shepherd Barron ...)

Let $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$ be odd, continuous, irreducible (over $\overline{\mathbb{F}}_p$), $p = \cancel{4}, 7, \cancel{9}$.
Under some "local conditions" on $\rho|_{D_3}, \rho|_{D_5}$, it is modular.

<u>Main point of proof</u>: close study of <u>geometry</u> of certain "twisted" Hilbert modular varieties that classify suitable abelian varieties with ρ identified in ~~a~~ torsion subgroup. Show by "weak approximation" that these have global points corresponding to abelian variety s.t. Wiles method applies to a particular member of resulting compatible fam

By strengthening the geometric input and automorphic input, as well as work of Skinner-Wiles (adapting Wiles' methods to certain reducible $\bar{\rho}: G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$), Taylor proved:

Thm Let $\bar{\rho}: G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ be any continuous, odd, irreducible rep. s.t. $\bar{\rho}|_{D_p}$ is "distinguished". There exists abelian variety $A_{/\mathbb{Q}}$ with action by integers of # field F of degree $\dim A$ over $\mathbb{Q}$ and prime $\mathfrak{p}$ of $\mathcal{O}_F$ (over p) s.t. $A[\mathfrak{p}] \simeq \bar{\rho}$ (over $\bar{\mathbb{F}}_p$).

Upshot: EVERY such $\bar{\rho}$ can be put in "compatible family" (or rather reduction of such). Can also control some local properties in family.

These advances on Serre's conjecture have provided key input ("$\bar{\rho}$ modular") to use Wiles' methods in weight 1 to attack Artin conjecture! We give one example:

<u>Thm</u> (Buzzard-Taylor). Let $\rho: G_{\mathbb{Q}} \to GL_2(\mathbb{C})$ be continuous, irreducible, odd. Subject to some "local conditions" at 3, 5, $\rho \simeq \rho_{f,\lambda}$ for some f of weight 1 (so $L(s,\rho) = L(s,f) =$ entire).

Geometric novelty in proof (beyond applying Wiles' deformation methods): the weight-1 form f is initially built over p-adic open in $X_1(N)^{an}_{/\mathbb{Q}_p}$, then analytically continued. Use rigid GAGA to relate to algebraic theory over $\mathbb{Q}_p \hookrightarrow \mathbb{C}$. !

§5. p-adic Hodge theory

$X=$ compact complex manifold

$$\begin{array}{ccc} H^i_{top}(X,\mathbb{C}) & & \mathbb{H}^i(\Omega^\bullet_{X/\mathbb{C}}) = H^i_{dR}(X) \\ \| & & \cong \\ H_i(X,\mathbb{C})^\vee & \xleftarrow[\sim]{\int_X} & H^i(\Gamma(X, A^\bullet_X)) \end{array}$$

$A^\bullet_X$: C^∞ $\mathbb{C}$-valued dR complex

$[K:\mathbb{Q}_p]<\infty$, $X=$ smooth proper K-scheme

$H^i_{et}(X_{\bar K},\mathbb{Q}_p)=$ finite-dimensional $\mathbb{Q}_p$-vector space with linear G_K-action

$H^i_{dR}(X/K)=\mathbb{H}^i(\Omega^\bullet_{X/K})=$ finite-dimensional K-vector space with Hodge filtration.

- Can these be related ?!?
- Are $H^i_{et}(X_{\bar K},\mathbb{Q}_p)$'s "special" as $\mathbb{Q}_p[G_K]$-mods

$$\begin{tikzcd}[column sep=small] & \widehat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p = \text{algebraically closed!} \\ \overline{\mathbb{Q}}_p \arrow[ur, dash] \arrow[d, dash] & \\ K \arrow[d, dash] & \\ \mathbb{Q}_p & \end{tikzcd}$$

By <u>continuity</u>, G_K acts on $\mathbb{C}_p$ (by isometries).

<u>Thm</u> (Tate) $\mathbb{C}_p^{G_K} = K$ ["no transcendental invariants"]

$\mathbb{Z}_p(1) = T_p(\mathbb{G}_m) = \varprojlim \mu_{p^n}(\overline{\mathbb{Q}}_p)$, a finite free $\mathbb{Z}_p$-module of rank 1 with continuous G_K-action.

Analogue for $K = \mathbb{C}$: for

$\mathbb{Z}(1) = \ker(\exp : \mathbb{C} \to \mathbb{C}^\times) = \pm 2\pi\sqrt{-1}\cdot\mathbb{Z}$,

$$\mathbb{Z}(1)/p^n\cdot\mathbb{Z}(1) \xrightarrow[e^{(\frac{\cdot}{p^n})}]{\sim} \mu_{p^n}(\mathbb{C})$$

so $\varprojlim \Rightarrow \mathbb{Z}_p \otimes \mathbb{Z}(1) \simeq \mathbb{Z}_p(1)$.

