Admissible Covers and Stable Maps

The Hurwitz scheme $\mathcal{H}_{d,g}$ parametrizes maps:

$$f : C \to \mathbf{P}^1; \text{genus}(C) = g, \deg(f) = d$$

with

$$b = 2g - 2 + 2d$$

simple ramification points.

It comes with a morphism (a covering map):

$$\pi_{d,g} : \mathcal{H}_{d,g} \to \mathbf{P}^b - \Delta$$

of degree equal to the "Hurwitz" number $h_{d,g}$.

Theme: Admissible covers and stable maps define related compactifications of $\mathcal{H}_{d,g}$. An explicit comparison of the intersection theories of the two would have important applications.

What to do when branch points collide?

• Stable Maps: The limiting object is a map

$$f : C \to \mathbf{P}^1$$

satisfying (Kontsevich-Manin) stability:

C has only simple nodes, and for each $p \in \mathbf{P}^1$,

$f^{-1}(p) =$ finite set $\cup$ stable marked curve

Examples: Limit points of:

$\mathcal{H}_{3,0}$

$\mathcal{H}_{2,1}$

• Admissible Covers: The limiting object is a finite map

$$f : C \to B(\to \mathbf{P}^1)$$

where B is a configuration of b points on $\mathbf{P}^1$: B is a rational curve with only simple nodes, the map to $\mathbf{P}^1$ has degree one, and the extra components of B are stable marked curves (markings are nodes and branch points).

Plus a "balanced" condition on f at the nodes.

Examples: Limits of points of:

$\mathcal{H}_{3,0}$

$\mathcal{H}_{2,1}$

Theorem: (Deligne-Mumford, Kontsevich-Manin) There is a proper (not smooth!) Deligne-Mumford stack of stable maps of genus g, degree d:

$$\overline{\mathcal{M}}_g(\mathbf{P}^1, d)$$

And(Fantechi-Pandharipande) $\pi_{d,g}$ extends to:

$$\overline{\pi}_{d,g} : \overline{\mathcal{M}}_g(\mathbf{P}^1, d) \to \mathbf{P}^b$$

Remark: This "compactification" of $\mathcal{H}_{d,g}$ has many extraneous components. For example:

$\mathcal{H}_{1,g} = \emptyset$ if $g > 0$, but $\overline{\mathcal{M}}_g(\mathbf{P}^1, 1)$ is covered by:

$$\{F_{\underline{g}} : \overline{\mathcal{M}}_{g_1,1} \times .... \times \overline{\mathcal{M}}_{g_n,1} \times (\mathbf{P}^1)^n - \Delta \to \mathcal{M}_g(\mathbf{P}^1, 1)$$

$$\mid g_1 + ... + g_n = g\}$$

and each of these corresponds to a component of the space of stable maps!

On the other hand:

Theorem: (Harris-Mumford, ℵ-Corti-Vistoli) There is a smooth stack of admissible covers:

$$\mathcal{A}dm_{g \xrightarrow{d} \mathbf{P}^1}$$

This is a quotient of the symmetric group $\mathcal{S}_b$ acting on the smooth stack:

$$\mathcal{A}dm_{g \xrightarrow{d} \mathbf{P}^1}(t_1, ..., t_b)$$

of admissible covers with labelled (simple) branch points, and the latter comes equipped with:

$$\widetilde{\pi}_{d,g} : \mathcal{A}dm_{g \xrightarrow{d} \mathbf{P}^1}(t_1, ..., t_b) \to \mathbf{P}^1[b]$$

to the (Fulton-MacPherson) configuration space of b-points on $\mathbf{P}^1$, which is even a covering map for a suitable "stack structure" on $\mathbf{P}^1[b]$.

Moreover, there is a natural map:

$$\Phi_{d,g} : \mathcal{A}dm_{g \xrightarrow{d} \mathbf{P}^1} \to \overline{\mathcal{M}}_g(\mathbf{P}^1, d)$$

All of this generalizes in many directions:

• $\mathbf{P}^1$ can be replaced with any non-singular projective "base" curve Σ of genus h.

• Points $x_1, ..., x_n \in \Sigma$ can be fixed, together with "profiles" (partitions) $\eta_1, ..., \eta_n$ of d over which we require each map: $f : C \to B$ to ramify according to η_i at x_i with the stable map (or admissible cover) conditions holding elsewhere over Σ. Denote the spaces:

$$\overline{\mathcal{M}}_g(\Sigma, \eta_1 x_1, ..., \eta_n x_n) \text{ and } \mathrm{Adm}_{g \xrightarrow{d} \Sigma}(\eta_1 x_1, ..., \eta_n x_n)$$

• Profiles $\eta_1, ..., \eta_n$ can be chosen at variable points of Σ with the admissible cover condition holding over those points:

$$\overline{\mathcal{M}}_g(\Sigma, \eta_1, ..., \eta_n) \quad \text{and} \quad \mathrm{Adm}_{g \xrightarrow{d} \Sigma}(\eta_1, ..., \eta_n)$$

• Σ can be allowed to vary in moduli:

$$\overline{\mathcal{M}}_g(h, \eta_1, ..., \eta_n) \quad \text{and} \quad \mathrm{Adm}_{g \xrightarrow{d} h}(\eta_1, ..., \eta_n)$$

Universal Families: Over each of the moduli spaces, there are universal maps:

$$\begin{array}{ccccc} \mathcal{C}_{\mathsf{Adm}} & \rightarrow & \mathcal{C}_{\mathsf{Stab}} & \xrightarrow{e} & \Sigma \\ \pi \downarrow & & \pi \downarrow & & \\ \mathsf{Adm}_{g \xrightarrow{d} \Sigma}(\eta_1, ..., \eta_n) & \rightarrow & \overline{\mathcal{M}}_g(\Sigma, \eta_1, ..., \eta_n) & & \end{array}$$

Emphasis: $\mathsf{Adm}_{g \xrightarrow{d} \Sigma}(\eta_1, ..., \eta_n)$ is smooth, of the "correct" dimension, desingularizing the "good" component of $\overline{\mathcal{M}}_g(\Sigma, \eta_1, ..., \eta_n)$.

