The Gromov–Witten Theory of a Point and KdV

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Outline

- Givental's geometric perspective on GW theory

  (symplectic linear algebra)

- The Sato/Segal–Wilson approach to KdV

  (via the geometry of a semi-infinite Grassmannian)

- A connection between the two pictures

Highlights

- A clean statement of Kontsevich–Witten
- A clean statement of Virasoro
- A more satisfactory understanding:

  Givental's Lagrangian cone ↔ the Baker function

  the dilaton shift ↔ circle-equivariant GW theory

The descendant potential for a point

$$\mathcal{D}(t_0, t_1, \ldots) = \exp\left(\sum_{g\geq 0}\sum_{n\geq 0}\frac{\lambda^{2g-2}}{n!}\left\langle\!\!\left\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n)\right\rangle\!\!\right\rangle_{g,n}\right)$$

where

$$\langle\!\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n)\rangle\!\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \mathbf{t}(\psi_1) \cdot \cdots \cdot \mathbf{t}(\psi_n)$$

$$\begin{aligned}\mathbf{t}(\psi) &= t_0 + t_1\psi + t_2\psi^2 + \ldots \\ &\in \mathbb{C}[\psi]\end{aligned}$$

$$\psi_i = c_1(L_i)$$

A symplectic vector space

Consider

$$\mathcal{H} = \mathbb{C}[z, z^{-1}]] \qquad \Omega(f, g) = \operatorname{Res}_{z=0} f(-z)g(z)\, dz$$

with the Lagrangian polarization

$$\mathcal{H}_+ = \mathbb{C}[z] \qquad \mathcal{H}_- = z^{-1}\mathbb{C}[[z^{-1}]]$$

Darboux co-ordinates:

$$\ldots + \frac{p_2}{(-z)^3} + \frac{p_1}{(-z)^2} + \frac{p_0}{(-z)} + q_0 + q_1 z + q_2 z^2 + \ldots$$

Givental's quantization formalism

Geometric quantization:

$$A \in \mathfrak{sp}(\mathcal{H}) \quad \rightsquigarrow \quad h_A(f) = \frac{1}{2}\Omega(Af, f)$$

Quantize:

$$\widehat{q_i q_j} = \frac{q_i q_j}{\lambda^2} \qquad \widehat{p_i q_j} = q_j \frac{\partial}{\partial q_i} \qquad \widehat{p_i p_j} = \lambda^2 \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j}$$

Quantized operators act on functions of

$$q_0, q_1, q_2, \ldots$$

The dilaton shift

Quantized operators act on functions of

$$\mathbf{q}(z) = q_0 + q_1 z + q_2 z^2 + \ldots$$

The descendant potential $\mathcal{D}(t_0, t_1, \ldots)$ is a function of

$$\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \ldots$$

Set

$$\mathbf{q}(z) = \mathbf{t}(z) - z$$

This is called the dilaton shift

$q_0 = t_0$

$q_1 = t_1 - 1$

$q_2 = t_2$

$\vdots$

Example: Virasoro constraints

The differential operators

$$
\begin{aligned}
L_{-1} &= -\frac{\partial}{\partial t_0} + \sum_{k \geq 1} t_k \frac{\partial}{\partial t_{k-1}} + \frac{t_0^2}{2\lambda^2} \\
L_0 &= -\frac{3}{2}\frac{\partial}{\partial t_1} + \sum_{k \geq 0} \left(k + \frac{1}{2}\right) t_k \frac{\partial}{\partial t_k} + \text{constant} \\
L_n &= -\frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \frac{\partial}{\partial t_{n+1}} + \sum_{k \geq 0} \frac{\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} t_k \frac{\partial}{\partial t_{k+n}} + \frac{\lambda^2}{2} \sum_{i=0}^{i=n-1} (-1)^{n+i} \frac{\Gamma\left(i+\frac{3}{2}\right)}{\Gamma\left(i-n+\frac{1}{2}\right)} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_{n-1-i}}
\end{aligned}
$$

annihilate the descendant potential $\mathcal{D}(t_0, t_1, \ldots)$

$L_n = \widehat{l_n} + \delta_{n,0} \cdot \text{constant}$ where $l_n = z^{-1/2} \left(z \frac{d}{dz} z \right)^{n+1} z^{-1/2}$