<u>Thm</u> (Tate) $(\mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes i})^{G_K} = 0 \ \forall i \neq 0$.

Let $M = \mathbb{Z}_p[G_K]$-module

$M(i) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes i}$. "Tate twist"

Thm (Tate, Serre) Let V be $\mathbb{Q}_p[G_K]$-mo[dule] with $\dim_{\mathbb{Q}_p} V < \infty$, continuous G_K-action. Then for

$$V\{i\} = ((\mathbb{C}_p \otimes V)(-i))^{G_K} = K\text{-vector } \underline{\underline{\text{space}}}$$

have

$$(*) \quad \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(-i) \otimes_K V\{i\} \longrightarrow \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$$

is injective (as $\mathbb{C}_p$-vector spaces with semilinear G_K-action)

Pf: Chase minimal tensor sums, use previous theorem of Tate.

Cor: $\dim_K V\{i\} < \infty$ $\forall i$, vanishes for all but finitely many i,

$$\sum \dim_K V\{i\} \le \dim_{\mathbb{Q}_p} V,$$

an equality $\Leftrightarrow$ $(*)$ is isomorphism.

Def: V is **Hodge-Tate** if equality holds. (preserved by $\otimes$, duality)
The i such that $V\{i\} \neq 0$ are Hodge-Tate wts of V.

Ex: X = abelian variety over K.

Using p-divisible groups over $\mathcal{O}_K$, Tate showed if X has good reduction then $\exists$ canonical $\mathbb{C}_p$-linear and G_K-linear isomorphism

$$(\mathbb{C}_p \otimes t_{X^\vee}) \oplus (\mathbb{C}_p(-1) \otimes t^*_X) \simeq \mathbb{C}_p \otimes H^1_{et}(X_{\bar{K}}, \mathbb{Q}_p)$$

$$\| \qquad\qquad\qquad \uparrow$$

$$(\mathbb{C}_p \otimes H^1(X, \mathcal{O}_X)) \oplus (\mathbb{C}_p(-1) \otimes H^0(X, \Omega^1_X))$$

HT weights $0, 1$.

Analogue of Hodge decomposition!

Thm (Faltings) $\exists$ canonical $\mathbb{C}_p$-linear and G_K-linear isomorphism [Hodge-Tate decomp

$$\bigoplus_{0 \le j \le n} \mathbb{C}_p(-j) \otimes H^{n-j}(X, \Omega^j_{X/K}) \simeq \mathbb{C}_p \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\bar{K}}, \ldots$$

HT weights in $[0, n]$

for X = smooth, proper over K.

Cor: $(\mathbb{C}_p(j) \otimes H^n_{ét}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K}$
$$\simeq H^{n-j}(X, \Omega^j_{X/K})$$
as K-vector spaces.

Q: i) Can we recover $H^n_{dR}(X/K)$ as filtered vector space/K from $H^n_{ét}(X_{\overline{K}}, \mathbb{Q}_p) = \mathbb{Q}_p[G_K]$-module?

ii) Better: can we go in other direction, reconstructing Galois module from simple linear algebra data?

A: i) Yes, using better viewpoint (of period rings)

ii) Not quite: need more structure on $H^n_{dR}(X/K)$ in general.

Let's restate Hodge-Tate decomposition (when it exists!) in a more convenient form.

$$B_{HT} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i) \quad (= \mathbb{C}_p[t, t^{-1}],) \quad [\mathbb{Z}_p(1) = \mathbb{Z}_p \cdot t]$$

= <u>graded</u> $\mathbb{C}_p$-algebra with semilinear G_K-action

Let V = finite dim $\mathbb{Q}_p$ vector space with continuous linear G_K-action

$$D_{HT}(V) = (\underbrace{B_{HT} \otimes_{\mathbb{Q}_p} V}_{\text{graded } G_K\text{-module}})^{G_K}$$

= graded K-vector space

Tate $\Rightarrow$ $\dim_K D_{HT}(V) \le \dim_{\mathbb{Q}_p} V$, with equality iff $B_{HT} \otimes_K D_{HT}(V) \to B_{HT} \otimes_{\mathbb{Q}_p} V$ as graded B_{HT}-mods w/ G_K-action is isom:

$$\begin{array}{ccc} B_{HT} \otimes_K D_{HT}(V) & \xrightarrow{\varphi} & \\ \| & & \\ B_{HT} \otimes_K (B_{HT} \otimes_{\mathbb{Q}_p} V)^{G_K} & \longrightarrow & B_{HT} \otimes_{\mathbb{Q}_p} V \\ b \otimes (b' \otimes v) & \longmapsto & bb' \otimes v \end{array}$$

