Outline

8/9

§0. Some concrete applications of modularity theorems [redacted]

§1. "Review" of modular forms and Galois representations, list some geometric methods used to prove modularity theorems. [redacted]

§2. Deformations and Hecke rings (+ modularity of elliptic curves) [redacted]

8/10

§3. p-divisible groups

§4. p-adic Hodge theory

§5. More modularity theorems.

2 We say nothing about techniques from Iwasawa theory, automorphic forms, ..., that have helped to extend Wiles' ide

§0. Some concrete applications of modularity theorems

Gauss' class number problem:

Let $h_D = \#\mathrm{Pic}(\mathrm{Spec}\, \mathcal{O}_{\mathbb{Q}(\sqrt{-D})})$ for sq.free $D>0$.

$$\mathcal{O}_{\mathbb{Q}(\sqrt{-D})} = \begin{cases} \mathbb{Z}[\sqrt{-D}] & D \not\equiv 3(4) \\ \mathbb{Z}[\frac{1+\sqrt{-D}}{2}] & D \equiv 3(4) \end{cases}.$$

This is computable!

Q1. Does $h_D \to \infty$ as $D \to \infty$?

(Analogue for real quadratic fields seems to be false.)

Q2 If so, for $h \geq 1$ find largest D so $h_D = h$.

Ex: Is $D = 163$ the "last" one for $h=1$?

Goldfeld proposed a "analytic" solution if can find suitable elliptic curves $E/\mathbb{Q}$.

(giving $h_D \geq c_\varepsilon (\log D)^{1-\varepsilon}$, $c_\varepsilon = c_{\varepsilon, E}$ explicit!)

What properties did Goldfeld require for $E/\mathbb{Q}$?

1. The L-function ("Euler product")

$$L(E/\mathbb{Q}, s) = \prod_{\substack{\text{good} \\ p}} (1 - a_p(E)p^{-s} + p^{1-2s})^{-1} \cdot \prod_{\substack{\text{bad} \\ p}} (1 - a_p(E)p$$

for $\mathrm{Re}(s) > \frac{3}{2}$ has analytic continuation and functional equation of "standard" type.

2. $\mathrm{ord}_{s=1} L(E/\mathbb{Q}, s) \geq 3$.

For any particular $E/\mathbb{Q}$, can check 1 by finite calculation, and use "sign of functional eqn" to determine parity of $\mathrm{ord}_{s=1} L(E/\mathbb{Q}, s)$. $\therefore$ need a way to PROVE $L'(E/\mathbb{Q}, 1) = 0 \ldots$

Gross-Zagier gave a method to study if $L'(E/\mathbb{Q}, 1) = 0$ or not by using points $P \in E(K)$ for certain $K/\mathbb{Q}$ — construct points as image under a finite map $/\mathbb{Q}$

$$X_0(N) \xrightarrow{\pi} E \quad \Big\} \text{ E is "modular"}$$

$X_0(N)$ = coarse moduli space of (E', C'), E' = ell. curve, C' = cyclic subgp of order N

1. Since $X_0(N)$ is a (coarse) moduli space, can make interesting points on it over specific fields (Heegner pts)
2. Existence of π is finite check! [can contr N
3. $Tr_{K/\mathbb{Q}}(P) \in E(\mathbb{Q})$. By G-Z/Kolyvagin if $L(E/\mathbb{Q},1)=0$, $L'(E/\mathbb{Q},1)\neq 0$, then $\mathrm{rk}\, E(\mathbb{Q}) = 1$ and $\Leftrightarrow$ $Tr_{K/\mathbb{Q}}(P)$ has infinite order.

Moral of story: The modularity property ($X_0(N) \xrightarrow[\text{finite}]{\pi} E_{/\mathbb{Q}}$) gives a METHOD to construct interesting points in $E(\mathbb{Q})$, and whether or not these have finite order is controlled by $L'(E/\mathbb{Q}, 1)$. Also, modularity is the "only" METHOD to show $L(E/\mathbb{Q}, s)$ has good analytic properties.

