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Moduli Spaces in the Derived Category of a K3 surface

Daniele Arcara
University of Utah
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(joint work with Aaron Bertram)

PLAN OF THE TALK

- Review of stability for vector bundles.

- Stability conditions on the derived category.

- Examples of stable objects.

- Moduli spaces of stable objects.

SLOPE-STABILITY FOR BUNDLES

The **slope** of a vector bundle E over a smooth curve C is

$$\mu(E) := \frac{\deg F}{\mathrm{rk} F}.$$

A vector bundle E over a smooth curve C is **stable** if

$$\mu(F) < \mu(E)$$

for every proper subbundle $F \subset E$.

There exists a similar definition for a torsion-free sheaf E over a smooth surface S with respect to the slope

$$\mu_H(E) := \frac{c_1(E) \cdot H}{\mathrm{rk} E},$$

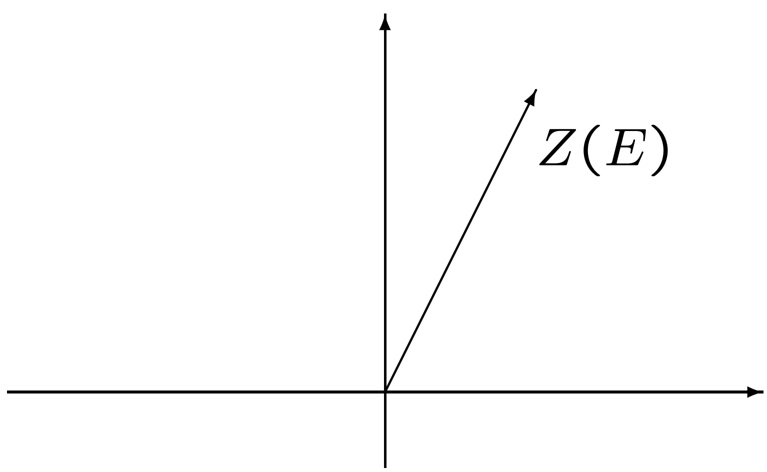
where H is a fixed ample divisor class on S.

CENTRAL CHARGE FOR CURVES

For a coherent sheaf E over a smooth curve C, define

$$Z(E) = -\deg E + i\ \mathrm{rk} E =: \rho e^{i\pi\phi} \in \mathbb{C},$$

where $\rho \geq 0$ and $\phi = \phi(Z(E)) \in [0, 2)$ is the "**phase**" of $Z(E)$.



Then E is stable if and only if

$$\phi(Z(F)) < \phi(Z(E))$$

for every proper subbundle $F \subset E$.

PROPERTIES OF $Z(E)$

(0) $Z(E \oplus F) = Z(E) + Z(F)$.

(1) $Z(E) = 0 \Leftrightarrow E = 0$ and $\phi(Z(E)) \in (0, 1]$ for all $E \neq 0$.

(2) If E, E' are stable of phases $\phi \geq \phi'$, then

$$\mathrm{Hom}_{\mathcal{O}_C}(E, E') = \begin{cases} \mathbb{C} & \text{if } E \simeq E' \\ 0 & \text{otherwise} \end{cases}$$

(3) **Harder-Narasimhan filtration**

$$0 \subset E_1 \subset \cdots \subset E_n = E,$$

$$\phi(Z(E_1/0)) > \cdots > \phi(Z(E_n/E_{n-1})).$$

(4) Moduli spaces $\mathcal{M}_C(r, d)$ which are projective when r and d are coprime.

NOTATION

From now on, S is a smooth K3 surface over $\mathbb{C}$. Recall that $\omega_S \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$.

We also assume that $\mathrm{Pic}(S) \simeq \mathbb{Z}$, generated by an ample line bundle $\mathcal{O}_S(1)$.

Let $H := c_1(\mathcal{O}_S(1))$. Then $H^2 = 2g-2$, where g is, by definition, the "**genus**" of S.

CENTRAL CHARGES FOR SURFACES (I)

Bridgeland introduced a notion of stability in the derived category of bounded complexes of coherent sheaves on S. It generalizes the notion of slope-stability for torsion-free sheaves.

