The Gromov–Witten Theory of a Point and KdV

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Outline

• Givental’s geometric perspective on GW theory
  (symplectic linear algebra)

• The Sato/Segal–Wilson approach to KdV
  (via the geometry of a semi-infinite Grassmannian)

• A connection between the two pictures
Highlights

• A clean statement of Kontsevich–Witten
• A clean statement of Virasoro
• A more satisfactory understanding:
  Givental’s Lagrangian cone ↔ the Baker function
  the dilaton shift ↔ circle-equivariant GW theory
The descendant potential for a point

\[ \mathcal{D}(t_0, t_1, \ldots) = \exp \left( \sum_{g \geq 0} \sum_{n \geq 0} \frac{\lambda^{2g-2}}{n!} \langle t(\psi_1), \ldots, t(\psi_n) \rangle_{g,n} \right) \]

where

\[ \langle t(\psi_1), \ldots, t(\psi_n) \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} t(\psi_1) \cdots t(\psi_n) \]

\[ t(\psi) = t_0 + t_1 \psi + t_2 \psi^2 + \ldots \]

\[ \in \mathbb{C}[\psi] \]

\[ \psi_i = c_1(L_i) \]
A symplectic vector space

Consider

\[ \mathcal{H} = \mathbb{C}[z, z^{-1}] \quad \text{and} \quad \Omega(f, g) = \text{Res}_{z=0} f(-z)g(z) \, dz \]

with the Lagrangian polarization

\[ \mathcal{H}_+ = \mathbb{C}[z] \quad \text{and} \quad \mathcal{H}_- = z^{-1}\mathbb{C}[z^{-1}] \]

Darboux co-ordinates:

\[ \ldots + \frac{p_2}{(-z)^3} + \frac{p_1}{(-z)^2} + \frac{p_0}{(-z)} + q_0 + q_1z + q_2z^2 + \ldots \]
Givental’s quantization formalism

Geometric quantization:

\[ A \in \mathfrak{sp}(\mathcal{H}) \sim h_A(f) = \frac{1}{2} \Omega(Af, f) \]

Quantize:

\[
\widehat{q_i q_j} = \frac{q_i q_j}{\lambda^2} \quad \widehat{p_i q_j} = q_j \frac{\partial}{\partial q_i} \quad \widehat{p_i p_j} = \lambda^2 \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j}
\]

Quantized operators act on functions of

\[ q_0, q_1, q_2, \ldots \]
The dilaton shift

Quantized operators act on functions of

\[ q(z) = q_0 + q_1 z + q_2 z^2 + \ldots \]

The descendant potential \( D(t_0, t_1, \ldots) \) is a function of

\[ t(z) = t_0 + t_1 z + t_2 z^2 + \ldots \]

Set

\[ q(z) = t(z) - z \]

This is called the dilaton shift
Example: Virasoro constraints

The differential operators

\[ L_{-1} = -\frac{\partial}{\partial t_0} + \sum_{k \geq 1} t_k \frac{\partial}{\partial t_{k-1}} + \frac{t_0^2}{2\lambda^2} \]

\[ L_0 = -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{k \geq 0} \left( k + \frac{1}{2} \right) t_k \frac{\partial}{\partial t_k} + \text{constant} \]

\[ L_n = -\frac{\Gamma \left( n + \frac{5}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \frac{\partial}{\partial t_{n+1}} + \sum_{k \geq 0} \frac{\Gamma \left( k + n + \frac{3}{2} \right)}{\Gamma \left( k + \frac{1}{2} \right)} t_k \frac{\partial}{\partial t_{k+n}} + \frac{\lambda^2}{2} \sum_{i=0}^{i=n-1} (-1)^{n+i} \frac{\Gamma \left( i + \frac{3}{2} \right)}{\Gamma \left( i - n + \frac{1}{2} \right)} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_{n-1-i}} \]

\[ L_n = \hat{L}_n + \delta_{n,0} \cdot \text{constant} \]

where

\[ l_n = z^{-1/2} \left( \frac{d}{dz} z \right)^{n+1} z^{-1/2} \]
KdV: Lax pair approach

The KdV hierarchy is an infinite family of commuting flows on the space of functions $u(\theta)$

Put

$$L = D^2 + u(\theta)$$

$$D = \frac{d}{d\theta}$$

$$\frac{\partial L}{\partial \theta_r} = [L, (L^{r/2})_+]$$

These flows commute $\leadsto u(\theta; \theta_1, \theta_2, \theta_3, \ldots)$
\( u(0, \theta, \phi, \ldots) \) only depends on odd times \( \theta \) or \( \phi \).

\[
\begin{align*}
\frac{\partial}{\partial \theta} u &= D_u + \frac{1}{6} D_u^3 \\
\frac{\partial}{\partial \phi} u &= 0
\end{align*}
\]
KdV: $\tau$-functions