The *local Gromov-Witten theory of curves* vastly generalizes the Aspinwall-Morrison formula:

$$\int_{\overline{\mathcal{M}}_0(\mathbf{P}^1,d)} e\left(R^1\pi_* e^*\left(\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)\right)\right)$$

$$= \frac{1}{d^3}$$

Choose line bundles L, M on Σ of degrees l, k and profiles (and points) $\eta_1 x_1, ..., \eta_n x_n$ on Σ

Then one (i.e. Bryan-Pandharipande) defines:

$$Z_d^g(h|l,m)_{\underline{\eta}} = \int_{[\overline{\mathcal{M}}_g(\Sigma,\eta_1 x_1,\ldots,\eta_n x_n)]} e_T(-R^\bullet \pi_* e^* L \oplus M)$$

where e_T is the equivariant Euler class for the standard action of $T = \mathbf{C}^* \times \mathbf{C}^*$ on $L \oplus M$. And:

$$Z_d(h|l,m)_{\underline{\eta}} = \sum_g u^{d(g)} Z_d^g(h|l,k)_{\underline{\eta}}$$

where $d(g) = \text{exp dim} - d(l+k)$.

These generating functions give the structure constants of a TQFT that deforms the center of the group ring of S_d (see Bryan's talk)

Renzo Cavalieri showed that the intersections:

$$A_d^g(h|l,m)_{\underline{\eta}} = \int_{\mathsf{Adm}_{g\to\Sigma(\eta_1 x_1,\ldots,\eta_n x_n)}} e_T(-R^\bullet \pi_* e^* L \oplus M)$$

while different from the Z-structure constants, determine essentially the same TQFT.

Namely, all of the difference between the two TQFT's is captured by the relation:

$$A_d(0|-1,0)_{(d)} = (2\sin(u/2))^d Z_d(0|-1,0)_{(d)}$$

Note: These are the basic calculations in the two TQFT's. They are explicitly calculated by (B-P) and by (Cav) entirely differently.

Moral: The contribution of the "extra" components of stable map spaces to the integrals of $e_T(-R^\bullet\pi_* e^* L \oplus M)$ for a given g are recursively determined by the corresponding integrals for admissible cover spaces for smaller g.

Question: Can we see this by analyzing the extra components of the stable map spaces?

Example: Let's return to the simplest case:

$$F_{\underline{g}} : \times_{i=1}^{n} \overline{\mathcal{M}}_{g_i,1} \times \mathbf{P}^1[n] \to \overline{\mathcal{M}}_g(\mathbf{P}^1, 1)$$

Here's a concrete version of the question above.

There is a "virtual class:"

$$[\mathcal{M}_g(\mathbf{P}^1, 1)] \in A_{2g}(\mathcal{M}_g(\mathbf{P}^1, 1))$$

arising from the deformation theory of stable maps. Can it be expressed in a natural way as a sum of push-forwards under the maps $F_{\underline{g}}$?

Remark: Pandharipande has shown that this is true generically. I.e. natural classes on open sets push forward to the virtual class away from intersections of the components.

Detailed knowledge of the decomposition of the virtual class would explain the relation above, but would also have more far-reaching applications. Consider again:

$$\Phi_{d,g} : \mathcal{A}dm_{g \xrightarrow{d} \mathbf{P}^1} \to \overline{\mathcal{M}}_g(\mathbf{P}^1, d)$$

thought of as a $\mathbf{C}^*$-equivariant map (for the natural scaling action of $\mathbf{C}^*$ on $\mathbf{P}^1$). The fixed loci on each side can be described, and there are interesting induced maps of fixed loci:

$$(i) \quad \mathcal{A}dm_{g \xrightarrow{d} 0}((d)) \to \overline{\mathcal{M}}_{g,1}$$

and

$$(ii) \quad \mathcal{A}dm_{g \xrightarrow{d} 0}(\mathbf{1}) \to \overline{\mathcal{M}}_{g,d}$$

The images are the loci of curves with:

(i) A (marked) Weierstrass point of order d,

and

(ii) A g^1_d and d (marked) points in a divisor of the linear series.

Thanks to Atiyah-Bott localization (and the virtual localization theorem of Graber-Pand.), an explicit decomposition of the virtual class along the lines described above in degree 1 would give an explicit formula for these classes in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ and $\overline{\mathcal{M}}_{g,d}$.

Example: In degree 2, the generic version of Pandharipande already gives formulas for the classes, valid on the locus of smooth curves. (i) recovers a formula in Mumford's original paper, but (ii) (derived by a gang of us in Utah) seems to be new.