For each real number $\alpha > 0$, define the **central charge** Z_α as

$$Z_\alpha(E) := -\int_S e^{-\left(\frac{H}{2}+i\sqrt{\frac{\alpha}{g-1}}H\right)}\mathrm{ch}(E)\sqrt{\mathrm{td}(S)},$$

which simplifies into

$$Z_\alpha(E) = r\left(\alpha - \frac{g+3}{4}\right) - \mathrm{ch}_2(E) + \frac{1}{2}c_1(E)\cdot H +$$

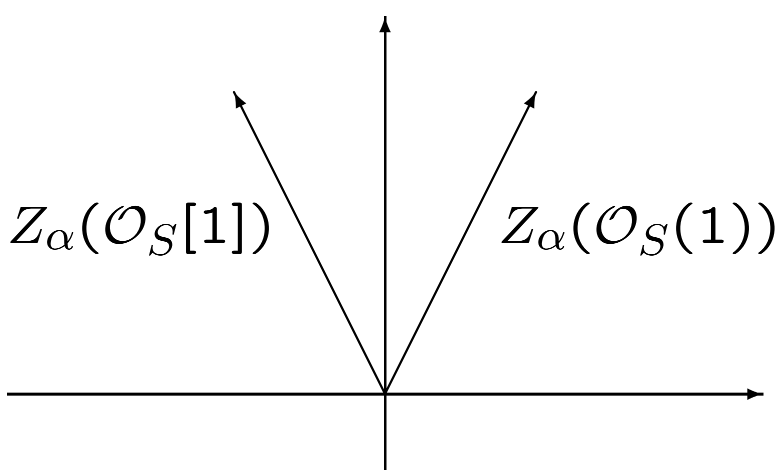
$$+i\sqrt{\frac{\alpha}{g-1}}(c_1(E)\cdot H - r(g-1)),$$

where $r = \mathrm{rk}E$.

CENTRAL CHARGES FOR SURFACES (II)

For example,

$$\begin{aligned} Z_\alpha(\mathcal{O}_S[1]) &= -Z_\alpha(\mathcal{O}_S) \\ &= \frac{g+3}{4} - \alpha + i\sqrt{\alpha(g-1)}, \\ Z_\alpha(\mathcal{O}_S(1)) &= \alpha - \frac{g+3}{4} + i\sqrt{\alpha(g-1)}. \end{aligned}$$



Note that $\alpha > (g+3)/4$ in this picture. When $\alpha < (g+3)/4$, $Z_\alpha(\mathcal{O}_S[1])$ has positive real part and $Z_\alpha(\mathcal{O}_S(1))$ has negative real part.

STABILITY CONDITIONS IN THE DERIVED CATEGORY

If we consider coherent sheaves as in the case of a smooth curve, the central charges Z_α do not satisfy the nice properties (1)-(4) we had.

The idea is to replace the abelian category of coherent sheaves with a "**tilting**" $\mathcal{A}$.

A **stability condition** in $\mathcal{D}(S)$ is a pair $(\mathcal{A}, Z_\alpha)$ of an abelian subcategory of $\mathcal{D}(S)$ (which is the core of a t-structure) together with a central charge Z_α satisfying certain conditions.

An object $E \in \mathcal{A}$ is then said to be α-**stable** if

$$\phi(Z_\alpha(F)) < \phi(Z_\alpha(E))$$

for all proper subobjects $F \subset E$ in $\mathcal{A}$.

DEFINITION OF $\mathcal{A}$

Every coherent sheaf E has a HN-filtration

$$\text{Tors}\,(E) = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

$$\mu_H(E_1/E_0) > \cdots > \mu_H(E_n/E_{n-1}),$$

where $\mu_H(E) := (c_1(E) \cdot H)/\,\text{rk}E$.

Let $\mathcal{T}$ consist of all coherent sheaves E such that $\mu_H(E/E_{n-1}) > g-1$ (this includes all torsion sheaves), and let $\mathcal{F}$ consist of all torsion-free sheaves such that $\mu_H(E_1) \le g-1$.

