Toric quotients and flips

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As my contribution to these proceedings, I will discuss the geometric invariant theory quotients of toric varieties. Specifically, I will show that quotients of the same problem with respect to different linearizations are typically related by a sequence of flips (or more precisely, log flips) in the sense of Mori, which in the quasi-smooth case can be characterized quite precisely. This seems to have little to do with the subject of the symposium, which was low-dimensional topology. However, it is intended as a model of a general theory, to be described elsewhere [11, 14], on the dependence of invariant theory quotients on linearizations. This theory was the subject of my talk in the symposium, and it can be applied to many of the moduli spaces employed in low-dimensional topology.

Since we will be studying geometric invariant theory, we will want all of our toric varieties to be quasi-projective. Accordingly, we will define them using polyhedra, which is essentially dual to the usual approach involving fans. This dual approach is much better suited to our own purposes, but it is not well documented in the literature. The whole of §2 is therefore expository, presenting well-known results in a polyhedral context. The usual approach, and its relationship with ours, are explained in the excellent new book of Fulton [3].

The bulk of the paper concerns the quotients of an arbitrary quasi-projective toric variety by a subtorus of the usual torus action. We show in §3 that the linearizations giving nonempty quotients are parametrized by a polyhedron (in fact, the projection on a subspace of the polyhedron defining the toric variety), and that the polyhedron is partitioned into polyhedral chambers (bounded by the projection of the appropriate skeleton of the polyhedron) inside which the quotient is essentially constant. Furthermore, we show that moving between adjacent chambers induces a birational map of the quotients which, in good cases, is a flip. Though they are important for us, these results are relatively easy. Indeed, though we prove them by toric methods, they also essentially follow from the descent lemma and the numerical criterion.

The main result, **(4.5)**, is stronger, and correspondingly harder, occupying most of §4. It gives an explicit description of the flip as a weighted blow-up and blow-down in the case when the toric variety is quasi-smooth, that is, a finite abelian quotient of a smooth variety. In the bad cases where the birational map is not a flip, either the blow-down or the blow-up is absent. These results are proved by a series of simplifications leading to an easy model case.

Three existing papers are very closely related to the present one. First, Kapranov, Sturmfels and Zelevinsky [7] have studied quotients of toric varieties by subtori, though their interest is more in the Chow quotients, and their relation to the inverse system of geometric invariant theory quotients, than in the matters we treat. Second, Guillemin and Sternberg [4] have carried out the whole program of §4 in the symplectic category, indeed treating arbitrary symplectic manifolds with torus actions, not just toric varieties. Finally, a recent paper of Hu [6] proves part of our **(4.5)**(a), in which we identify the exceptional loci of the birational maps between quotients, for torus actions on arbitrary smooth varieties. The special interest of the present paper, then, is to demonstrate in the algebraic category that the birational map is a weighted blow-up and blow-down at these loci. The general theory mentioned above will provide a similar result for arbitrary geometric invariant theory problems, even those involving nonabelian groups.

We work over an arbitrary algebraically closed field k, and denote $k^\times$ the multiplicative group of k.

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1 Preliminaries on geometric invariant theory and flips

(1.1) First, a few generalities on geometric invariant theory. For more details, see [9, 10]. Let X be a projective variety over k, and let G be a reductive group acting on X. We assume that the action is *effective*, that is, every nontrivial element of G acts nontrivially somewhere. A *linearization* v of the action is an ample line bundle $\mathcal{O}(1) \to X$ together with a lifting of the G-action to a linear action on $\mathcal{O}(1)$. More generally, we will allow $\mathcal{O}(1)$ to be a formal rational power of an ample bundle; this is what Kapranov, Sturmfels and Zelevinsky [7] call a *fractional linearization.* A linearized G-action on X is equivalent to a homogeneous action of G on the graded algebra $R = \bigoplus_n H^0(\mathcal{O}(n))$, where $\mathcal{O}(n)$ is deemed to have sections only if it is a bundle. According to Nagata's theorem [10, 3.4], G reductive implies that the subalgebra R^G of G-invariants is finitely generated over k. Hence $\operatorname{Proj} R^G = X/\!\!/G$ is a projective variety over k. When our focus is on the choice of the linearization v, we will write $X/\!\!/v$ in place of $X/\!\!/G$.

More generally, X may be a variety over k, projective over an affine variety. The quotient $X/\!\!/G$ is then a variety of the same kind; otherwise, however, the above discussion applies verbatim.

In any case, the inclusion $R^G \hookrightarrow R$ defines a dominant rational map $\pi : X \dashrightarrow X/\!\!/G$. It is defined at $x \in X$ if and only if, for some n, $\mathcal{O}(n)$ has a G-invariant section which does not vanish at x; then x is said to be *semistable.* Clearly π is G-invariant, so its fibres are unions of orbits; those semistable x such that $\pi^{-1}(\pi(x))$ consists of a single orbit are said to be *stable.* Both the stable and semistable sets are open; note this implies that, if x is stable, then its stabilizer is discrete. In good cases, the stable set is nonempty; the quotient is then nonempty of dimension $\dim X - \dim G$.

(1.2) Second, we will state the definition of a flip. Luckily, this definition is all we will need from Mori theory. For more details, see [1, 8].

Let $X_- \to X_0$ be a *contraction.* This means a small birational projective morphism of integral varieties over k; *small* means that the exceptional set has dimension greater than 1. Let $L \to X_-$ be a line bundle, or a formal rational power of one, such that L^{-1} is relatively ample over X_0. Then the *flip* (or more properly, the L-*flip*) is an integral variety X_+, with a small birational morphism $X_+ \to X_0$, such that, if $f : X_- \dashrightarrow X_+$

is the induced birational map, then the divisor class f_*L is $\mathbb{Q}$-Cartier, that is, a formal rational power of a line bundle, and is relatively ample over X_0. We emphasize the shift between ampleness of L^{-1} and that of f_*L. If a flip exists it is easily seen to be unique.

Two things should be mentioned. First, a point of terminology: in the literature, the unmodified word *flip* usually denotes a K-flip where K is the canonical bundle of X_-. However, to avoid awkwardness, we will use it to mean an L-flip for general L. Second, several authors, including Mori [8], require that each contraction reduce the Picard number by exactly 1. As we will see, this will be not be true of our flips, so we do not require it.

2 Preliminaries on toric varieties

Let V be a vector space over $\mathbb{Q}$ of dimension n. For us, a *polyhedral cone* $P \subset V$ will simply be a subset defined by finitely many weak linear inequalities over $\mathbb{Q}$. We will take for granted the basic fact [12, 19.1] that any such cone equals $\{\sum x_i v_i \mid x_i \in \mathbb{Q}_{\geq 0}, v_i \in S\}$ for some finite set $S \subset V$. A *face* of P is a subset $F \subset P$ obtained by replacing a subset of the inequalities with the corresponding equalities; its *interior* $\operatorname{int} F$ is the complement in F of all smaller faces.

Let $\Lambda \subset V$ be a lattice which spans V.

