Definition of Néron model

\[ k = \overline{k} \quad B \text{ nonsingular curve } / k \quad K = k(B) \]

\[ \text{basic case: } B = \text{Spec } R \quad R \text{ a D.V.R.} \]

\[ \text{Spec } k = B \text{ the "generic point"} \]

\[ A_k = \text{abelian variety } / K \quad A_k \rightarrow \text{Spec } k \]

or

a torsor under such an abelian variety

The Néron model \( N(A_k) \) of \( A_k \) is a smooth scheme \( /B \) having \( A_k \) as generic fiber and satisfying the Néron Mapping Property (N.M.P.)

\[ \forall Z \rightarrow B \text{ smooth} \]

\[ \forall \varphi_k : Z_k \rightarrow A_k \]

\[ \exists ! \varphi : Z \rightarrow N(A_k) \text{ extending } \varphi_k \]
Thm (Néron '60): \( N(A_k) \) exists.

Properties:

1. Uniqueness

2. Separation (\( N(A_k) \) is not proper in general)

3. Local \( \leftrightarrow \) Global (Hence \( B = \text{Spec} \))

4. The group/torsor structure of \( A_k \) naturally extends to \( N(A_k) \)

- Bad functorial properties

\[ A_k(k) \xrightarrow{1-1} N(A_k)(\mathbb{R}) \]
OUR SET UP

\( \mathcal{X}_K \) smooth proj. curve / \( K \)

\[ A_K = \text{Pic}^0 \mathcal{X}_K \]

The Jacobian of \( \mathcal{X}_K \)

\[ f : \mathcal{X} \longrightarrow B \]

The rel. minimal model of \( \mathcal{X}_K \) over \( B \)

\[ X := \text{the special fiber} \]

(a curve / \( K \) possibly singular non integral)

\[ \text{Our case} \quad X = \bigcup_{i=1}^{\ell} C_i \]

\( X \) stable curve

\[ \text{Pic}^0 \longrightarrow B \]

the (relative) degree 0

\text{Picard scheme} \ (\text{Pic}^0_{/B})

\[ \uparrow \text{a smooth model of } \text{Pic}^0 \mathcal{X}_K \text{ but NOT SEPARATE} \]
Special fiber of $\text{Pic}^0$ is

$$\text{Pic}^0 X = \prod_{d \in \mathbb{Z}} \frac{\text{Pic} d}{\text{Pic}^0 X}$$

$$l d l = 0$$

$$\Theta_{\mathit{d} \mathit{r}} \quad \longmapsto \quad \left\{ \Theta_{\mathit{d} \mathit{r}}(\mathbb{Z} m \cdot \mathbb{C} \cdot \mathbb{i}) \right\} / X \quad \{ \text{Twist}\} \quad X$$

$m \in \mathbb{Z}$

"$\mathit{d}$-Twisters"

$N(\text{Pic}^0 X_{/ \mathit{k}})$ is the largest separated quotient of $\text{Pic}^0 X$

Special fiber of $N(\text{Pic}^0 X_{/ \mathit{k}})$ is

$$\text{Pic}^0 X \quad \sim \quad \prod_{\delta \in \Delta_X} \text{Pic} d \delta \quad X$$

$$\Delta_X = \left\{ d \in \mathbb{Z} : \begin{array}{l} ld l = 0 \\ \{ \text{multidegrees of all} \} \\ \{ \text{twisters} \} \end{array} \right\}$$

$A$ finite group

$A$ combinatorial invariant of $X$
Notation: \[ N^0_g := N(Pic^0 \mathcal{L}_k) \]

RK: its special fiber does not depend on \( f \)

\[ \Rightarrow \]

\[ N^0_x := (N^0_g)_k \]

Similarly, \( \forall d \in \mathbb{Z}, A_k = Pic^d \mathcal{L}_k \) is a torsor under \( Pic^0 \mathcal{L}_k \) ....

\[ N^d_g := N(Pic^d \mathcal{L}_k) \quad N^d_x := (N^d_g)_k \]

Q: Do the \( N^d_x \)'s glue together over \( \overline{M}_g \)?