Every coherent sheaf E is an extension

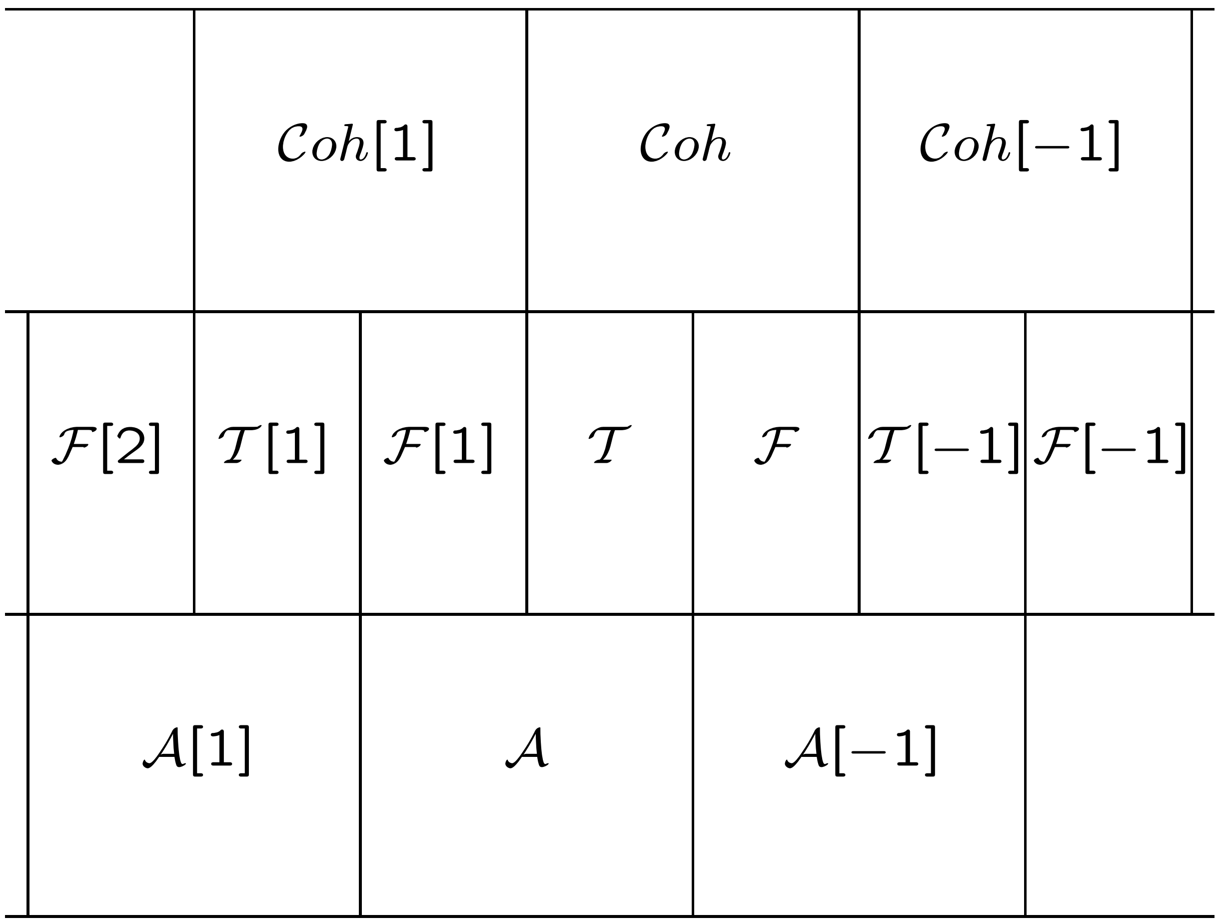
$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Let $\mathcal{A}$ be the set of all objects $E \in \mathcal{D}(S)$ such that

$\mathcal{H}^0(E) \in \mathcal{T}, \mathcal{H}^{-1}(E) \in \mathcal{F}, \mathcal{H}^i(E) = 0$ otherwise.

IDEA OF TILTING



EXAMPLES (I)

- $\mathcal{O}_S(1) \in \mathcal{A}$ because $\mu_H(\mathcal{O}_S(1)) = 2(g-1) > g-1$.

- $\mathcal{O}_S \notin \mathcal{A}$ because $\mu_H(\mathcal{O}_S) = 0 \leq g-1$. We have that $\mathcal{O}_S[1] \in \mathcal{A}$.

- All torsion sheaves are in $\mathcal{A}$.