For ANY scheme Z of finite type over $\mathbb{Z}$, can define an L-function as Euler product in half-plane. Meromorphic continuation to $\mathbb{C}$ still not known in general, except for Z over $\operatorname{Spec} \mathbb{F}_p$ (Grothendi

For an "abstract" $E/\mathbb{Q}$, how to find non constant

$$X_0(N) \longrightarrow E \quad (\infty \mapsto 0)$$

for some $N \geq 1$?

By Albanese property of Jacobian,

$$\mathrm{Map}((X_0(N), \infty), (E, 0)) = \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Jac}(X_0(N)), E)$$

$\overset{?}{\neq} 0$ $\qquad$ $\parallel$ finite free $\mathbb{Z}$-module

<u>Faltings isogeny thm.</u>: For abelian varieties A, A' over $K = \#$ field,

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}_K(A, A') \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}_\ell[G_K]}(T_\ell A, T_\ell A'),$$

where $T_\ell A = \varprojlim A[\ell^n](\overline{K})$ ("$=$" $H_1(A, \mathbb{Z}_\ell)$)

<u>Key point:</u> $T_\ell(\mathrm{Jac}(X_0(N)))$ as $\mathbb{Z}_\ell[G_{\mathbb{Q}}]$-module can be understood by using arithmetic of modular forms (e.g., to describe "semisimplification")

<u>Wiles' discovery:</u> "Pieces" of $T_\ell(\mathrm{Jac}(X_0(N)))$ provide certain kinds of universal deformations of their "mod ℓ" reduction (as $G_{\mathbb{Q}}$-module over finite field).

$\therefore$ if (for some N) can show $T_\ell E$ is also such a deformation, then $\mathrm{Hom}_{\mathbb{Z}_\ell[G_{\mathbb{Q}}]}(T_\ell(\mathrm{Jac}(X_0(N))), T_\ell E) \neq 0$!

<u>Q:</u> How to identify a universal deformat

§1. "Review" of Galois repns and (classical) modular forms

For any field k, separable closure k_s,

$$\mathrm{Gal}(k_s/k) = G_k = \varprojlim_{\substack{k'/k \\ \text{finite, Galois}}} \mathrm{Gal}(k'/k)$$

$= \text{profinite}.$

Def. An Artin representation of a number field F is

$$\rho : G_F \xrightarrow{\text{cont.}} GL_{\mathbb{C}}(V) \simeq GL_n(\mathbb{C})$$

$$G_F \twoheadrightarrow \underbrace{\mathrm{Gal}(F'/F)}_{\text{finite}} \to GL_n(\mathbb{C})$$

Lie groups have "no small subgroups"

$$L(s,\rho) = \prod_{\mathfrak{p}} \det\left(1 - \rho(\mathrm{Frob}_{\mathfrak{p}}) q_{\mathfrak{p}}^{-s} \mid V^{I_{\mathfrak{p}}}\right)^{-1}$$

$(\mathrm{Re}(s) > 1)$ $\quad (q_{\mathfrak{p}} = \# \mathcal{O}_F/\mathfrak{p})$

Artin conjecture ρ irred, $\rho \neq 1 \Rightarrow L(s,\rho)$ entire

Evidence for Artin conjecture:

1. $\dim V = 1$ — follows from class field theory + Hecke/Tate.

2. Analogue for global function field $\mathbb{F}_q(C)$ is true (Grothendieck), even polynomial in q^{-s}.

3. Results of Taylor, et al., using Wiles' techniques for $F = \mathbb{Q}$, $n = 2$. (building on Langlands, ...) (+ some local constraints)

Def. A p-adic Galois representation is

$$\rho : G_k \xrightarrow{\text{cont}} GL_n(\mathbb{Q}_p) \quad \left(\text{or } GL_n(K),\ [K:\mathbb{Q}_p] < \infty\right)$$

Ex: 1) $A_{/k}$ abelian variety, $\mathrm{char}(k) \nmid p$. Use $V_p A = \mathbb{Q}_p \otimes T_p A$.

2) $X_{/k}$ separated, f.type. Use $H^*_{c,et}(X_{k^s}, \mathbb{Q}$ ($\mathrm{char}\, k \nmid p$)

Some properties of p-adic Galois representations:

Suppose $F = k$ is a number field.