φ in degree r is: $\mathbb{C}_p$-linear, G_K-linear

$$\bigoplus_j \mathbb{C}_p(j) \otimes_K (\mathbb{C}_p(r-j) \otimes_{\mathbb{Q}_p} V)^{G_K} \longrightarrow \mathbb{C}_p(r) \otimes_{\mathbb{Q}_p} V$$

This is $\mathbb{Q}_p(r) \otimes_{\mathbb{Q}_p} (\cdot)$ applied to

$$\bigoplus_{j' (= r-j)} \mathbb{C}_p(-j') \otimes_K (\underbrace{\mathbb{C}_p(j') \otimes_{\mathbb{Q}_p} V}_{V\{j'\}})^{G_K} \longrightarrow \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$$

This has NOTHING to do with r!

<u>Conclusion</u> For study of $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ as semilinear G_K-module, when V is HT it is "equivalent" to work with $D_{HT}(V)$ = f. dim graded K vectorspa

Indeed, as $\mathbb{C}_p$-v.spaces with G_K-action,

$$gr^0(B_{HT} \otimes_K D_{HT}(V)) \simeq gr^0(B_{HT} \otimes_{\mathbb{Q}_p} V)$$
$$= \mathbb{C}_p \otimes_{\mathbb{Q}_p} V.$$

Though $\mathbb{C}_p^{G_K} = K$, no way we can extract V from $D_{HT}(V)$.

Ex: $\dim V = 1$, G_K acts by finite-order non-trivial character. Then $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{G_K} \neq 0$, so $\mathbb{C}_p \otimes V \simeq \mathbb{C}_p$ as $\mathbb{C}_p$-vector space with G_K-action. $\therefore D_{HT}(V) = (B_{HT} \otimes V)^{G_K}$

$$= (\oplus \mathbb{C}_p(i) \otimes V)^{G_K}$$
$$= (\oplus \mathbb{C}_p(i))^{G_K}$$
$$= K \; (= gr^0).$$
$$= D_{HT}(\mathbb{Q}_p).$$

Ex: $\rho_{f,\lambda}$ is HT with wts $0, k-1$ for f of weight k (up to coefficients...)

Ex: Weird "non-geometric" p-adic operations (exp...) make some non-HT rep

Ex: Let $\chi : G_K \to \mathbb{Z}_p^\times$ be action of G_K on $\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}(\bar{K})$; ie, $g(\zeta) = \zeta^{\chi(g)}$ for p-power roots of unity ζ. Using p-adic log/exp between neighborhoods of $1, 0$ in $\mathbb{Z}_p^\times, \mathbb{Z}_p$ resp, can define χ^r for any $r \in \mathbb{Z}_p$ if $K \supseteq \mathbb{Q}_p(\zeta_{p^2})$. This is Hodge-Tate iff $r \in \mathbb{Z}$!

Rem: For K'/K finite, V = repn of G_K is HT $\iff$ V as repn of $G_{K'}$ is HT (same wts)

Fontaine defined a remarkable field $B_{dR} = \mathrm{Frac}(B_{dR}^+)$ (field of p-adic periods) ↳ dvr, complete where

- B_{dR}^+ is topological $\mathbb{Q}_p[G_K]$-algebra, residue field $\mathbb{C}_p$ as topological ring with G_K-action
- $\exists$ canonical $\mathbb{Z}_p(1) \hookrightarrow B_{dR}^+$ giving basis of $\mathfrak{m}/\mathfrak{m}^2$ over $\mathbb{C}_p$.

For example, what really happens in construction of B_{dR}^+ is that one makes some very non-trivial extensions of $\mathbb{C}_p(r)$ by $\mathbb{C}_p(s)$ for $r \neq s$:

$$
\begin{array}{ccccccccc}
0 \to & \mathfrak{m}/\mathfrak{m}^2 & \to & B_{dR}^+/\mathfrak{m}^2 & \to & B_{dR}^+/\mathfrak{m} & \to 0 \\
 & \parallel & & & & \wr\!\mid \text{ canonical} & \\
 & (B_{dR}^+/\mathfrak{m})\cdot \mathbb{Z}_p(1) & & & & \mathbb{C}_p & \\
 & \wr\!\mid & & & & & \\
 & \mathbb{C}_p(1) & & & & &
\end{array}
$$

is ~~[illegible]~~ exact sequence of <u>topological</u> $\mathbb{Q}_p[G_K]$-modules, NOT split as such.

By commutative algebra, $B_{dR}^+ \simeq \mathbb{C}_p[[T]]$. However, there is <u>no</u> structure on topological ring B_{dR}^+ of $\mathbb{C}_p$-algebra respecting G_K-action and residue field identification with $\mathbb{C}_p$.