(2.1) Lemma (Gordan). *$P \cap \Lambda$ is a finitely-generated semigroup.*

Proof. Without loss of generality, we may suppose that the set S mentioned above consists of points in Λ. Let $Q = \{\sum x_i v_i \mid x_i \in [0,1], v_i \in S\}$; then Q is a bounded set, so $Q \cap \Lambda$ is finite. But writing any element of $P \cap \Lambda$ as $\sum x_i v_i = \sum (x_i - [x_i])v_i + \sum [x_i] v_i$ expresses it as a sum of elements in $Q \cap \Lambda$. □

(2.2) With this understood, the triple $\phi = (V, \Lambda, P)$ defines an n-dimensional affine *toric variety* $\mathbb{V}\phi$ as follows. Let $k[\phi]$ be the associated finitely-generated algebra over k, that is, the free k-vector space on $P \cap \Lambda$, endowed with a multiplication induced by the semigroup structure. Then $\mathbb{V}\phi = \operatorname{Spec} k[\phi]$.

The group $T = \operatorname{Hom}(\Lambda, k^\times)$ acts naturally on $k[\phi]$ and hence on $\mathbb{V}\phi$. If P is not contained in a proper subspace of V, then the action is *effective*, that is, every nontrivial element of T acts nontrivially somewhere. Notice that there is a natural correspondence between m-dimensional subspaces of V and $n-m$-parameter subgroups of T, induced by stabilizers.

(2.3) Examples. Let $V = \mathbb{Q}^n$, $\Lambda = \mathbb{Z}^n$, and $P = \mathbb{Q}^n_{\geq 0}$. Then $P \cap \Lambda = \mathbb{N}^n$ and $k[\phi] = k[z_1, \ldots, z_n]$, so that $\mathbb{V}\phi = k^n$. If instead we take $P = \mathbb{Q}^n$, then $P \cap \Lambda = \mathbb{Z}^n$ and $k[\phi] = k[z_1, z_1^{-1} \ldots, z_n, z_n^{-1}]$, so that $\mathbb{V}\phi = k^{\times n}$. Crossing these two examples together shows that $k^{\times n-p} \times k^p$ is also a toric variety.

(2.4) Example. A simple example of a singular toric variety is furnished by $V = \mathbb{Q}^2$, $\Lambda = \mathbb{Z}^2$, P generated by $(1,1)$ and $(-1,1)$. The semigroup $P \cap \Lambda$ is free abelian on three generators $v_1 = (-1,1)$, $v_2 = (0,1)$, $v_3 = (1,1)$ satisfying $v_1 + v_3 = 2v_2$, so $k[\phi] \cong k[z_1, z_2, z_3]/(z_1 z_3 - z_2^2)$, and $\mathbb{V}\phi$ has a rational double point.

Let $\phi = (V, \Lambda, P)$ and $\psi = (W, \Gamma, Q)$, and let $f : V \to W$ be a linear map such that $f(\Lambda) \subset \Gamma$ and $f(P) \subset Q$. Then f induces a morphism $\mathbb{V}f : \mathbb{V}\psi \to \mathbb{V}\phi$ intertwining the torus actions. We call $\mathbb{V}f$ an affine *toric morphism.* An affine *toric subvariety* is a subvariety of a toric variety whose inclusion is a toric morphism.

(2.5) Lemma. *If f induces isomorphisms $V \cong W$ and $\Lambda \cong \Gamma$, and P is contained in no proper subspace, then $\mathbb{V}f$ is birational.*

Proof. In this case f is essentially an inclusion $P \subset Q$. Composing it with the inclusion $Q \subset V$ induces a toric morphism $T \to \mathbb{V}\psi \to \mathbb{V}\phi$, which is dominant since the corresponding homomorphism of algebras is injective. But since P is contained in no proper subspace, T acts effectively on $\mathbb{V}\phi$, so the morphism $T \to \mathbb{V}\phi$ is a birational embedding. Hence the morphism $\mathbb{V}\psi \to \mathbb{V}\phi$ is birational. $\square$

Notice that in the course of the proof, we constructed an open T-orbit in $\mathbb{V}\phi$ which is isomorphic to T. Consequently, $\mathbb{V}\phi$ is n-dimensional whenever P is contained in no proper subspace.

(2.6) Proposition. (a) *There is a bijective correspondence $F \mapsto \operatorname{orb} F$ between m-dimensional faces of P and m-dimensional orbits of $\mathbb{V}\phi$;* (b) *the stabilizer of* $\operatorname{orb} F$ *corresponds to the subspace of V generated by F;* (c) *the closure of* $\operatorname{orb} F$ *is the toric subvariety $\mathbb{V}\psi$ for $\psi = (V, \Lambda, F)$;* (d) $\operatorname{orb} G \subset \mathbb{V}\psi$ *if and only if $G \subset F$;* (e) $\Im_{\mathbb{V}\psi} \subset k[\phi]$ *is the free vector space on $(P \backslash F) \cap \Lambda$.*

We leave the details of the proof to the reader, contenting ourselves with the following.

Sketch of proof. There is a natural surjection $k[\phi] \to k[\psi]$ whose kernel is the free vector space mentioned, hence a toric embedding $\mathbb{V}\psi \hookrightarrow \mathbb{V}\phi$ with image corresponding to that ideal. By the remark just before the proposition, this image is m-dimensional and contains an open T-orbit.

Conversely, any T-orbit of dimension m has a stabilizer $S \subset T$ which corresponds to an m-dimensional subspace $U \subset V$. It is not hard to check that $U \cap P$ is the corresponding m-dimensional face of P. $\square$

In particular, if F is a *facet*, that is, a face of codimension one, then $\mathbb{V}\psi \subset \mathbb{V}\phi$ is a Weil divisor. In this case, given any regular function f on $\mathbb{V}\phi$, let $\rho_F(f)$ be the *order of vanishing* of f on $\mathbb{V}\phi$, that is, the maximum integer p such that $f \in \Im^p_{\mathbb{V}\psi}$. This determines a map $\rho_F : P \cap \Lambda \to \mathbb{N}$ which vanishes on F.

(2.7) Lemma. *ρ_F is an epimorphism of semigroups.*

Note that together with the vanishing, this completely determines ρ_F.

Proof. Without loss of generality suppose that P is contained in no proper subspace. Let $U \subset V$ be the hyperplane containing F, let H be the half-space bounded by U containing P, and let $\phi' = (V, \Lambda, H)$ and $\psi' = (V, \Lambda, U)$. By **(2.5)** the square of morphisms

$$\begin{array}{ccc} \mathbb{V}\psi' & \hookrightarrow & \mathbb{V}\phi' \\ \downarrow & & \downarrow \\ \mathbb{V}\psi & \hookrightarrow & \mathbb{V}\phi \end{array}$$

has birational columns. Hence ρ_F is the restriction of ρ_U to $P \cap \Lambda$, so it suffices to check the claim for $U \subset H$. But this is easy, because $\mathbb{V}\phi' \cong k^{\times n-1} \times k$, and under this isomorphism $\mathbb{V}\psi \cong k^{\times n-1} \times 0$. □

In fact, any element of Λ in the subspace generated by P, being the difference of two elements in $P \cap \Lambda$, determines a well-defined rational function on $\mathbb{V}\phi$, and the unique extension of ρ_F to an epimorphism $\Lambda \to \mathbb{Z}$ gives the order of vanishing of this rational function at $\mathbb{V}\psi$.