1. $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p A$ for $A_{/F}$ abelian variety
   - unramified at almost all places [because A extends to abelian scheme over $\mathcal{U} \subseteq \operatorname{Spec} \mathcal{O}_F[\frac{1}{p}]$ some dense open]
   - If $F = \mathbb{Q}_p$ and A has "good reduction" at p (i.e., extends to abelian scheme $\mathcal{A}$ over $\mathbb{Z}_{(p)}$) then each $A[p^n] =$ finite $\mathbb{Q}_p$-group scheme must extend to finite flat group scheme over $\mathbb{Z}_{(p)}$ (e.g., $\mathcal{A}[p^n]$). Converse also true! (Grothendieck/Raynaud)
   - Can similarly encode "potentially good reduction" at all $\mathfrak{p} | p$ on F'/F.

2. $V = H^n_{c,ét}(X_{/\overline{F}}, \mathcal{F})$ "geometric" for $X_{/F}$ separated f.type, $\mathcal{F}$ = constructible p-adic sheaf on X (such as $\mathcal{F} = \mathbb{Q}_p$).

Again, unramified at almost all places of F.

What can be said about $H^n_{c,ét}(X_{\overline{F}}, \mathbb{Q}_p)$ as representation of $D_\mathfrak{p} \subseteq G_F$ for $\mathfrak{p}|p$? Need p-adic Hodge theory (later).

Reduction of representations

For $\rho : G_k \xrightarrow{\text{cont.}} GL_n(K)$ $([K:\mathbb{Q}_p] < \infty)$

Can conjugate so $\rho : G_k \to GL_n(\mathcal{O}_K)$.

This gives $\overline{\rho} : G_k \xrightarrow[\text{cont.}]{} GL_n(\mathcal{O}_K/\mathfrak{m})$.

(open kernel!) $\quad \mathcal{O}_K/\mathfrak{m}$ finite field.

[really $\overline{\rho}^{ss}$ is well defined]

$$\begin{tikzcd}
G_K \arrow[r, "\rho"] \arrow[ddr, bend right, "\bar{\rho}"'] & GL_n(\mathcal{O}_K) \arrow[d, dotted] \\
 & GL_n(\mathcal{O}_K/\mathfrak{m}^2) \arrow[d] \\
 & GL_n(\mathcal{O}_K/\mathfrak{m})
\end{tikzcd}$$

Consider ρ as "deformation" of $\bar{\rho}$; passing to $\bar{\rho}$ can <u>lose</u> a lot of (ramification) information.

<u>Q:</u> Given continuous $\bar{\rho}: G_F \to GL_n(\kappa)$ for $F = \#$field, $\kappa =$ finite, can we "deform" $\bar{\rho}$ to characteristic 0 satisfying prescribed local properties at places of F? (e.g., unramified almost everywhere)

<u>Ex:</u> $F = \mathbb{Q}$, $n = 2$: yes! (Ramakrishna)

We'll return to deformation theory of Galois representations in §2.

Compatible families (vary p!)

Let F = # field, $\rho: G_F \to GL_n(K)$, $[K:\mathbb{Q}_p]<\infty$ a p-adic Galois rep.

Can we vary K?

1. Consider $A_{/F}$ abelian variety.
For each p get $V_p(A) = \mathbb{Q}_p \otimes T_pA \circlearrowleft G_F$

Let $S = \{\text{bad places for } A_{/F}\} \cup \{\text{arch. places}\}$.

By Weil, $V_p(A)$ is unramified at all places v of F with $v \notin S$, $v \nmid p$, and $\mathrm{Frob}_v \circlearrowleft V_p(A)$ has characteristic polynomial in $\mathbb{Q}[T]$ independent of p. Thus, $\{V_pA\}_p$ is "compatible family"

2. $X_{/F}$ smooth, proper. By RH (Deligne)

"$\mathbb{Q}_p \otimes H^i(X,\mathbb{Q})$" $\simeq$ $\{H^i_{et}(X_{\bar{F}}, \mathbb{Q}_p)\}_p$ is "compatible family"!