- If Z is a 0-dimensional subscheme of S, the short exact sequence

  $$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_Z \longrightarrow 0 \tag{1}$$

  of coherent sheaves becomes the short exact sequence

  $$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{I}_Z[1] \longrightarrow \mathcal{O}_S[1] \longrightarrow 0$$

  in $\mathcal{A}$, where the map $\mathcal{O}_Z \to \mathcal{I}_Z[1]$ is exactly the extension (1) in the sense that

  $$\mathcal{O}_Z \simeq (\mathcal{I}_Z \to \mathcal{O}_S) \longrightarrow (\mathcal{I}_Z \to 0) = \mathcal{I}_Z[1].$$

EXAMPLES (II)

- Given a section $s \in H^0(S, \mathcal{O}_S(1))$, the short exact sequence

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\cdot s} \mathcal{O}_S(1) \longrightarrow \omega_C \longrightarrow 0$$

of coherent sheaves becomes the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(1) \longrightarrow \omega_C \longrightarrow \mathcal{O}_S[1] \longrightarrow 0$$

in $\mathcal{A}$.

- If Z is a 0-dimensional subscheme of S, then $\mathcal{I}_Z^\vee[1] \in \mathcal{A}$, where $E^\vee$ denotes the derived dual $R\mathcal{H}om(E, \mathcal{O}_S)$. Indeed,

$$\mathcal{H}^{-1}(\mathcal{I}_Z^\vee[1]) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_Z, \mathcal{O}_S) \simeq \mathcal{O}_S \in \mathcal{F}$$

and

$$\mathcal{H}^0(\mathcal{I}_Z^\vee[1]) = \mathcal{E}xt_S^1(\mathcal{I}_Z, \mathcal{O}_S) \simeq \mathcal{O}_Z \in \mathcal{T}.$$

STABLE OBJECTS

For each real number $\alpha > 0$ (or $\alpha > 1/4$ if g is even), an object $E \in \mathcal{A}$ is α-**stable** if

$$\phi(Z_\alpha(F)) < \phi(Z_\alpha(E))$$

for every proper subobject $F \subset E$ in $\mathcal{A}$.

Recall that

$$Z_\alpha(E) = r\left(\alpha - \frac{g+3}{4}\right) - \mathrm{ch}_2(E) + \frac{1}{2}c_1(E)\cdot H +$$

$$+i\sqrt{\frac{\alpha}{g-1}}(c_1(E)\cdot H - r(g-1)),$$

and $\mathcal{A}$ consists of objects such that

$\mathcal{H}^0(E) \in \mathcal{T}, \mathcal{H}^{-1}(E) \in \mathcal{F}, \mathcal{H}^i(E) = 0$ otherwise, with

- $\mathcal{T}$ "=" $\{\mu_H(E/E_{n-1}) > g-1\}$ and
- $\mathcal{F}$ "=" $\{\mu_H(E_1) \leq g-1\}$.

PROPERTIES OF Z_α

Proposition (Bridgeland, A.-Bertram). *For each $\alpha > 0$, or $\alpha > 1/4$ if g is even, the central charge Z_α satisfies the following properties as a function on $\mathcal{A}$:*

(1) $Z_\alpha(E) = 0 \Leftrightarrow E = 0$ *and* $\phi(Z_\alpha(E)) \in (0, 1]$ *for all* $E \neq 0$.

(2) *If* E, E' *are* α*-stable of phases* $\phi \geq \phi'$, *then*

$$\operatorname{Hom}_{\mathcal{A}}(E, E') = \begin{cases} \mathbb{C} & \textit{if } E \simeq E' \\ 0 & \textit{otherwise} \end{cases}$$

(3) **Harder-Narasimhan filtration**

$$0 \subset E_1 \subset \cdots \subset E_n = E,$$

$$\phi(Z_\alpha(E_1/0)) > \cdots > \phi(Z_\alpha(E_n/E_{n-1})).$$

EXAMPLES OF α-STABILITY

- If E is a semi-stable sheaf such that

  $$c_1(E) \cdot H = \mathrm{rk}E(g-1),$$

  then $E \in \mathcal{A}$ and E has maximal phase for all α. Note that $\mathrm{rk}E$ is even in this case.

- If Z is a 0-dimensional subscheme of S, $\mathcal{I}_Z[1]$ is not α-stable because of the short exact sequence

  $$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{I}_Z[1] \longrightarrow \mathcal{O}_S[1] \longrightarrow 0$$

  in $\mathcal{A}$.

Proposition (A.-Bertram '05). *The following objects of $\mathcal{A}$ are α-stable for all α:*

$$\mathcal{O}_S(1), \quad \mathcal{O}_S[1], \quad \mathcal{I}_Z(1), \quad \mathcal{I}_Z^\vee[1],$$

where Z is a 0-dimensional subscheme of S.

SAMPLE PROOF (I)

Let $p \in S$, and let us prove that $\mathcal{I}_p^\vee[1]$ is α-stable for all α.