(2.8) We will actually be more interested in the following generalization. Let ϕ be a triple (V, Λ, P), where V and Λ are as before, but $P \subset V$ is now a *convex polyhedron*, that is, a subset defined by finitely many affine inequalities over $\mathbb{Q}$. (Faces and their interiors are defined as before.) Then ϕ defines a quasi-projective toric variety $\mathbb{V}\phi$ as follows. Let $C(P) \subset \mathbb{Q} \times V$ be the associated polyhedral cone, that is, the closure in V of $\{(\lambda, \lambda x) \mid \lambda \in \mathbb{Q}_{\geq 0}, x \in P\}$; and let $C(\phi) = (\mathbb{Q} \times V, \mathbb{Z} \times \Lambda, C(P))$. Then $k[C(\phi)]$ is a finitely generated k-algebra by **(2.1)**, but now give it the grading induced by the 0th coordinate. Then $\mathbb{V}\phi = \operatorname{Proj} k[C(\phi)]$.

An action of $T = \operatorname{Hom}(\Lambda, k^\times)$ is constructed as in the affine case, except that the 0th coordinate is acted on trivially. Since it is induced from an action on $k[C(\phi)]$, the action on $\mathbb{V}\phi$ is automatically linearized on $\mathcal{O}(1)$.

If P is actually a cone, then $C(P) = \mathbb{Q}_{\geq 0} \times P$ naturally, so $C(P) \cap (\mathbb{Z} \times \Lambda) = \mathbb{N} \times (P \cap \Lambda)$ and $\operatorname{Proj} k[C(\phi)] = \operatorname{Proj} k[\phi][z] = \operatorname{Spec} k[\phi]$. Hence our new definition is consistent with that in **(2.2)**. On the other hand, if P is a *polytope*, that is, a bounded polyhedron, then the part of $k[C(\phi)]$ graded by 0 is just k and $\mathbb{V}\phi$ is projective. In fact, the general $\mathbb{V}\phi$ is projective over an affine toric variety, as we now prove.

(2.9) Lemma. *If $P \neq \varnothing$, let $P^0 = C(P) \cap (0 \times V)$, and let $\phi^0 = (V, \Lambda, P^0)$. Then there is a natural toric surjection $\mathbb{V}\phi \to \mathbb{V}\phi^0$ with projective fibres.*

Proof. The degree 0 part of $k[C(\phi)]$ is exactly $k[\phi^0]$. □

(2.10) Example. Let $\phi = (\mathbb{Q}^n, \mathbb{Z}^n, P)$ where P is the simplex $\{(x_i) \in \mathbb{Q}^n \mid x_i \geq 0, \sum x_i \leq 1\}$. Then $k[C(\phi)] \cong k[z_0, \ldots, z_n]$ with the usual grading, so $\mathbb{V}\phi = \mathbb{P}^n$. More generally, let $w_1, \ldots, w_n$ be positive integers, and let $\phi = (\mathbb{Q}^n, \mathbb{Z}^n, P)$ where P is the "weighted" simplex $\{(x_i) \in \mathbb{Q}^n \mid x_i \geq 0, \sum w_i x_i \leq 1\}$. Then $k[C(\phi)] \cong k[z_0, \ldots, z_n]$ again, but with z_i graded by w_i for $i > 0$, 1 for $i = 0$. Hence $\mathbb{V}\phi$ is the weighted projective space $W\mathbb{P}(1, w_1, \ldots, w_n)$.

(2.11) Example. Let $V = \mathbb{Q}^n$, $\Lambda = \mathbb{Z}^n$, and $P = \mathbb{Q}^n_{\geq 0}$. Then $\mathbb{V}\phi = k^n$; indeed, the lattice point $(x_i) \in \mathbb{Q}^n_{\geq 0} \cap \mathbb{Z}^n = \mathbb{N}^n$ represents the function $\prod z_i^{x_i}$. Hence if $Q = \{(x_i) \in P \mid \sum x_i \geq 1\}$ and $\psi = (V, \Lambda, Q)$, then $k[C(\psi)] = \bigoplus_k \Im^k$ where $\Im$ is the ideal sheaf of the origin in $k[\phi] = k[z_1, \ldots, z_n]$. So $\mathbb{V}\psi = \operatorname{Proj} \bigoplus_k \Im^k$, the blow-up of k^n at the origin. It is projective over $\mathbb{V}\psi^0 = \mathbb{V}\phi$.

We state without proof the following analogues of **(2.5)**, **(2.6)**, and **(2.7)** for the quasi-projective toric varieties defined in **(2.8)**.

Let $\phi = (V, \Lambda, P)$ and $\psi = (W, \Gamma, Q)$, where P and Q are now polytopes, and let $f : V \to W$ be a linear map such that $f(\Lambda) \subset \Gamma$ and $f(P) \subset Q$. Then f induces a rational map $\mathbb{V}f : \mathbb{V}\phi \dashrightarrow \mathbb{V}\psi$ intertwining the torus actions. We call $\mathbb{V}f$ a *toric rational map.* A *toric subvariety* is a subvariety of a toric variety whose inclusion is such a map.

(2.12) Lemma. *If f induces isomorphisms $V \cong W$ and $\Lambda \cong \Gamma$, and P is contained in no proper subspace, then $\mathbb{V}f$ is birational.* □

(2.13) Proposition. (a) *There is a bijective correspondence $F \mapsto \operatorname{orb} F$ between m-dimensional faces of P and m-dimensional orbits of $\mathbb{V}\phi$;* (b) *the stabilizer of* $\operatorname{orb} F$ *corresponds to the vector subspace of V generated by (differences of elements of) F;* (c) *the closure of* $\operatorname{orb} F$ *is the toric subvariety $\mathbb{V}\psi$ for $\psi = (V, \Lambda, F)$;* (d) $\operatorname{orb} G \subset \mathbb{V}\psi$ *if and only if $G \subset F$;* (e) $\Im_{\mathbb{V}\psi}(n) \subset k[\phi]$ *is the free vector space on $(C(P) \backslash C(F)) \cap (\{n\} \times \Lambda)$.* □

In particular, if F is a *facet*, then given any section f of $\mathcal{O}(d)$ on $\mathbb{V}\phi$, let $\rho_F(f)$ be the *order of vanishing* on $\mathbb{V}\psi$. This determines a map $\rho_F : C(P) \cap (\mathbb{Z} \times \Lambda) \to \mathbb{N}$ which vanishes on F.

(2.14) Lemma. *ρ_F is an epimorphism of semigroups.* □

In fact, any element of Λ in the subspace generated by P determines a rational section of $\mathcal{O}(1)$ on $\mathbb{V}\phi$, and the unique extension of ρ_F to an epimorphism $\Lambda \to \mathbb{Z}$ gives its order of vanishing at $\mathbb{V}\psi$.

For convenience we will in future make two abuses of terminology. Given a face F of P, we refer to the subspace of V generated by differences of elements of F as the subspace generated by F; and we refer to the triple (V, Λ, F) as a face of (V, Λ, P).

(2.15) Warning. On $\mathbb{V}\phi$, $\mathcal{O}(1)$ may be not a bona fide ample line bundle, though $\mathcal{O}(d)$ will be for some d. Somewhat inaccurately, we therefore regard $\mathcal{O}(1)$ as a formal rational power of an ample bundle, and refer to such a structure as a *fractional polarization*. For example, the lattice $\mathbb{Z} \subset \mathbb{Q}$ together with the line segment $P = [0, c/d]$ gives the toric variety $\mathbb{P}^1$, but with $\mathcal{O}(d)$ equal to the cth power of the usual hyperplane bundle. Those who find this alarming may dilate P by an integer d; this picks out the part of $k[C(\phi)]$ graded by a multiple of d, and so replaces $\mathcal{O}(1)$ by $\mathcal{O}(d)$, which for suitable d will be a line bundle.