(use arith./geom. Frob action)

Modular forms $\rightsquigarrow$ compatible families of p-adic Galois repns of $G_{\mathbb{Q}}$

(Deligne construction via étale cohomology)

For $N \geq 5$, let $\begin{array}{c} E \\ f \downarrow \circlearrowright P \\ Y = Y_1(N) \end{array}$ be universal elliptic curve, equipped with section with order N on fibers (all done over $\mathbb{Q}$.)

This Y is smooth affine curve$/\mathbb{Q}$.

For $X = X_1(N) =$ compactification, there is a universal "generalized elliptic curve" $\bar{E} \xrightarrow{\bar{f}} X$ extending $E \to Y$; $X - Y =$ cusps

$\bar{\omega} := \bar{f}_*(\omega_{\bar{E}/X}) =$ invertible sheaf on X,

$\bar{\omega}|_Y = f_*(\Omega^1_{E/Y})$; $\bar{\omega}^{\otimes 2} \overset{KS}{\simeq} \Omega^1_X(\text{cusps})$

$M_k = H^0(X, \bar{\omega}^{\otimes k})$

= weight-k modular forms$/\mathbb{Q}$ of level N

$S_k = H^0(X, \bar{\omega}^{\otimes k}(-\text{cusps}))$

= weight-k cuspidal forms$/\mathbb{Q}$ of level N

Over $\mathbb{C}$, have (for fixed $i = \sqrt{-1}$)

$$\begin{tikzcd}
(\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}z, \tfrac{1}{N}) \arrow[d, mapsto] & \mathcal{E} \arrow[r] \arrow[d] & E_{\mathbb{C}} \arrow[d, shift right] \\
z & \mathfrak{h} \arrow[r] & Y_{\mathbb{C}} \arrow[u, shift right, "P"']
\end{tikzcd}$$

$\mathfrak{h} = \{\operatorname{Im} z > 0\}$

Trivialize $\Omega^1_{\mathcal{E}/\mathfrak{h}}$ via $2\pi i\, dt$. (t = coordinate on $\mathbb{C}$)

This identifies

$$\mathbb{C} \otimes_{\mathbb{Q}} H^0(X, \bar{\omega}^{\otimes k}) = H^0(X_{\mathbb{C}}, \bar{\omega}_{\mathbb{C}}^{\otimes k}) \hookrightarrow \mathcal{O}_{\mathfrak{h}}(\mathfrak{h})$$

= holomorphic functions on $\mathfrak{h}$

onto "classical" modular forms.

There are lots of correspondences between modular curves, $\ldots$ giving rise to many interesting endomorphisms of M_k, in fact large commutative subring $\mathbb{T} \subset \operatorname{End}_{\mathbb{Q}}(M_k)$. An eigenform $f \in \mathbb{C} \otimes M_k$ is simultaneous eigenvector for $\mathbb{T}$. Eigenforms give rise to Euler products with good analytic properties

Key point: for cuspidal eigenform $f \in S_k$, wish to understand its $\mathbb{T}$-eigenvalues, and these are algebraic integers in a number field $K_f \subseteq \mathbb{C}$.

$$f(z) = \sum' \underbrace{a_n(f)} e^{2\pi i n z}, \quad \underline{a_1(f) = 1}$$

$L(s,f) = \sum a_n(f) n^{-s}$
= Euler product!
with good analytic properties

↳ eigenvalue against operator $T_n \in \mathbb{T}$.

A deRham calculation (Deligne)

We work analytically: $\begin{array}{c} E \\ f\downarrow \\ Y \end{array} /\mathbb{C}$

Fix $k \geq 2$.

$$\omega = f_* \Omega^1_{E/Y} \xrightarrow{\text{edge}} \mathcal{O}_Y \otimes_{\mathbb{C}} R^1 f_* \mathbb{C} \quad \text{[Hodge to dR]}$$

$$\qquad\qquad\qquad \mathcal{O}_Y \otimes_{\mathbb{Z}} R^1 f_* \mathbb{Z}$$

so

$$\omega^{\otimes(k-2)} = \mathrm{Sym}^{k-2}\omega \longrightarrow \mathcal{O}_Y \otimes \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z}$$

Using KS: $\omega^{\otimes 2} \simeq \Omega^1_Y$, get

$$\omega^{\otimes k} \longrightarrow \Omega^1_Y \otimes \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z}$$