\( d, g \geq 3 \) fixed
Thm (05). Assume \((d-g+1, 2g-2) = 1\)

1) \exists a modular \(D-M\) stack \(\overline{\mathcal{P}_d,g}\)
with a strongly representable natural map
\[
\overline{\mathcal{P}_d,g} \to \overline{M_g}
\]
s.t. \(\forall f: X \to B\) (\(X\) regular, \(X\) stable)
there is a natural isomorphism of schemes
\[
N^d_g \cong \overline{\mathcal{P}_d,g} \times \overline{M_g} B
\]

2) \(\mathcal{P}_d,g\) is completed by a modular \(D-M\) stack \(\overline{\mathcal{P}_d,g} \to \overline{M_g}\)

strongly representable \(\overline{M_g}\)
Theorem: Assume \((d-g+1, 2g-2) = 1\).

There exists a smooth scheme

\[
P^d_{g-g} \longrightarrow \overline{M}_g^0
\]

which is modular (i.e., a fine moduli) scheme such that \(\forall f: X \longrightarrow B\) family of automorphism-free stable curves with \(X\) regular, there is a natural isomorphism

\[
N^d_g \cong P^d_{g-g} \times \overline{M}_g^0 \, B.
\]
1) (Scheme version) \( P^d_g \) is a suitable subscheme in a compactification of the Universal Picard over \( \overline{M}_g \) (P.D thesis 93)

Modularity = \( P^d_g \) parametrizes line bundles on stable curves

2) (Stackification) Abramovich-Vistoli (01) Edidin (00) Deligne-Mumford Artin-Vistoli others

3) \((d-g+1, 2g-2) = 1\) is nec. and suff. for

(3.a) Poincaré Line Bundle (M.R.)
(3.b) Neron N.M.P.
(3.c) Stackification
Further motivations

1. Comparing Completions

a) Different compactifications of Picard

\[ d = g - 1 \quad \text{Alexeev 04} \]

b) Completions of Picard with completions of remarkable subfunctors

\[ \text{e.g. Spin, Prym} \]

\[ \text{Casagrande Cornalba 04} \]

lots of mysteries!

2. Towards Brill-Noether Theory of Stable Curves
Recall: \( C \) smooth projective curve \( d > 0 \)

The \( d \)-th \textbf{Abel Map} of \( C \):

\[
\begin{align*}
C^d & \xrightarrow{\alpha_C^d} \Pi \ell^d C \\
(P_1, \ldots, P_d) & \mapsto \Theta_C \left( \sum_i P_i \right)
\end{align*}
\]

\( \alpha_C^d \) is the moduli map of a suitable line bundle on \( C^d \times C \)

Brill-Noether varieties (rough)

\[
W^0_d(C) = \text{Im} \alpha_C^d
\]

\[
W^r_d(C) = \{ L : \dim (\alpha_C^d)(L) \geq r \}
\]
What if $X$ is singular (i.e., a reducible stable curve)?

If $X$ is integral $\rightarrow$ Altman - Kleiman

Set $X \hookrightarrow \mathcal{X} \xrightarrow{f} B$

$x$, $x_k$ regular

\[
\begin{array}{ccc}
\mathcal{X}_k & \xrightarrow{\alpha_k} & \operatorname{Pic} d \mathcal{X}_k \\
\downarrow & & \downarrow \\
\text{Smooth} \left( \mathcal{F} \right) =: \mathcal{X}_B & \xrightarrow{\alpha_f} & \mathcal{N}_d \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & \mathcal{X}_B & \xrightarrow{N_d} & \mathcal{N}_d \\
\end{array}
\]

$\leftarrow$ N.M.P.

$\xrightarrow{\text{Thm 2)}}$
Thm (05) 1) $\chi^1_g$ extends to a regular map

\[ \chi^1_g : \mathcal{X} \rightarrow \overline{\mathcal{N}}_g \]

2) $\chi^1_g / \mathcal{X}$ does not depend on $g$

(i.e. $\overline{\chi}^1_x : \mathcal{X} \rightarrow \overline{\mathcal{N}}_x$)

3) $\chi^1_g$ is an IMMERSION $\iff$
the normalization of $\mathcal{X}$ at its
separating nodes contains no
rational curve. Ens. patched here.

IN PROGRESS