(2.16) Any translate of P by an element $\lambda \in \Lambda$ determines the same graded k-algebra as P itself, with the T-action tensored by the character $T \to k^\times$ determined by λ. This does not affect the action on the homogeneous ideals, so the toric variety $\mathbb{V}\phi$ is unaffected by translation. However, the linearization of the T-action on $\mathcal{O}(1)$ is tensored by the character. Likewise, translating P by any element of V does not affect $\mathbb{V}\phi$, but may be regarded formally as tensoring the linearization by a *fractional character*. This can be seen by dilating P as in **(2.15)** above, and then translating by an element of Λ.

Similarly, if $P \subset Q$ and $(P - v) \subset Q$, then the two inclusions induce the same rational map of toric varieties.

In fact, quasi-projective toric varieties of the kind defined in **(2.8)** can be covered by affine toric varieties, as follows. For any $x \in P$, the unbounded convex polyhedron

$$P_F = \{x + \lambda(y - x) \mid \lambda \in \mathbb{Q}_{\geq 0}, y \in P\} \subset V$$

contains P, but depends only on the minimal face F containing x; call it the *localization of* P *at* F, and let $\phi_F = (V, \Lambda, P_F)$. Notice that the inclusion $P \subset P_F$ induces a bijection between the faces of P containing F and the faces of P_F. In particular, F itself corresponds to the maximal affine subspace of V contained in P_F.

(2.17) Proposition. *The set* $\{\mathbb{V}\phi_F \mid F \text{ is a face of } P\}$ *is an affine toric cover of* $\mathbb{V}\phi$. *If* $H \subset P$ *is the smallest face containing the two faces* $F, G \subset P$, *then* $\mathbb{V}\phi_H = \mathbb{V}\phi_F \cap \mathbb{V}\phi_G$.

Consequently, we may define a *toric morphism* as a morphism of toric varieties which is locally an affine toric morphism.

Proof. First of all, P_F is a translate of a cone, so by **(2.16)**, $\mathbb{V}\phi_F$ is affine. By **(2.12)** there is a birational map $\mathbb{V}\phi_F \dashrightarrow \mathbb{V}\phi$. We claim that it is actually an embedding.

First, to prove that it is a morphism, it suffices to show that no homogeneous prime ideal in $k[C(\phi_F)]$ restricts to zero in $k[C(\phi)]$. Since $\operatorname{Proj} k[C(\phi_F)] = \operatorname{Spec} k[\phi_F]$, the homogeneous primes in $k[C(\phi_F)]$ are all of the form $\mathfrak{p}[z]$ for $\mathfrak{p}$ a prime in $k[\phi_F]$. But the semigroup homomorphism $C(P) \cap (\mathbb{Z} \times \Lambda) \to C(P_F) \cap (\mathbb{Z} \times \Lambda) \to P_F \cap \Lambda$ induced by projection is surjective, so the algebra homomorphism $k[C(\phi)] \to k[C(\phi_F)] \to k[\phi_F]$ given by $z \mapsto 1$ is surjective as well. The restriction of $\mathfrak{p}[z]$ to $k[C(\phi)]$ is exactly the inverse image of $\mathfrak{p}$ by this map, so it is nonzero and $\mathbb{V}\phi_F \to \mathbb{V}\phi$ is a morphism.

Any section of $\mathcal{O}(k) \to \mathbb{V}\phi$ pulls back to a section of $\mathcal{O}(k) \to \mathbb{V}\phi_F$; since $\mathbb{V}\phi_F$ is affine, this is just a regular function on $\mathbb{V}\phi_F$. The pullback map is the map $k[C(\phi)] \to k[\phi_F]$ constructed above. Since it is surjective, the morphism $\mathbb{V}\phi_F \to \mathbb{V}\phi$ is an embedding. In light of the remarks before the statement of the proposition, its image consists of those orbits in $\mathbb{V}\phi$ corresponding to faces of P containing F.

Let $H \subset P$ be any face containing F. If $S \subset T$ is the subgroup corresponding to the subspace it generates, and $\psi = (V, \Lambda, H)$, then by **(2.13)**(b,c) and the above, there is a square of embeddings

$$\begin{array}{ccc} T/S & \longrightarrow & \mathbb{V}\phi_F \\ \downarrow & & \downarrow \\ \mathbb{V}\psi & \longrightarrow & \mathbb{V}\phi. \end{array}$$

Hence the image of $\mathbb{V}\phi_F$ in $\mathbb{V}\phi$ contains $\operatorname{orb} H$. In particular, it contains $\operatorname{orb} F$, so we have found in $\mathbb{V}\phi$ an open toric affine containing any orbit, and hence an affine toric cover of $\mathbb{V}\phi$. If H is a face containing F, then $P_H \supset P_F \supset P$, so the embedding $\mathbb{V}\phi_H \hookrightarrow \mathbb{V}\phi$ is a composition of embeddings $\mathbb{V}\phi_H \hookrightarrow \mathbb{V}\phi_F \hookrightarrow \mathbb{V}\phi$. Hence if $H \subset P$ is the smallest face containing the two faces $F, G \subset P$, there is a square of embeddings

$$\begin{array}{ccc} \mathbb{V}\phi_H & \longrightarrow & \mathbb{V}\phi_F \\ \downarrow & & \downarrow \\ \mathbb{V}\phi_G & \longrightarrow & \mathbb{V}\phi. \end{array}$$

Indeed, $\mathbb{V}\phi_H = \mathbb{V}\phi_F \cap \mathbb{V}\phi_G$, since both sides consist of those orbits corresponding to faces in P containing F and G. $\square$

3 Quotients of toric varieties

Since the T-action on $\mathbb{V}\phi$ has a dense orbit, the quotient $\mathbb{V}\phi /\!\!/ T$ will be either a point or empty. But the actions of subgroups will be more interesting. We will warm up by considering quotients by finite subgroups of T, and then proceed to the case of real interest, namely quotients by subtori.

(3.1) Proposition. *Let $G \subset T$ be a finite subgroup, let $\Gamma = \{\lambda \in \Lambda \mid g(\lambda) = 1 \ \forall\, g \in G\}$, and let $\phi/G = (V, \Gamma, P)$. Then $(\mathbb{V}\phi)/G = \mathbb{V}(\phi/G)$.*

Proof. The G-action on $k[C(\phi)]$ is restricted from the canonical action on $k[C(\psi)]$ where $\psi = (V, \Lambda, V)$. The algebra of G-invariants of $k[C(\psi)]$ is clearly $k[C(\psi/G)]$, so the algebra of G-invariants of $k[C(\phi)]$ is $k[C(\phi)] \cap k[C(\psi/G)] = k[C(\phi/G)]$. Hence the quotient $(\mathbb{V}\phi)/G$ is $\operatorname{Proj} k[C(\phi/G)] = \mathbb{V}(\phi/G)$. $\square$

Now we proceed to consider quotients by subtori. Assume that the polytope P is contained in no proper subspace of V, so that T acts effectively. For $m \leq n$, let W be a $\mathbb{Q}$-vector space of dimension m, and let $M : V \to W$ be a surjection. The kernel U determines an m-parameter subgroup $S \subset T$.