Passing to H^0, get

$$\mathbb{C}\otimes M_k = H^0(X_{\mathbb{C}}, \overline{\omega}^{\otimes k}) \to H^0(Y_{\mathbb{C}}, \omega^{\otimes k})$$

$$\to H^0(Y_{\mathbb{C}}, \Omega^1_Y \otimes_{\mathbb{Z}} \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z})$$

$$\xrightarrow{\text{edge}} H^1(Y_{\mathbb{C}}, \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z})$$

This carries $\mathbb{C}\otimes S_k \subseteq \mathbb{C}\otimes M_k$ into

$\tilde{H}^1(Y_{\mathbb{C}}, \mathbb{C}\otimes_{\mathbb{Z}} \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z})$

$:= \mathrm{image}\,(H^1_c(Y_{\mathbb{C}}, \mathbb{C}\otimes \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z})$

$\to H^1(Y_{\mathbb{C}}, \mathbb{C}\otimes \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z}))$

Can define $\mathbb{T}$-action <u>directly</u> on

$\tilde{H}^1(Y_{\mathbb{C}}, \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z})$, respects

$$(\mathbb{C}\otimes S_k) \oplus (\overline{\mathbb{C}\otimes S_k}) \xrightarrow{\sim} \mathbb{C}\otimes \tilde{H}^1(Y_{\mathbb{C}}, \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z})$$

Eichler-Shimura

(so $\mathbb{T}$ is $\mathbb{Z}$-<u>finite</u>!)

For cuspidal eigenform f, $\mathbb{C}f \oplus \mathbb{C}\overline{f}$ is motivically "cut out" by knowledge of Hecke eigenvalues. (almost...), so try RHS

<u>Better:</u> use comparison isomorphism

$$\overline{\mathbb{Q}}_p \otimes_{\mathbb{Z}} \widetilde{H}^1(Y_{\mathbb{C}}, \mathrm{Sym}^{k-2} R^1 f_* \mathbb{Z})$$

$$\simeq \widetilde{H}^1_{et}(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1 f_* \overline{\mathbb{Q}}_p)$$

Using T-action that is $\overline{\mathbb{Q}}_p[G_{\mathbb{Q}}]$-linear on RHS (all correspondences defined over $\mathbb{Q}$) we can cut out a piece of p-adic cohomology $V_{f,\lambda}$ that is <u>2-dimensional</u> over $K_{f,\lambda}$ ($K_f \subseteq \mathbb{C}$ field of T-eigenvalues of f)

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow GL(V_{f,\lambda}) \simeq GL_2(K_{f,\lambda})$$

is unramified at all $\ell \nmid Np$,

$$\mathrm{trace}(\rho_{f,\lambda}(\mathrm{Frob}_\ell^{-1})) = \underset{T_\ell\text{-eigenvalue.}}{\overset{\shortparallel}{a_\ell(f)}} \in K_f \subseteq K_{f,\lambda}$$

$\{\rho_{f,\lambda}\}_\lambda$ is a "compatible family."

Using congruences, Deligne-Serre
(with wt ≥ 2)
made such $\rho_{f,\lambda}$ for weight 1;
these are Artin representations
(not so for weight > 1).

Such $\rho_{f,\lambda}$ are irreducible over $\overline{\mathbb{Q}}_p$,
and define the continuous

$$\overline{\rho}_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$$

to be its reduction (lands in GL_2(finite)),

Fact: $\det \overline{\rho}_{f,\lambda}$(complex conjugation) $= -1$
"$\overline{\rho}_{f,\lambda}$ is odd".

Def. Say $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ (resp. $\overline{\rho} : G_{\mathbb{Q}} \to GL_2$
is modular if arises as some
$\rho_{f,\lambda}$ (resp. $\overline{\rho}_{f,\lambda}$).

*Ex: $E_{/\mathbb{Q}}$ is modular in "geometric sense"
if and only if some (or all) $V_p E$ is modul

Serre's conjecture:

Any continuous, irreducible, odd $\rho: G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is modular.