KdV: Lax pair approach

The KdV hierarchy is an infinite family of commuting flows on the space of functions $u(\theta)$

Put

$$L = D^2 + u(\theta) \qquad\qquad D = \frac{d}{d\theta}$$

$$\frac{\partial L}{\partial \theta_r} = \left[L, \left(L^{r/2}\right)_+\right]$$

These flows commute $\quad \rightsquigarrow \quad u(\theta; \theta_1, \theta_2, \theta_3, \ldots)$

For r even:

$$\frac{\partial u}{\partial \theta_r} = 0$$

Also:

$$\frac{\partial u}{\partial \theta_1} = Du \qquad \theta \leftrightarrow \theta_1$$

$$\frac{\partial u}{\partial \theta_3} = u\,Du + \frac{1}{6} D^3 u$$

$$\vdots$$

$u(\theta_1, \theta_3, \theta_5, \ldots)$ only depends on odd times

KdV: τ-functions

Set

$$u(\theta_1, \theta_3, \ldots) = 2D^2 \log \tau(\theta_1, \theta_3, \ldots)$$

Witten–Kontsevich: If we set

$$t_j = \begin{cases} (2j+1)!!\, \theta_{2j+1} & j \neq 1 \\ 3\theta_3 + \text{shift} & j = 1 \end{cases}$$

then $\mathcal{D}(t_0, t_1, \ldots)$ becomes a τ-function for the KdV hierarchy

Sato/Segal–Wilson approach I

Idea: commuting flows on a Grassmannian

Set

$$V = \mathbb{C}[y, y^{-1}]]$$

$$V_+ = \mathbb{C}[y]$$

$$\mathrm{Gr}(V) = \{\text{subspaces of } V \text{ commensurable with } V_+\}$$

$$g(\theta_1, \theta_2, \ldots) = \exp\left(\sum_{j>0} \theta_j y^j\right) \in T$$

T acts by multiplication on $\mathrm{Gr}(V)$

Sato/Segal–Wilson approach II

$$\begin{array}{c} \mathrm{Det}^\vee \\ \big\downarrow \;\; \big\uparrow \sigma \\ \mathrm{Gr}(V) \end{array}$$

$\sigma(W) = \det \pi_+$ where $\pi_+ : W \to V_+$ is the projection

σ is not equivariant. Define

$$\tau_W(\theta_1, \theta_2, \ldots) = \frac{\sigma(g^{-1}W)}{g^{-1}\sigma(W)}$$

KdV condition: $y^2 W \subset W$

CONNECTION TO PREVIOUS APPROACH :