The quotient $(\mathbb{V}\phi) /\!\!/ S$ by this action, with respect to the canonical linearization on $\mathcal{O}(1)$, has a residual action of the $(n-m)$-parameter torus T/S, so it ought to be a toric variety. Indeed, we find the following.

(3.2) Proposition. *Let $\phi /\!\!/ S = (U, U \cap \Lambda, U \cap P)$. Then $(\mathbb{V}\phi) /\!\!/ S = \mathbb{V}(\phi /\!\!/ S)$.*

Proof. The S-action on $k[C(\phi)]$ is restricted from the canonical action on $k[C(\psi)]$ where $\psi = (V, \Lambda, V)$. According to the correspondence between subspaces of V and subgroups of T, the algebra of S-invariants of $k[\psi]$ is $k[\psi /\!\!/ S]$. Hence the algebra of S-invariants of $k[\phi]$ is $k[\phi] \cap k[\psi /\!\!/ S] = k[\phi /\!\!/ S]$, and the geometric invariant theory quotient is $(\mathbb{V}\phi) /\!\!/ S = \operatorname{Proj} k[\phi /\!\!/ S] = \mathbb{V}(\phi /\!\!/ S)$. $\square$

In fact, with a little more work the stable and semistable points can be determined. We write $F /\!\!/ S$ for $U \cap F$.

(3.3) Lemma. (a) *The S-stable and S-semistable sets are unions of T-orbits;* (b) $\operatorname{orb} F$ *is S-semistable if and only if $F /\!\!/ S \neq \varnothing$;* (c) *the image in $\mathbb{V}\phi /\!\!/ S$ of an S-semistable orbit* $\operatorname{orb} F$ *is* $\operatorname{orb} F /\!\!/ S$*;* (d) $\operatorname{orb} F$ *is S-stable if and only if U meets the interior of F transversely.*

Proof. Part (a) is easy, since the linearized S-action extends to a linearized T-action. Part (b) then follows from part (a), **(3.2)** and **(2.13)**(e). It also follows from **(2.13)**(e) that an invariant section of $\mathcal{O}(n) \to \mathbb{V}\phi$ vanishes on $\operatorname{orb} F$ if and only if its descent vanishes on $\operatorname{orb} F /\!\!/ S$; this proves (c).

A T-orbit $\operatorname{orb} F$ is certainly S-unstable if it is stabilized by a subtorus in S. By **(2.13)**(b) this happens if and only if U is not transverse to the affine subspace generated by F. If it is transverse, however, by part (c) it is stable if and only if $G/\!\!/S \neq F/\!\!/S$ for all faces $G \neq F$. This is true if and only if U meets the interior of F. □

(3.4) Lemma. *The set* $\{\mathbb{V}\phi_F/\!\!/S \mid F \text{ is a face of } P,\ U \cap \operatorname{int} F \neq \varnothing\}$ *is the affine toric cover of* $\mathbb{V}\phi/\!\!/S$*, and stability and semistability under* S *are preserved by restriction from* $\mathbb{V}\phi$ *to such an* $\mathbb{V}\phi_F$*.*

Proof. If F is a face of P whose interior meets U, taking the point x in the definition of the localization P_F to lie in $U \cap \operatorname{int} F$ shows that intersecting with U commutes with localization at F. On the other hand, every face of $U \cap P$ is of the form $U \cap F$ for an unique face F whose interior meets U. The second statement follows easily from **(3.3)** and the remarks preceding **(2.17)**. □

Now for $v \in V$, let $P - v = \{x - v \mid x \in P\}$, and let $\phi - v = (V, \Lambda, P - v)$. By **(2.16)**, $\mathbb{V}(\phi - v) = \mathbb{V}\phi$, but with a different linearization of the torus action on $\mathcal{O}(1)$. We will write $P/\!\!/v$ for $(P - v)/\!\!/S$ and $\phi/\!\!/v$ for $(\phi - v)/\!\!/S$, so that **(3.2)** becomes $(\mathbb{V}\phi)/\!\!/v = \mathbb{V}(\phi/\!\!/v)$. The polyhedra $P/\!\!/v$ are the intersections with P of parallel affine subspaces, as if a polyhedral sausage was being put through a meat-slicer; we will sometimes call them *slices*.

(3.5) By **(2.9)**, the quotient $\mathbb{V}\phi/\!\!/v$, if nonempty, has a natural projective surjection to the affine $\mathbb{V}(\phi/\!\!/v)^0$. In $\mathbb{Q} \times V$, certainly intersecting with $\mathbb{Q} \times U$ and intersecting with $0 \times V$ commute, so $(\phi/\!\!/v)^0 = ((\phi - v)/\!\!/S)^0 = (\phi - v)^0/\!\!/S$. But $(\phi - v)^0 = \phi^0$ for any v, so $(\phi - v)^0/\!\!/S = \phi^0/\!\!/0$. Hence for all $v \in M(P)$ the quotients $\mathbb{V}\phi/\!\!/v$ are all projective over the single affine $\mathbb{V}\phi^0/\!\!/0$.

(3.6) Lemma. *The quotient* $\mathbb{V}\phi/\!\!/v$ *depends only on* $M(v) \in W$*, up to a linearization of the residual* T/S*-action; it is nonempty if and only if* $M(v) \in M(P)$*.*

Proof. The first statement follows from **(2.16)**, since if $M(v_1) = M(v_2)$, then $P/\!\!/v_1$ is a translate of $P/\!\!/v_2$. As for the second, both statements are equivalent to $P/\!\!/v \neq \varnothing$. □

So if we choose a right inverse for M, regarding W as contained in V, by the first part of the corollary, any quotient $\mathbb{V}\phi/\!\!/v$ is isomorphic to $\mathbb{V}\phi/\!\!/M(v)$, except for the linearization of the residual torus action. Since this is not very interesting, we will suppose in future that $v \in W$.

Let $\operatorname{sk}_i P$ be the *i-skeleton* of P, that is, the union of the i-dimensional faces. Then $M(P) = M(\operatorname{sk}_m P) \neq M(\operatorname{sk}_{m-1} P)$. In fact, $M(\operatorname{sk}_{m-1} P)$ is the union of a set of codimension 1 *walls*—each contained in an affine hyperplane—dividing $M(P)$ into closed *chambers*. A wall is called *external* if it is on the boundary of $M(P)$, *internal* otherwise. Notice that any face of dimension $\geq m$ projects to a union of chambers. Also, if $v \in W$ is in the interior $\operatorname{int} C$ of a chamber C, then every face of $P - v$ meeting U meets it transversely. Hence the i-dimensional faces of $P/\!\!/v$ correspond to the $(m + i)$-dimensional faces of $P - v$ meeting U, in a manner preserving intersections.

Before proving the main results of the section, we pause to prove the following, which will be needed in §4.