**Ex: $G_{\mathbb{Q}} \to GL_2$! by L-T.

Strong Artin conjecture for $GL_2(\mathbb{Q})$:

Any continuous irreducible Artin representation

$$\rho: G_{\mathbb{Q}} \to GL_2(\mathbb{C}) \simeq GL_2(\overline{\mathbb{Q}}_p)$$

is $\rho_{f,\lambda}$ for some f of weight 1.

This implies $L(s,\rho) = L(s,f)$
= good analytic properties (e.g., entire!)

Which continuous $\rho: G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ are modular? (Fontaine-Mazur conj...)

Wiles program: Given modularity of $\overline{\rho}$, develop methods to show "all" deformations are modular.

There are other p-adic methods (not using algebraic geometry) to build p-adic representations "associated" to (Hilbert) modular forms, in sense

Frobenius trace = Hecke eigenvalue.

Advantage of algebro-geometric construction

This provides information on "p-adic Hodge theory" properties of

$$\rho_{f,\lambda}|_{D_p} : D_p = \left\{\begin{matrix}\text{decomposition} \\ \text{group at } p\end{matrix}\right\} \to GL_2(\overline{\mathbb{Q}}_p).$$

Strategy for Serre's conjecture:

Given $\bar{\rho}: G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$, say, find "nice" lift to $\rho: G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)$ fitting into "compatible family" — maybe another residue characteristic ($< p$?) is easier.

Ex: Does $\bar{\rho} = E[p]$? If so, use $\{V_\ell E\}_\ell$!

Ex: The 3-5 trick. Wiles' method originally required $\bar{\rho}$ to be irred.

Say $E_{/\mathbb{Q}}$ has $E[3]$ reducible as $G_{\mathbb{Q}}$-module. Let $\bar{\rho} = E[5]$.

$X_{\bar{\rho}}$ = modular curve classifying $(E', E'[5] \simeq \bar{\rho})$.

= genus 0 curve with $\mathbb{Q}$-point

$= \mathbb{P}^1_{\mathbb{Q}}$.

Can use Hilbert irreducibility to "often" find $x \in X_\rho(\mathbb{Q})$ s.t. $E_x[3]$ is irreducible.

E_x modular $\Rightarrow E_x[5]$ modular

$\Rightarrow \bar{\rho}$ modular,

so can try Wiles program at 5 rather than 3 for E.

This case is geometrically trivial. Taylor et al. has vastly extended scope of this idea using geometry of higher-dimensional Hilbert modular varieties, and "weak approx." for global pts.

Some relevant geometric techniques

- étale cohomology to construct/study Galois rep
- deformation theory for local/global Galois representations; use Galois cohomology (eg, Tate duality ...) to control deformation ring.
- p-divisible groups/group schemes, p-adic Hodge theory is used in study of "local at p" aspects of p-adic representations (viewed on $D_p \subseteq G_{\mathbb{Q}}$).
- rigid geometry allows to build some modular forms by p-adic analytic methods + rigid GAGA
- geometry of modular varieties: this is the key to putting representations into compatible families, so allows to sometimes switch to a better residue ch
- connectivity properties of Spec(A)−{m} allows to "move" on deformation ring.

§2. Deformation theory and Hecke rings

Let k = finite field.

$$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_n(k) \quad \text{absolutely irreducible.}$$

This is unramified outside finite set of places $S \ni \{\infty\}$, so can view as

$$\bar{\rho} : G_{\mathbb{Q},S} = \pi_1^{et}(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{S}]) \longrightarrow GL_n(k).$$

Key: $G_{\mathbb{Q},S}$ has good finiteness properties for its Galois cohomology, whereas $G_{\mathbb{Q}}$ does not.

$$\mathcal{C}_k = \left\{ \begin{array}{l} \text{complete local noetherian} \\ \text{rings with residue field } k \end{array} \right\}$$

$$F_{\bar{\rho}} : \mathcal{C}_k \longrightarrow \underline{Set}$$

$$\underset{(R,m)}{R} \longmapsto \left\{ \begin{array}{ccc} G_{\mathbb{Q},S} & \xrightarrow[\text{cont}]{\rho} & GL_n(R) \\ & \searrow_{\bar{\rho}} & \downarrow \\ & & GL_n(k) \end{array} \right\} \Big/ \begin{array}{c}\text{conj. by} \\ 1+M_n(m)\end{array}$$

This "classifies" deformations with no "extra ramification.