<u>Idea</u> : recall $L = D^2 + u$

$$W = \{ f(y;\Theta_1,\Theta_2 \dots) \mid Lf = y^2 f \}$$

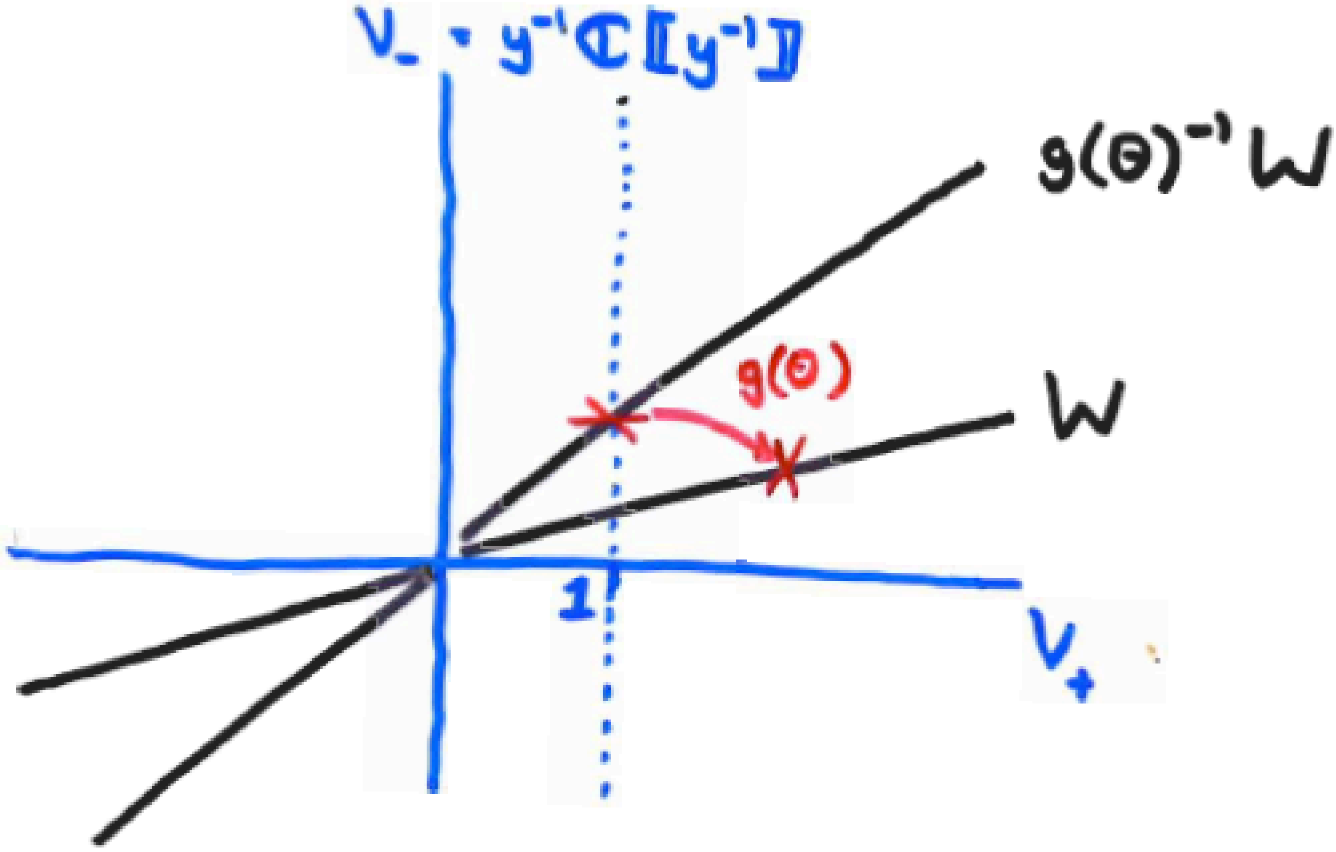
<u>Define</u> :

$$u_W(\Theta_1, \Theta_2, \dots) = 2 D^2 \log \tau_W$$

<u>Claim</u> : u_W satisfies differential equations

The Baker Function

$$\psi_W(y;\theta_1,\theta_3,\dots) \in W$$



Bases for W:

$$\left\{\frac{\partial\psi_W}{\partial\theta_1}, \frac{\partial\psi_W}{\partial\theta_3}, \dots\right\} \text{ and } \left\{D\psi_W, D^2\psi_W, \dots\right\}$$

The Baker function

W determines τ_W. In fact, τ_W determines W too.

Intermediate object: the Baker function

$$\psi_W(\theta_1, \theta_3, \ldots; y) = g(\theta)\Big(g^{-1}(\theta)W \cap (1 + V_-)\Big)$$

Derivatives of ψ_W span W

Sato:

$$\psi_W(\theta_1, \theta_3, \ldots; y) = \exp\left(\sum_{j\geq 0} \theta_{2j+1} y^{2j+1}\right) \frac{\tau_W\left(\theta_1 - \frac{1}{y}, \theta_3 - \frac{1}{3y^3}, \theta_5 - \frac{1}{5y^5}, \ldots\right)}{\tau_W(\theta_1, \theta_3, \theta_5, \ldots)}$$