(3.7) Proposition. *Let $F \subset P$ be a face, C a chamber containing $M(F)$, and $v \in \operatorname{int} C$. Then there is a natural embedding $\mathbb{V}\phi_F /\!\!/ v \hookrightarrow \mathbb{V}\phi /\!\!/ v$, and stability and semistability under S are preserved by restriction from $\mathbb{V}\phi$ to $\mathbb{V}\phi_F$.*

Proof. It follows easily from the hypotheses and the definition of localization at F that for any face G of P containing F, $G_F /\!\!/ v = G /\!\!/ v$ and $\operatorname{int} G_F /\!\!/ v = \operatorname{int} G /\!\!/ v$. It therefore follows from **(3.3)** that stability and semistability under S are preserved by restriction to $\mathbb{V}\phi_F$. Moreover, since $(P_F)_G = P_G$, by **(3.4)** every element in the affine toric cover of $\mathbb{V}\phi_F /\!\!/ v$ is in that of $\mathbb{V}\phi /\!\!/ v$. The overlaps just come from intersections, so they are certainly the same. Hence $\mathbb{V}\phi_F /\!\!/ v \hookrightarrow \mathbb{V}\phi /\!\!/ v$. □

It is also worth mentioning the following interesting fact (compare [2]), which we will not need in the sequel at all, but which follows immediately from what we already know.

(3.8) Proposition. *Every quasi-projective toric variety is a geometric invariant theory quotient of k^n.*

Proof. Every polyhedron with n facets is the intersection in $\mathbb{Q}^n$ of $\mathbb{Q}^n_{\geq 0}$ with some affine subspace. □

(3.9) Theorem. *Let C be a chamber of $M(P)$, and let v vary in $\operatorname{int} C$. If any $x \in \mathbb{V}\phi$ is semistable for some v, then it is stable for all v. The quotient $\mathbb{V}\phi /\!\!/ v$ and the surjection to $\mathbb{V}\phi^0 /\!\!/ 0$ are independent of v, but the fractional polarization $\mathcal{O}(1)$ depends affinely on v.*

Proof. Since any $(m+i)$-dimensional face of P projects to a union of chambers, U meets the same faces of $P - v$ for any $v \in \operatorname{int} C$, and meets their interiors transversely. By **(3.3)**, this proves the first statement.

It also shows that the i-dimensional faces of $P /\!\!/ v$ for different v correspond in a manner preserving intersections. However, for any face $F \subset P$ such that $F - v$ meets U, $P_F /\!\!/ v$ is independent of $v \in C$, up to translation. So by **(3.4)** the affines in the toric cover of $\mathbb{V}\phi /\!\!/ v$ and their inclusions in one another (which after all are induced by inclusions of cones) are independent of v. In particular, the overlaps are independent of v, since if $H \subset P$ is the smallest face containing $F, G \subset P$, then $H /\!\!/ v$ is the smallest face containing $F /\!\!/ v$ and $G /\!\!/ v$. Since the toric affine cover and the overlaps are thus independent of v, as a variety with torus action $\mathbb{V}\phi /\!\!/ v$ is independent of v.

Likewise, the restriction to each affine $\mathbb{V}\phi_F /\!\!/ v$ of the surjection $\mathbb{V}\phi /\!\!/ v \to \mathbb{V}\phi^0 /\!\!/ 0$ of **(3.5)** is induced by the inclusion $P^0 /\!\!/ 0 \subset P_F /\!\!/ v$, which is independent of v up to translation. Hence by **(2.16)** the surjection is independent of v.

We now turn to the polarization. As we saw just after **(2.14)**, any fixed lattice point $u \in U \cap \Lambda$ determines a rational section of $\mathcal{O}(1)$ on each $\mathbb{V}\phi /\!\!/ v$ with zeroes and poles only at the Weil divisors corresponding to the facets of $P /\!\!/ v$. The latter are all of the form $F /\!\!/ v$ for F a facet of P, so $\mathcal{O}(1) = \mathcal{O}(\sum_\psi \rho_{F /\!\!/ v}(u) \mathbb{V}\psi /\!\!/ v)$, where ψ runs over the facets (V, Λ, F) of ϕ, and ρ is the order of vanishing homomorphism from **(2.14)**. But each $F-v$ is contained in an affine hyperplane in V which depends affinely on v, so the same is true of $F/\!/v \subset U$. Consequently, every $\rho_{F/\!/v}$ depends affinely on v, and hence so does $\mathcal{O}(1)$. $\square$

(3.10) Now let C_0 be a wall, and $C_\pm$ the chambers it bounds. (If C_0 is an external wall, substitute $W\backslash M(P)$ for the missing chamber.) Choose $v_+ \in \operatorname{int} C_+$, $v_0 \in \operatorname{int} C_0$, and $v_- \in \operatorname{int} C_-$. Let

$$\begin{aligned} \mathbb{V}_-\phi &= \{x \in \mathbb{V}\phi \mid x \text{ stable for } v_-, \text{ unstable for } v_+\}, \\ \mathbb{V}_0\phi &= \{x \in \mathbb{V}\phi \mid x \text{ semistable for } v_0, \text{ unstable for } v_\pm\}, \\ \mathbb{V}_+\phi &= \{x \in \mathbb{V}\phi \mid x \text{ stable for } v_+, \text{ unstable for } v_-\}, \end{aligned}$$

and for any v, let $\mathbb{V}_-\phi/\!/v$, $\mathbb{V}_0\phi/\!/v$, $\mathbb{V}_+\phi/\!/v$ be their images in the rational map $\mathbb{V}\phi \dashrightarrow \mathbb{V}\phi/\!/v$. By **(3.3)**,

$$\begin{aligned} \mathbb{V}_-\phi &= \cup\{\operatorname{orb} F \mid v_- \in M(F), v_+ \notin M(F)\}, \\ \mathbb{V}_0\phi &= \cup\{\operatorname{orb} F \mid v_0 \in M(F), v_\pm \notin M(F)\}, \\ \mathbb{V}_+\phi &= \cup\{\operatorname{orb} F \mid v_+ \in M(F), v_- \notin M(F)\}. \end{aligned}$$

(3.11) Theorem. *Let v_-, v_0, v_+ be as above. Then there are toric morphisms $\mathbb{V}\phi/\!/v_\pm \to \mathbb{V}\phi/\!/v_0$, projective over $\mathbb{V}\phi^0/\!/0$, which send $\operatorname{orb} F/\!/v_\pm$ to $\operatorname{orb} F/\!/v_0$. They are isomorphisms except possibly over $\mathbb{V}_0\phi/\!/v_0$, whose preimages are $\mathbb{V}_\pm\phi/\!/v_\pm$.*

Proof. Without loss of generality, concentrate on v_+. We will construct a toric morphism from each affine in the toric cover of $\mathbb{V}\phi/\!/v_+$ to some affine in the toric cover of $\mathbb{V}\phi/\!/v_0$, in a manner compatible with overlaps and the maps to $\mathbb{V}\phi^0/\!/0$.

By **(3.4)** the affine toric cover of $\mathbb{V}\phi/\!/v_+$ is

$$\{\mathbb{V}\phi_F/\!/v_+ \mid \ F \text{ is a face of } P, \, (\operatorname{int} P)/\!/v_+ \neq \varnothing\},$$

and similarly for $\mathbb{V}\phi/\!/v_0$. For any F such that $(\operatorname{int} F)/\!/v_+ \neq \varnothing$, let $\delta(F)$ be the minimal face such that $\delta(F)/\!/v_0 = F/\!/v_0$; then $\delta(F) \subset F$, and the following are clearly equivalent: (a) $\delta(F) \neq F$, (b) $\delta(F) = M^{-1}(C_0) \cap F$, (c) $(\operatorname{int} F)/\!/v_0 = \varnothing$, (d) $M(F)$ does not meet both C_+ and C_-.