Set

$$u(\theta_1, \theta_3, \ldots) = 2D^2 \log \tau(\theta_1, \theta_3, \ldots)$$

Witten–Kontsevich: If we set

$$t_j = \begin{cases} (2j + 1)!! \theta_{2j+1} & j \neq 1 \\ 3\theta_3 + \text{shift} & j = 1 \end{cases}$$

then $D(t_0, t_1, \ldots)$ becomes a $\tau$-function for the KdV hierarchy
Sato/Segal–Wilson approach I

Idea: commuting flows on a Grassmannian

Set

\[ V = \mathbb{C}[y, y^{-1}] \]
\[ V_+ = \mathbb{C}[y] \]
\[ \text{Gr}(V) = \{ \text{subspaces of } V \text{ commensurable with } V_+ \} \]

\[ g(\theta_1, \theta_2, \ldots) = \exp \left( \sum_{j>0} \theta_j y^j \right) \in T \]

\( T \) acts by multiplication on \( \text{Gr}(V) \)
Sato/Segal–Wilson approach II

\[
\sigma(W) = \det \pi_+ \quad \text{where} \quad \pi_+ : W \to V_+ \text{ is the projection}
\]

\[\sigma\] is not equivariant. Define

\[
\tau_W(\theta_1, \theta_2, \ldots) = \frac{\sigma(g^{-1}W)}{g^{-1}\sigma(W)}
\]

KdV condition: \[y^2 W \subset W\]
Claim: \( u \) satisfies differential equations

\[
\begin{align*}
\left\{ f(y_1, y_2, \ldots) \mid Lf = g \right\} &= M \\
\text{Approach:} & \quad \text{Recall} \quad L = D_x^n \\
\text{Idea:} & \quad \text{Connection to Previous}
\end{align*}
\]
\[ \{ z_i, \overline{z_i}, \ldots \} \quad \text{and} \quad \{ \frac{\partial z_i}{\partial \bar{z}_i}, \ldots \} \]

Bases for \( \Lambda \):

\[ \Lambda = \{ \theta, \theta^2, \ldots \} \]

\[ T \]

\[ M \]

\[ M_{\lambda}(\theta) \]

\[ g(\theta) \]

\[ \varphi(y) \]

\[ \varphi(y; \theta, \theta^2, \ldots) \in M \]

The Baker Function
The Baker function

$W$ determines $\tau_W$. In fact, $\tau_W$ determines $W$ too.

Intermediate object: the Baker function

$$\psi_W(\theta_1, \theta_3, \ldots; y) = g(\theta) \left( g^{-1}(\theta) W \cap (1 + V_\cdot) \right)$$

Derivatives of $\psi_W$ span $W$

Sato:

$$\psi_W(\theta_1, \theta_3, \ldots; y) = \exp \left( \sum_{j \geq 0} \theta_{2j+1} y^{2j+1} \right) \frac{\tau_W \left( \theta_1 - \frac{1}{y}, \theta_3 - \frac{1}{3y^3}, \theta_5 - \frac{1}{5y^5}, \ldots \right)}{\tau_W(\theta_1, \theta_3, \theta_5, \ldots)}$$
Another symplectic space

The Abelian Lie algebra

$$\sum_{j>0} \tilde{q}_j y^j - \sum_{k>0} \frac{\tilde{p}_k}{k} y^{-k}$$

acts on $Gr(V)$, so a central extension acts on $\text{Det}^\vee$

$$\tilde{\Omega}(f, g) = \text{Res}_{y=0} g \, df$$

$\tilde{q}_j$ and $\tilde{p}_k$ give Darboux co-ordinates

$\tilde{q}_j \leftrightarrow \theta_j$ (the time for the $j$th KdV flow)

Set $y = \sqrt{2x}$
A connection between the pictures

\[ \text{Res}_{z=0} f(-z)g(z) \, dz \]

\[ \begin{array}{ccc}
C[z, z^{-1}] & \xrightarrow{T} & C[x^{1/2}, x^{-1/2}] \\
\quad z^k & \mapsto & \frac{\sqrt{\pi}}{2 \Gamma(k + \frac{3}{2})} x^{k+\frac{1}{2}} \\
2 \text{Res}_{x=0} g \, df
\end{array} \]

T is a symplectic embedding.