Another symplectic space

The Abelian Lie algebra

$$\sum_{j>0} \tilde{q}_j y^j - \sum_{k>0} \frac{\tilde{p}_k}{k} y^{-k}$$

acts on $\mathrm{Gr}(V)$, so a central extension acts on $\mathrm{Det}^\vee$

$$\widetilde{\Omega}(f,g) = \mathrm{Res}_{y=0}\, g\, df$$

$\tilde{q}_j$ and $\tilde{p}_k$ give Darboux co-ordinates

$\tilde{q}_j \longleftrightarrow \theta_j$ (the time for the *j*th KdV flow)

Set $y = \sqrt{2x}$

A connection between the pictures

$$\mathrm{Res}_{z=0}\, f(-z)g(z)\,dz \qquad \begin{array}{ccc} \mathbb{C}[z,z^{-1}]\!] & \xrightarrow{T} & \mathbb{C}[x^{1/2},x^{-1/2}]\!] \\ z^k & \longmapsto & \sqrt{\frac{\pi}{2}}\,\frac{x^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})} \end{array} \qquad 2\,\mathrm{Res}_{x=0}\, g\,df$$

T is a symplectic embedding.

Also

$$T^\star(\tilde{q}_{2j+1}) = \frac{q_j}{(2j+1)!!}$$

Kontsevich–Witten: there is a subspace $W \in \mathrm{Gr}$ such that the pullback of $\tau_W(\tilde{q}_1, \tilde{q}_3, \tilde{q}_5, \ldots)$ via T is $\mathcal{D}(t_0, t_1, \ldots)$.

T is an inverse Laplace transform

$$T^{-1}\left(x^{k+\frac{1}{2}}\right) = z^k\sqrt{\frac{2}{\pi}}\int_0^\infty t^{k+\frac{1}{2}}\,e^{-t}\,dt$$

[0,∞] should be replaced by an appropriate contour C

$$T^{-1}(f) = \frac{1}{z^{3/2}}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,e^{-x/z}\,dx$$

$$= \frac{1}{z^{3/2}}\sqrt{\frac{2}{\pi}}\mathcal{L}(f)\Big|_{z\mapsto 1/z}$$

Virasoro constraints again

T intertwines the Virasoro operators l_n with the standard ones:

$$z^{-1/2}\left(z\frac{d}{dz}z\right)^{n+1} z^{-1/2}\, T^{-1}(f) = T^{-1}\left[x^{n+1}\frac{d}{dx}\, f\right]$$

The Virasoro conjecture becomes

$$\left(\widehat{x^{n+1}\frac{d}{dx}}\right)\mathcal{D} = 0 \qquad\qquad n \geq -1$$

$$z^{-1/2}\left(z\frac{d}{dz}z\right)^{n+1} z^{-1/2}\ \sqrt{\frac{2}{\pi}}\ \frac{1}{z^{3/2}}\int_0^\infty f(x)\, e^{-x/z}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\ z^{-3/2}\left(z^2\frac{d}{dz}\right)^{n+1}\frac{1}{z}\int_0^\infty f(x)\, e^{-x/z}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\ z^{-3/2}\left(z^2\frac{d}{dz}\right)^{n+1}\int_0^\infty \frac{df}{dx}\, e^{-x/z}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\ z^{-3/2}\int_0^\infty x^{n+1}\frac{df}{dx}\, e^{-x/z}\, dx$$

What is x?

A good interpretation of z:

- S^1-equivariant Floer homology of loop space
- $H^\bullet_{S^1}(pt) = \mathbb{C}[z]$

But what is x:

- geometrically?
- in terms of moduli space?

The Baker function

Kontsevich–Witten says $\mathcal{D} = \tau_W$ for some subspace W

What is the corresponding Baker function?

$$\psi_W(\tilde{q}_1, \tilde{q}_3, \ldots; x) = \frac{\exp\left(\sum_{j\geq 0} \hat{\tilde{q}}_{2j+1} \sqrt{2x}^{2j+1}\right) \exp\left(-\sum_{k\geq 0} \frac{1}{(2k+1)\sqrt{2x}^{2k+1}} \hat{\tilde{p}}_{2k+1}\right) \tau_W(\tilde{q}_1, \tilde{q}_3, \ldots)}{\tau_W(\tilde{q}_1, \tilde{q}_3, \ldots)}$$