Let $F \subset \mathcal{I}_p^\vee[1]$ be a subobject in $\mathcal{A}$, and let Q be the quotient. The short exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{I}_p^\vee[1] \longrightarrow Q \longrightarrow 0$$

in $\mathcal{A}$ induces a long exact sequence of cohomologies

$$0 \longrightarrow \mathcal{H}^{-1}(F) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{H}^{-1}(Q) \longrightarrow$$

$$\longrightarrow \mathcal{H}^0(F) \longrightarrow \mathcal{O}_p \longrightarrow \mathcal{H}^0(Q) \longrightarrow 0$$

with $\mathcal{H}^{-1}(F), \mathcal{H}^{-1}(Q) \in \mathcal{F}$ and $\mathcal{H}^0(F), \mathcal{H}^0(Q) \in \mathcal{T}$.

Since $\mathcal{H}^{-1}(Q) \in \mathcal{F}$ is torsion-free, we have that either $\mathcal{H}^{-1}(F) = \mathcal{O}_S$ or $\mathcal{H}^{-1}(F) = 0$.

Also, since $\mathcal{O}_p$ surjects onto $\mathcal{H}^0(Q)$, we have that either $\mathcal{H}^0(Q) = \mathcal{O}_p$ or $\mathcal{H}^0(Q) = 0$.

SAMPLE PROOF (II)

Case I: $\mathcal{H}^{-1}(F) = \mathcal{O}_S$. Then $\mathcal{H}^{-1}(Q) \in \mathcal{F}$ and $\mathcal{H}^0(F) \in \mathcal{T}$ have the same μ_H, which forces $\mathcal{H}^{-1}(Q) = 0$ and $\mathcal{H}^0(F)$ is a torsion sheaf. Since $F \neq E$, $\mathcal{H}^0(F) = 0$, and F does not destibilize E.

Case II: $\mathcal{H}^{-1}(F) = 0$. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{H}^{-1}(Q) \longrightarrow \mathcal{H}^0(F) \longrightarrow T \longrightarrow 0$$

with T either $\mathcal{O}_p$ or 0, and therefore

$$\begin{aligned}(r_Q - 1)(g-1) &= r_F(g-1) < c_1(\mathcal{H}^0(F)) \cdot H \\ &= c_1(\mathcal{H}^{-1}(Q)) \cdot H \leq r_Q(g-1).\end{aligned}$$

Since $c_1(\mathcal{H}^{-1}(Q)) \cdot H \in 2(g-1)\mathbb{Z}$, this implies that

$$c_1(\mathcal{H}^{-1}(Q)) \cdot H = r_Q(g-1)$$

and r_Q is even, making Q an object of maximal slope.

[NOTE: We skipped a step]

FLAT FAMILIES

A **flat family** of objects in $\mathcal{A}$ over a base B is an object $\mathcal{E} \in \mathcal{D}(S \times B)$ such that

$$Li^*_{S\times\{x\}}\mathcal{E} \in \mathcal{A}$$

for all $x \in B$.

If B is smooth and projective, Abramovich and Polishchuk constructed (a t-structure whose core is) an abelian subcategory $\mathcal{A}_B$ of $\mathcal{D}(S \times B)$ which contains all of the flat families.

MODULI SPACES

We are interested in the **moduli spaces**

$$\mathcal{M}_\alpha := \mathcal{M}_\alpha(0, H, g-1)$$

of α-stable objects $E \in \mathcal{A}$ of fixed invariants

$$\mathrm{rk} E = 0, c_1(E) = H, \mathrm{ch}_2(E) = g - 1.$$

These invariants are chosen in such a way that $\phi(Z_\alpha(E)) = 1/2$ for all α, and therefore E is α-stable if and only if

$$Re(Z_\alpha(F)) > 0$$

for all proper subobjects $F \subset E$ in $\mathcal{A}$.

For $\alpha >> 0$, an object $E \in \mathcal{A}$ is α-stable if and only if it is a torsion-free sheaf L_C of rank 1 supported on a curve $C \in |\mathcal{O}_S(1)|$ of degree $2g-2$, i.e.,

$$\mathcal{M}_\alpha \simeq \mathrm{Pic}_{2g-2} \to |\mathcal{O}_S(1)|,$$

the relative Picard variety over $|\mathcal{O}_S(1)|$.