Using absolute irreducibility, Mazur checked Schlessinger's criterion.

Thm $\exists$ universal deformation (depends on S)

$$\bar{\rho}^{univ}: G_{\mathbb{Q},S} \to GL_n(R_{\bar{\rho}}),$$

tangent space to deformation functor is $H^1(G_{\mathbb{Q},S}, \mathrm{End}\,\bar{\rho})$.

In practice, want to impose "local conditions" at places in S too, so as to cut down size of $R_{\bar{\rho}}$.

Ex: $E/\mathbb{Q}$. For T_pE, encode local properties of E as representation-theoretic conditions, study all deformations of $E[p]$ with such properties. [is there a modular one?]

Say $E/\mathbb{Q}_p$ gets good reduction over some finite Galois extension $K/\mathbb{Q}_p$. How to encode in terms of ρ's?

By Grothendieck/Raynaud:

$E/\mathbb{Q}_p$ becomes good reduction $/K$ $\iff$ $\forall\, n \geq 1$, $E[p^n]|_K$ extends to finite flat group scheme over $\mathcal{O}_K$

Say deformation $\rho : G_{\mathbb{Q}} \to GL_2(R)$ of $\bar{\rho} = E[p]$

is <u>K-flat</u> if $\forall\, n \geq 1$,

$$\rho \bmod m_R^{n}: G_{\mathbb{Q}} \to \underbrace{GL_2(R/m^n)}_{\underline{\text{finite}}\ G_{\mathbb{Q}}\text{-module}}$$

has K-fiber that extends to finite flat gp scheme $\mathcal{G}_n$ over $\mathcal{O}_K$.

<u>Big problem:</u> if $e(K/\mathbb{Q}_p) \geq p-1$, such $\mathcal{G}_n$ are generally <u>not</u> unique. To make this a workable notion, need to impose stronger conditions on $\mathcal{G}_n$'s.

[Also need to track descent data – <u>ignore</u> it!]

What kinds of "local properties" $\mathbb{P}$ can we impose on $\rho|_{D_\ell}$'s and still retain representability? ($\mathbb{P}$ = property of $G_{\mathbb{Q}}$-modules)

Ramakrishna's criterion: If $\mathbb{P}$ stable under $\oplus$, subobject, quotient, then defines representable subfunctor.

Ex: This applies to "K-flat", but non-uniqueness makes it hard to control size of tangent space, etc.

For carefully-chosen collection $\mathcal{D} = \{\mathcal{D}_\ell\}$ of "local conditions" at all places, get universal deformation $\bar{\rho}_{\mathcal{D}}^{univ} : G_{\mathbb{Q}} \to GL_2(R_{\mathcal{D}})$ and get $\mathbb{T}_{\mathcal{D}}$ = completed localization of Hecke ring (= finite flat / $\mathbb{Z}_p$) and $\rho_{\mathcal{D}}^{mod} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}_{\mathcal{D}})$ such that

- for any $\mathbb{T}_{\mathcal{D}} \to \bar{\mathbb{Q}}_p$, pushforward of $\rho_{\mathcal{D}}^{mod}$ is some $\rho_{f,\lambda}$, and $\underline{\mathbb{T}_{\mathcal{D}} \neq 0}$!

- $R_{\mathcal{D}} \to \mathbb{T}_{\mathcal{D}}$ via universality is a surjection. If an isomorphism, every "type $\mathcal{D}$" deformation of $\bar{\rho}$ to a p-adic integer ring is modular

[A priori, NOT clear if $R_{\mathcal{D}}$ is $\mathbb{Z}_p$-finite OR $\mathbb{Z}_p$-flat

Study of $R_{\mathcal{D}} \to \mathbb{T}_{\mathcal{D}}$ rests on a lot of number theory, but geometry enters forcefully in study of local deformation problem $\mathcal{D}_p$ imposed at p.