In any case, $(\operatorname{int} \delta(F))/\!/v_0 \neq \varnothing$; as we observed earlier, this implies that $P_{\delta(F)}/\!/v_0 = (P/\!/v_0)_{\delta(F)}$. Choose $u_0 \in \delta(F)/\!/v_0$, and $u_+ \in F/\!/v_+$. Then $P_F + u_+ - u_0 = P_F$, and $P_{\delta(F)} \subset P_F$, so $P_{\delta(F)} + u_+ - u_0 \subset P_F$; moreover, it is easy to check that $P_{\delta(F)} + u_+ - u_0$ is independent of the choices of u_+ and u_0. Hence there is a natural inclusion of cones $((P_{\delta(F)})/\!/v_0) + u_+ - u_0 \subset (P_F)/\!/v_+$; this induces the desired toric morphism $\mathbb{V}\phi_F/\!/v_+ \to \mathbb{V}\phi_{\delta(F)}/\!/v_+$. Since an inclusion of cones induces an inclusion of their maximal affine subspaces, by the remark preceding **(2.17)** the toric morphism sends $\operatorname{orb} F/\!/v_+$ to $\operatorname{orb} \delta(F)/\!/v_0 = \operatorname{orb} F/\!/v_0$. Notice that if $\delta(F) = F$, then the inclusion of cones, and hence the morphism, is an isomorphism.

Since both the morphisms of the above paragraph and the inclusions of the affines of the toric cover in one another are defined by inclusions of cones, they commute. But toric affine covers are closed with respect to intersections, so all the above morphisms glue to give a morphism $\mathbb{V}\phi/\!/v_+ \to \mathbb{V}\phi/\!/v_0$, as desired.

By **(2.9)** and **(3.5)**, each of the above affines has a natural morphism to $\mathbb{V}\phi^0/\!/0$. Indeed, they are induced by the inclusions $P^0/\!/0 \subset P_F/\!/v$, which are compatible for all

F and v. All of the affine toric varieties therefore map compatibly to $\mathbb{V}\phi^0/\!\!/0$, and hence the morphism is a morphism over $\mathbb{V}\phi^0/\!\!/0$. As a morphism of projective varieties over an affine base, it is automatically projective.

The rest of the proposition follows from **(3.10)** and from the remarks at the ends of the second and third paragraphs. □

(3.12) Corollary. *If C_0 is an interior wall, then the morphisms $\mathbb{V}\phi/\!\!/v_\pm \to \mathbb{V}\phi/\!\!/v_0$ are birational.*

Proof. Since $v_\pm \in M(P)$, $\mathbb{V}_0\phi/\!\!/v_0$ does not contain the open orbit $\operatorname{orb} P/\!\!/v_0$, so the morphism is an isomorphism on an open set. □

(3.13) Theorem. *If C_0 is an interior wall, and the birational morphisms of* **(3.11)** *above are small, then $\mathbb{V}\phi/\!\!/v_+$ is a flip of $\mathbb{V}\phi/\!\!/v_-$.*

Proof. By **(3.9)**, we may choose $v_\pm$ anywhere inside their chambers, so may suppose that $v_0 - v_- = v_+ - v_0$. Let $X_- = \mathbb{V}\phi/\!\!/v_-$, $X_0 = \mathbb{V}\phi/\!\!/v_0$, and $X_+ = \mathbb{V}\phi/\!\!/v_+$. Since the morphisms $X_\pm \to X_0$ of **(3.11)** are toric, the only divisors which can possibly be contracted are those corresponding to the facets of $P/\!\!/v_\pm$. Hence the morphisms are small if and only if δ induces a bijection between the facets of $P/\!\!/v_\pm$ and those of $P/\!\!/v_0$.

Consider the line bundle $L \to X_-$ given by $\mathcal{O}_{X_0}(1) \otimes \mathcal{O}_{X_-}(-1)$, where by abuse of notation we write $\mathcal{O}_{X_0}(1)$ for its pullback by the morphism $X_- \to X_0$. Since $\mathcal{O}_{X_-}(1)$ is ample on X_-, it is relatively ample over X_0; hence so is L^{-1}. We will show that the push-forward of L to X_+ is relatively ample over X_0.

We saw just after **(2.14)** that any fixed lattice point $u \in U \cap \Lambda$ determines rational sections of $\mathcal{O}(1)$ on $X_\pm$ and X_0. Hence $L = \mathcal{O}(\sum_\psi (\rho_{F/\!\!/v_0} - \rho_{F/\!\!/v_-})(u)\mathbb{V}\psi/\!\!/v_-)$, where ψ runs over the facets (V, Λ, F) of ϕ, and ρ is the homomorphism of **(2.14)**. But since $v_0 - v_- = v_+ - v_0$ and $\rho_{F/\!\!/v}$ depends affinely on v, $\rho_{F/\!\!/v_0} - \rho_{F/\!\!/v_-} = \rho_{F/\!\!/v_+} - \rho_{F/\!\!/v_0}$. Also, by **(3.11)** the birational map $f : X_- \dashrightarrow X_+$ sends $\psi/\!\!/V_-$ to $\psi/\!\!/V_+$. Hence

$$f_*L = \mathcal{O}(\sum_\psi (\rho_{F/\!\!/v_+} - \rho_{F/\!\!/v_0})(u)\mathbb{V}\psi/\!\!/v_+) = \mathcal{O}_{X_+}(1) \otimes \mathcal{O}_{X_0}(-1).$$

But $\mathcal{O}_{X_+}(1)$ is ample, so relatively ample over X_0; hence so is f_*L. Hence all the requirements of **(1.2)** for a flip are satisfied. □

(3.14) Counterexample. A flip is uniquely determined by one contraction. But if one or both of our birational morphisms is not small, the morphism $\mathbb{V}\phi/\!\!/v_- \to \mathbb{V}\phi/\!\!/v_0$ may not uniquely determine $\mathbb{V}\phi/\!\!/v_+$, because of the appearance of new faces in the polyhedron. To see this, let $P \subset V$ be a polyhedron, and $\mu : V \to \mathbb{Q}$ a linear functional such that $\mu(P) = [0,1]$. Let $Q \subset V \times \mathbb{Q}$ be the polyhedron $\{(v,t) \mid v \in P, \mu(v) - t \geq 0\}$, let $\phi = (V \times Q, \Lambda \times \mathbb{Z}, Q)$, and let $M : V \times \mathbb{Q} \to \mathbb{Q}$ be the projection. Then $(-\infty, 0]$ is a chamber; the cross-sections $Q/\!\!/-1$ and $Q/\!\!/0$ are both P, so the morphism $\mathbb{V}\phi/\!\!/-1 \to \mathbb{V}\phi/\!\!/0$ is an isomorphism. However, for $\epsilon > 0$ $\mathbb{V}\phi/\!\!/\epsilon$ depends strongly on the choice of μ.

4 Quotients of quasi-smooth toric varieties

In this section we concentrate on the case when our toric variety is smooth, or at least quasi-smooth, which is a natural generalization to be defined shortly. In this good case, we will be able to describe the flips much more sharply, and obtain better results in the divisorial case. Our statements and proofs will be local in nature, so we will begin by characterizing smooth affine toric varieties.