Take this across the Laplace transform:

$$\log \psi_W(q_0, q_1, \ldots; z) = \frac{1}{\lambda}\left(\mathbf{q}(z) + \sum_{n\geq 0} \frac{1}{n!} \left\langle\!\!\left\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \frac{1}{-z - \psi_{n+1}} \right\rangle\!\!\right\rangle_{0,n+1}\right) + \ldots$$

Givental's Lagrangian cone

Genus-zero Gromov–Witten invariants of a point are encoded by the Lagrangian cone

$$\mathcal{L} = \left\{ \sum_{j\geq 0} q_j z^j + \sum_{k\geq 0} \frac{\partial \mathcal{F}^0(q_0, q_1, \ldots)}{\partial q_k} \frac{1}{(-z)^{k+1}} \right\} \subset \mathbb{C}[z, z^{-1}]\!]$$

This is the image of the semi-classical limit of $\log \psi_W$

Circle-equivariant GW theory

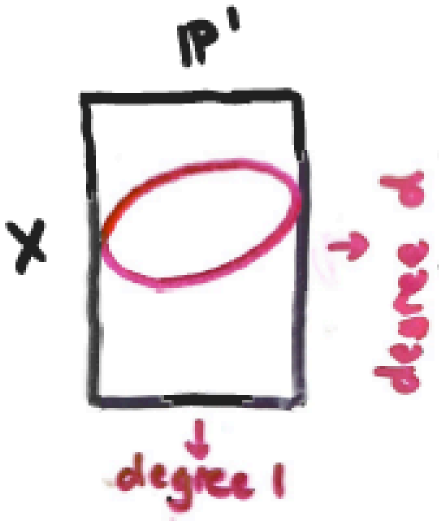
Our formula for $\log \psi_W$ is very suggestive:

$\mathcal{L} \quad \leftrightarrow \quad S^1$-equivariant GW theory of $X \times \mathbb{C}$

More precisely...

This gives a conceptual explanation of the dilaton shift.

Consider the **GRAPH SPACE** $(\mathcal{X}\times\mathbb{P}^1)_{0,n,(d,1)}$



w/ BRAVERMAN

Evaluation map:

$$ev_\infty : (\mathcal{X}\times\mathbb{P}^1)^{open}_{0,n,(d,1)} \longrightarrow X$$

no bubbling or markings at ∞

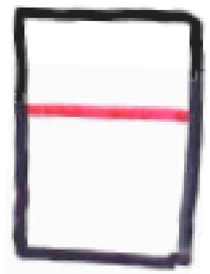
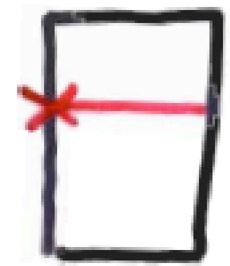
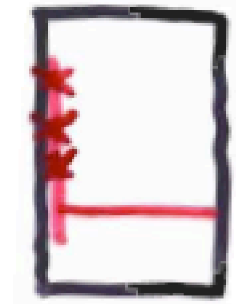
Form

$$(-z)\sum_{n,d} \frac{Q^d}{n!}\, ev_{\infty *}\left(\mathbf{t}(\psi_1)\wedge\ldots\wedge\mathbf{t}(\psi_n)\right)$$

$(-z)$: really a "twist"

$ev_{\infty *}(\ldots)$: in $H^*(X)$

where we define the push-forward by localization $(S^1 \circlearrowright \mathbb{P}^1)$

	$n=0$ $d=0$	$n=1$ $d=0$	rest
FIXED LOCUS			
	$\cong X$	$\cong X$	$\cong X_{0,n+1,d}$
NORMAL BUNDLE	empty	$\frac{1}{-z}$	$\frac{1}{(-z)(-z-\psi_{n+1})}$
CONTRIBUTION	$-z$	$t(z)$	

$$\sum \frac{Q^d}{n!} ev_{n+1\,*}\left(t(\psi_1)\wedge\cdots\wedge t(\psi_n)\wedge\frac{1}{-z-\psi_{n+1}}\right)$$

assemble to give the Lagrangian cone $\mathcal{L}$

Thank you for your time