Key pt in "p aspect" for modularity of elliptic curves: must find $\mathcal{D}_p$ so that

1. T_pE satisfies $\mathcal{D}_p$?
2. $R_{\mathcal{D}_p} \twoheadleftarrow \mathbb{Z}_p[[t]]$ — i.e., $\dim H^1_{\mathcal{D}_p}(\mathbb{Q}_p, \mathrm{End}(\bar{\rho})) \leq$

$\ast\ast$ 3. $\mathbb{T}_{\mathcal{D}} \neq 0$

Main problem post-Wiles for case of elliptic curves: allow for nasty reduction type of $E_{/\mathbb{Q}_p}$; e.g., E acquires good reduction only over a wildly ramified extension $K/\mathbb{Q}_p$ (i.e., $p \mid e(K/\mathbb{Q}_p)$).

Typically, $\bar{\rho}|_K$ admits many integral models $\mathcal{G}/\mathcal{O}_K$, so for K-flat deformation problem the tangent space is too big.

[To actually calculate, one uses semilinear algebra objects of Dieudonné sort that "classify" finite group schemes; work of Fontaine, Breuil ...

Possible refinement: fix <u>one</u> model $\mathcal{G}_0$ for $\bar{\rho}|_K$, and for deformation $\rho: G_{\mathbb{Q}} \longrightarrow GL_2$ (R = artin) require $\rho|_K$ admits integral model $\mathcal{G}$ such that for <u>some</u> filtration of ρ with successive quotient $\bar{\rho}$ get induced filtration on $\mathcal{G}$ (by sch. closure) with successive quotient $\mathcal{G}_0$.

<u>Fact</u>: This uniquely determines such a $\mathcal{G}$, but may depend on choice of filtration of ρ.

Is there a criterion that makes such a property hold for all such filtrations of ρ if for on

There is a criterion in terms of vanishing of an $\mathrm{Ext}^1_{\mathbb{G}_p}$-map that suffices. In specific situations for various $\bar{\rho}|_{\mathbb{G}_p}$ it can be checked, gives 1-dim tangent sp

Big problem #2:

Which $\mathcal{G}_0$ to choose?!?

- for deformation $T_p E$, should use $\mathcal{E}[p]$ for $\mathcal{E} =$ Néron (E/K).

But are there any modular deformations of "type $\mathcal{E}[p]$"?

Not hard to check $\bar{\rho} = E[p]$ has modular deformations (if it's modular!) of type $\mathcal{G}_0$ for some $\mathcal{G}_0$, but can we get $\mathcal{G}_0 = \mathcal{E}[p]$?

In cases relevant to modularity of elliptic curves, can describe $K|\mathbb{Q}_p$, $\bar{\rho}|_{\mathbb{Q}_p}$ rather explicitly, so find there are just a few $\mathcal{G}_0$'s.

Let $R_{\mathcal{G}_0}$ be corresponding deformation ring (for $G_{\mathbb{Q}_p}$-modules)

Goal: Want only one of these to have points with values in p-adic integer ring.

Idea: Can we perhaps show $p=0$ in $R_{\mathcal{G}_0}$ in all but one case?

Main crutch: for exactly one $\mathcal{G}_0$ on list have Frob: $\bar{\mathcal{G}}_0 \to \bar{\mathcal{G}}_0^{(p)}$ is NONZERO ($\bar{\mathcal{G}}_0 = \mathcal{G}_0 \bmod m_{\mathcal{O}_K}$).

Thus, wish to show $p = 0$ in $R_{\mathfrak{g}_0}$ if Frob $= 0$ on $\overline{\mathfrak{g}}_0$.

Step 1 First compute equicharacteristic deformation ring $R_{\mathfrak{g}_0}/(p) \simeq \mathbb{F}_p[[T]]$.

Just requires:

i) $\dim(\underbrace{\text{tangent space}}_{\text{some concrete } H^1}) \leq 1$

ii) Make one ~~[illegible]~~ deformation to $\mathbb{F}_p[t]/(t^n)$ that is nontrivial mod t^2 $\forall n \geq 2$

Step 2 By Step 1, have written down explicit univ. deformation. By inspection, Frob $= 0$ on closed fiber for these deformations, hence for all equichar. d

Step 3 For general deformation, use Step 2 to show p kills deformation mod p^2, so $pR \subseteq p^2 R$, so $pR = 0$ ∎