The field ~~[illegible]~~ B_{dR} is <u>filtered</u> by powers of max. ideal of B_{dR}^+,

$$gr^{\cdot}(B_{dR}) \simeq \bigoplus_{i \in \mathbb{Z}} \underset{\substack{\shortparallel \\ \mathfrak{m}^i/\mathfrak{m}^{i+1}}}{\mathbb{C}_p(i)} = B_{HT}$$

as graded $\mathbb{C}_p$-algebras with G_K-action. By Tate, $(B_{dR})^{G_K} = K$.

~~(By comm~~

$$D_{dR}(V) = (\underbrace{B_{dR} \otimes_{\mathbb{Q}_p} V}_{})^{G_K}$$

filtered K-vector space with compatible G_K-action.

$$\dim_K D_{dR}(V) \leq \dim_{\mathbb{Q}_p} V.$$

<u>Def</u> V is <u>deRham</u> if equality holds (preserved by $\otimes$, duality)

Another viewpoint:

$$D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

$$= \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V^*, B_{dR})$$

is finite-dimensional K-vector space, so using basis $\{\varphi_i\}$ get finite-dimensional K-subspace

$$\sum_i \varphi_i(V^*) \subseteq B_{dR}.$$

More canonically, this is image of canonical map

$$V^* \otimes_{\mathbb{Q}_p} D_{dR}(V) = V^* \otimes_{\mathbb{Q}_p} (B_{dR} \otimes V)^{G_K} \to B_{dR}.$$

Elements of image are the p-adic periods of V. The dR repns are those with the "most" such periods, given their $\mathbb{Q}_p$-dimension.

There is always a canonical map

$$B_{dR} \otimes_K D_{dR}(V) \to B_{dR} \otimes_{\mathbb{Q}_p} V$$

This is isomorphism iff V is deRham (always injective)

<u>Thm</u> (Faltings, t. ...) X = smooth, proper $/K$. Then $V = H^m_{ét}(X, \mathbb{Q}_p)$ is deRham, have canonically $D_{dR}(V) \simeq H^m_{dR}(X/K)$ as filtered K-vector spaces. Thus, have B_{dR}-linear isomorphism

$$B_{dR} \otimes_K H^m_{dR}(X/K) \simeq B_{dR} \otimes_{\mathbb{Q}_p} H^m_{ét}(X_{\bar{K}}, \mathbb{Q}_p)$$

as filtered K-vector spaces with G_K-action (also respects cycle maps, Poincaré duality, cup products)

Still insufficient to extract $H^m_{ét}(X_{\bar{K}}, \mathbb{Q}_p)$ from "linear algebra" (via B_{dR}, etc.)

Fontaine defined more intricate subrings

$$B_{cris} \subseteq B_{st} \subseteq B_{dR}$$

with more "linear algebra" structure (Frobenius operator, monodromy operator...)

Defined functors D_{cris}, D_{st} into suitable "linear algebra" categories, again "dim" $D_* \leq \dim_{\mathbb{Q}_p} V$, say V is crystalline, semistable (resp.) when equality holds.

<u>Thm</u> D_{cris}, D_{st} are fully faithful on crystalline, sst repns. respectively

<u>Thm</u> (Faltings, Tsuji...) $H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ as above is crystalline (resp. sst) if X admits proper smooth (resp. proper sst) model over O_K

Thm (Fontaine–Colmez) The essential image of D_{st} can be described in terms of "linear algebra" structures alone! (in terms of Newton and Hodge polygons)

Consequence: to "deform" a semistable Galois repn, can try to deform the associated linear algebra datum! (Kisin uses this viewpoint)

There is a (reasonable) notion of "potentially semistable", D_{pst}-functor.

Thm (Berger, Kedlaya, ...)
deRham = pst !

By deJong's alterations, $H^m_{ét}(X_{\bar{K}}, \mathbb{Q}_p)$ is pst for ANY proper K-scheme X.

Significance: when formulating global deformation problems for p-adic representations, for local condition "at p" need to impose some property relate to p-adic period rings.

<u>Fontaine-Mazur Conjecture</u>: If $\rho : G_F \to GL_n(K)$, $[K:\mathbb{Q}_p] < \infty$, is a continuous irred. representation and

- ρ is unramified at all but finitely many places
- $\forall\, v \mid p$, $\rho|_{D_v}$ is pst

then ρ "arises from algebraic geometry" (= subquotient of some $H^m_{et}(X_{\bar{F}}, \mathbb{Q}_p(r))$).

For $F = \mathbb{Q}$, $n=2$, Taylor proved many cases (using Wiles, etc...)