Also

\[ T^*(q_{2j+1}) = \frac{q_j}{(2j + 1)!!} \]

Kontsevich–Witten: there is a subspace \( W \in Gr \) such that the pullback of \( \tau_W(q_1, q_3, q_5, \ldots) \) via T is \( D(t_0, t_1, \ldots) \).
$T_{-1} \left( x^{k+\frac{1}{2}} \right) = \sqrt[\pi]{\frac{2}{\pi}} \int_0^\infty t^{k+\frac{1}{2}} e^{-t} \, dt$

$[0, \infty]$ should be replaced by an appropriate contour $C$

$\frac{1}{z^{3/2}} \sqrt{\frac{2}{\pi}} \int_0 \infty f(x) e^{-x/z} \, dx$

$= \frac{1}{z^{3/2}} \sqrt{\frac{2}{\pi}} \mathcal{L}(f) \bigg|_{z \to 1/z}$
Virasoro constraints again

T intertwines the Virasoro operators $l_n$ with the standard ones:

$$z^{-1/2} \left( z \frac{d}{dz} z \right)^{n+1} z^{-1/2} T^{-1}(f) = T^{-1} \left[ x^{n+1} \frac{d}{dx} f \right]$$

The Virasoro conjecture becomes

$$\left( x^{n+1} \frac{d}{dx} \right) \mathcal{D} = 0 \quad n \geq -1$$
\[ z^{-\frac{1}{2n}} (z \frac{d}{dz})^{n+1} z^{-\frac{1}{2n}} \sqrt{\frac{2}{\pi}} \frac{1}{2^{3n}} \int_0^\infty f(x) e^{-\frac{x}{2}} dx \]

\[ = \sqrt{\frac{2}{\pi}} z^{-\frac{1}{2n}} (z^2 \frac{d}{dz})^{n+1} \frac{1}{2} \int_0^\infty f(x) e^{-\frac{x}{2}} dx \]

\[ = \sqrt{\frac{2}{\pi}} z^{-\frac{3}{2n}} (z^2 \frac{d}{dz})^{n+1} \int_0^\infty \frac{df}{dx} e^{-\frac{x}{2}} dx \]

\[ = \sqrt{\frac{2}{\pi}} z^{-\frac{3}{2n}} \int_0^\infty x^{n+1} \frac{df}{dx} e^{-\frac{x}{2}} dx \]
What is $x$?

A good interpretation of $z$:

- $S^1$-equivariant Floer homology of loop space

- $H^*_{S^1}(pt) = \mathbb{C}[z]$

But what is $x$:

- geometrically?

- in terms of moduli space?
The Baker function

Kontsevich–Witten says $D = \tau_W$ for some subspace $W$

What is the corresponding Baker function?

$$\psi_W(\tilde{q}_1, \tilde{q}_3, \ldots; x) = \frac{\exp\left(\sum_{j \geq 0} \hat{q}_{2j+1} \sqrt{2x}^{2j+1}\right) \exp\left(-\sum_{k \geq 0} \frac{1}{(2k+1)\sqrt{2x}^{2k+1}} \hat{p}_{2k+1}\right) \tau_W(\tilde{q}_1, \tilde{q}_3, \ldots)}{\tau_W(\tilde{q}_1, \tilde{q}_3, \ldots)}$$

Take this across the Laplace transform:

$$\log \psi_W(q_0, q_1, \ldots; z) = \frac{1}{\lambda} \left( q(z) + \sum_{n \geq 0} \frac{1}{n!} \left\langle t(\psi_1), \ldots, t(\psi_n), \frac{1}{-z - \psi_{n+1}} \right\rangle_{0,n+1} \right) + \ldots$$
Givental’s Lagrangian cone

Genus-zero Gromov–Witten invariants of a point are encoded by the Lagrangian cone

\[ L = \left\{ \sum_{j \geq 0} q_j z^j + \sum_{k \geq 0} \frac{\partial F^0(q_0, q_1, \ldots)}{\partial q_k} \frac{1}{(-z)^{k+1}} \right\} \subset \mathbb{C}[z, z^{-1}] \]

This is the image of the semi-classical limit of \( \log \psi_W \)
Circle-equivariant GW theory

Our formula for $\log \psi_W$ is very suggestive:

$$L \leftrightarrow S^1\text{-equivariant GW theory of } X \times \mathbb{C}$$

More precisely...

This gives a conceptual explanation of the dilaton shift.
by localization (S,ū, π₁)

where we define the push-forward

\[ \text{Form} \]

\[ \exists \frac{p_i}{\text{lim}} \]

no bubbling or markings at \( \infty \)

\[ X \leftarrow \text{Evaluation map:} \]

\( u/\text{EVEN} \)

Consider the graph space \((X \times \mathbb{P}^1, g_{\mathbb{P}^1})\)
**FIXED LOCUS**

\[ n=0 \quad d=0 \]

\[ \cong X \]

\[ \cong X \]

**NORMAL BUNDLE**

empty

\[ \frac{1}{-2} \]

\[ \frac{1}{(-2)^2-X_{0,n+1,d}} \]

**CONTRIBUTION**

assemble to give the Lagrangian cone \( \mathcal{L} \)

\[ \sum_{i=1}^{p} \text{ev}_{n+1} \ast (H(y_1) \ast \ldots \ast H(y_d)) \ast \frac{1}{-2} \]
Thank you for your time