FIRST CRITICAL VALUE

Recall that we have the short exact sequences

$$0 \longrightarrow \mathcal{O}_S(1) \longrightarrow \omega_C \longrightarrow \mathcal{O}_S[1] \longrightarrow 0$$

in $\mathcal{A}$.

Since

$$Re(Z_\alpha(\mathcal{O}_S(1))) = \alpha - \frac{g+3}{4},$$

the sheaves of the form ω_C are not α-stable if $\alpha \leq (g+3)/4$.

Proposition (A.-Bertram '05). *For all $\alpha > (g+3)/4$,*

$$\mathcal{M}_\alpha \simeq \mathrm{Pic}_{2g-2} \to |\mathcal{O}_S(1)|.$$

Moreover, the sheaves of the form ω_C are the only ones that become α-unstable on the other side of this critical value.

MUKAI FLOPS (I)

Theorem (A.-Bertram '05). *Choose* $\alpha_0 > (g+3)/4$ *and* $\alpha_1 < (g+3)/4$ *(with* α_1 *close enough to* $(g+3)/4$*). There exists a Mukai flop*

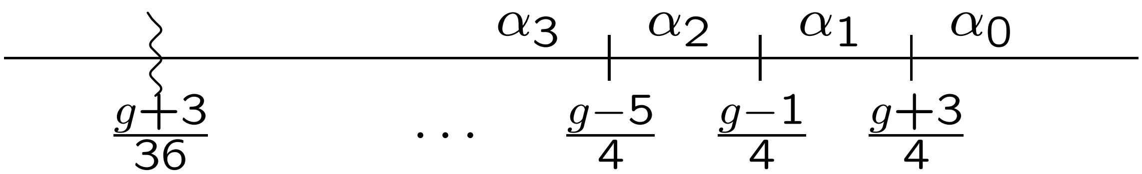
$$\begin{array}{ccc} & \widetilde{\mathcal{M}}_{\alpha_0} & \\ \swarrow & & \searrow \\ \mathcal{M}_{\alpha_1} & <--- & \mathcal{M}_{\alpha_0}, \\ \bigcup & & \bigcup \\ \mathbb{P}^\vee & & \mathbb{P} \end{array}$$

where $\mathbb{P} := \mathbb{P}(H^0(S, \mathcal{O}_S(1)))$*. Note that* $\mathbb{P}^\vee \simeq \mathbb{P}(\mathrm{Ext}^2_S(\mathcal{O}_S(1), \mathcal{O}_S)) \simeq \mathbb{P}(\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{O}_S(1), \mathcal{O}_S[1]))$*. Moreover, there exists a universal family in* $\mathcal{D}(S \times \mathcal{M}_{\alpha_1})$*. It is a sheaf, but it is not flat over* $\mathcal{M}_{\alpha_1}$*.*

Note that the moduli space $\mathcal{M}_{\alpha_1}$ can be identified with the moduli space $\mathcal{M}(g-1, H, 1-g)$ of stable sheaves of rank $g-1$, first Chern class H, and second Chern character $1-g$, via a Fourier-Mukai transform.

MUKAI FLOPS (II)

Critical values:



Theorem (A.-Bertram '05). *There exist Mukai flops*

$$
\begin{array}{ccccccccc}
 & & \widetilde{\mathcal{M}}_{\alpha_2} & & \widetilde{\mathcal{M}}_{\alpha_1} & & \widetilde{\mathcal{M}}_{\alpha_0} & & \\
\ldots & \swarrow & & \searrow \swarrow & & \searrow \swarrow & & \searrow & \\
\mathcal{M}_{\alpha_3} & & \mathcal{M}_{\alpha_2} & & \mathcal{M}_{\alpha_1} & & \mathcal{M}_{\alpha_0} & & \\
\bigcup & & \bigcup & & \bigcup & & \bigcup & & \\
\mathbb{P}_3, \mathbb{P}_2^\vee & & \mathbb{P}_2, \mathbb{P}_1^\vee & & \mathbb{P}_1, \mathbb{P}_0^\vee & & \mathbb{P}_0 & &
\end{array}
$$

where each of the $\mathbb{P}_i$*'s is a projective bundle over* $\text{Hilb}^i(S) \times \text{Hilb}^i(S)$ *with fibers isomorphic to* $\text{Ext}^1_{\mathcal{A}}(\mathcal{I}_Z(1), \mathcal{I}_W^\vee[1])$ *over a point* (Z, W)*.*