(4.1) Proposition. *Any smooth affine toric variety is isomorphic to* $k^{\times n-p} \times k^p$ *for some* $p \leq n$.

Proof. Suppose $\mathbb{V}\phi$ is smooth for $\phi = (V, \Lambda, P)$ where $P \subset V$ is a cone. Without loss of generality suppose P is contained in no proper subspace of V. Let $Q \subset V$ be the maximal linear subspace contained in P, and call its codimension p. Choosing a splitting compatible with Λ of the inclusion $Q \hookrightarrow V$ yields an isomorphism $\phi \cong (Q, Q \cap \Lambda, Q) \times (V/Q, \Lambda/Q, P/Q)$, where the product of two triples is defined in the obvious way, and Λ/Q, P/Q are the images of Λ, P in V/Q. Notice that P/Q is a cone containing no subspace of V/Q, so it has a 0-dimensional face, namely its vertex. Let $\psi = (V/Q, \Lambda/Q, P/Q)$. Then $\mathbb{V}\phi \cong k^{\times n-p} \times \mathbb{V}\psi$; we will show that $\mathbb{V}\psi \cong k^p$.

Certainly $\mathbb{V}\psi$ is smooth, so in particular it is smooth at the point $\operatorname{orb} 0$ corresponding to the vertex of P. Hence if $\mathfrak{m}$ is the ideal of this point in $k[\psi]$, then $\dim \mathfrak{m}/\mathfrak{m}^2 = p$. But by **(2.6)**(e) $\mathfrak{m}$ is the free k-vector space on $P \cap \Lambda \backslash 0$, so $\mathfrak{m}^2$ is the free k-vector space on $(P \cap \Lambda \backslash 0)^2$ and $\mathfrak{m}/\mathfrak{m}^2$ is the free k-vector space on the set of points in $P \cap \Lambda \backslash 0$ which are not sums of two points in $P \cap \Lambda \backslash 0$. This certainly includes the first lattice point on every edge (that is, of P, so P must have at most p edges, hence exactly p edges. The images in $k[\psi]$ of these p lattice points consequently generate $\mathfrak{m}/\mathfrak{m}^2$, and hence $\mathfrak{m}$. So these p lattice points generate the semigroup $P \cap \Lambda$. This induces an isomorphism $V/Q \cong \mathbb{Q}^p$ identifying Λ with $\mathbb{Z}^p$ and P with $\mathbb{Q}^p_{\geq 0}$. So $\mathbb{V}\psi \cong k^p$ as desired. □

A polyhedron P is said to be *simple* if each face of codimension i is contained in exactly i faces of codimension $i+1$.

(4.2) Proposition. *A cone* P *is simple if and only if* $\mathbb{V}\phi$ *is a quotient of a smooth affine toric variety by a finite subgroup* $G \subset T$.

Proof. By **(4.1)** the cones which determine smooth affine toric varieties are certainly simple. But **(3.1)** shows that dividing by a finite subgroup of T has no effect on the polyhedron, only on the lattice.

On the other hand, suppose that P is simple, and let p be the largest codimension of a face of P. Then there exists a lattice $\Gamma \supset \Lambda$ and an isomorphism $V \to \mathbb{Q}^n$ sending Γ to $\mathbb{Z}^n$ and P to $\mathbb{Q}^{n-p} \times \mathbb{Q}^p_{\geq 0}$. Let $G = \operatorname{Hom}(\Gamma/\Lambda, k^\times)$; then $\mathbb{V}\phi$ is the quotient k^n/G in the obvious way. □

A toric variety $\mathbb{V}\phi$ is called *quasi-smooth* if the polyhedron P is simple. Notice that P is simple if and only if the (translated) cones P_F are for all faces F, so that a toric variety is quasi-smooth if and only if its affine toric cover is. Also, if P is simple, then so is every codimension m slice of P not meeting the m-skeleton of P. Hence if $\mathbb{V}\phi$ is quasi-smooth, so is the quotient $\mathbb{V}\phi/\!\!/v$ for any v in the interior of a chamber.

On a quasi-smooth affine toric variety $(k^{\times n-p} \times k^p)/G$, a *quasi-bundle* is defined to be the quotient of a bundle over $k^{\times n-p} \times k^p$ by a lifting of the G-action. On a general toric variety $\mathbb{V}\phi$, a quasi-bundle is a variety over $\mathbb{V}\phi$ whose restriction to every affine in the toric cover is a quasi-bundle in the sense above.

We will encounter some quasi-weighted projective bundles shortly, but for now, our chief example is a quasi-vector bundle, the *quasi-tangent bundle* Θ. This is defined for a quasi-smooth affine toric variety as the quotient of the tangent bundle to a smooth covering by the natural lifting of the G-action. It is independent of the choice of a smooth covering, as can be seen by passing to a common covering of any two, and hence is well-defined. For a general quasi-smooth toric variety, it is defined locally as above. The torus action on $\mathbb{V}\phi$ lifts naturally to an action on Θ. This lifting is linear in the sense that on each affine where the quasi-bundle is covered by a bundle, the action lifts to a linear action on the bundle of a finite covering of the group. (We remark that, since quasi-smooth toric varieties have only finite abelian quotient singularities, they are examples of what in manifold topology are called *orbifolds*, and the quasi-tangent bundle is nothing but the orbifold tangent bundle.)

Let us return to the situation studied in **(3.13)**: C_0 is an interior wall separating chambers C_+ and C_-, and $v_- \in \operatorname{int} C_-$, $v_0 \in \operatorname{int} C_0$, $v_+ \in \operatorname{int} C_+$. From now on, however, we assume that $\mathbb{V}\phi$ is quasi-smooth.

(4.3) Lemma. *The stabilizer in S of every point in $\mathbb{V}_0\phi$ is the one-parameter subgroup of S corresponding to C_0.*

Proof. First of all, by **(2.13)**(b) the stabilizer in T of any point in a T-orbit $\operatorname{orb} F$ corresponds to the subspace of V generated by F. Hence the stabilizer in S corresponds to the subspace of W generated by $M(F)$.

Since v_0 is on the single wall C_0, it is in $M(\operatorname{sk}_{m-1} P)$ but not $M(\operatorname{sk}_{m-2} P)$. Hence if F is any face such that $v_0 \in M(F) \subset C_0$, then $M(F)$ has dimension $m-1$ and v_0 is in its interior. In particular, $M(F)$ generates the same codimension 1 subspace as C_0. Hence for any such F, $\operatorname{orb} F$ has the same one-parameter stabilizer in S. By **(3.10)**, this includes all of $\mathbb{V}_0\phi$. □

An orientation of C_0 thus induces a fixed isomorphism of the stabilizer with $k^\times$: so choose the orientation directed toward C_+. Now restrict the quasi-tangent bundle Θ to $\mathbb{V}_0\phi$; the tangent spaces to the S-orbits form, in the obvious sense, a quasi-subbundle $E \subset \Theta|_{\mathbb{V}_0\phi}$, which has rank $m-1$ since the stabilizer is $k^\times$. Moreover, this $k^\times$ acts naturally on $\Theta|_{\mathbb{V}_0\phi}$ and on E, hence on the quotient quasi-bundle $D = \Theta|_{\mathbb{V}_0\phi}/E$ over $\mathbb{V}_0\phi$. There are well-defined weight spaces D_-, D_0, D_+ which are quasi-subbundles of D. The quotients by $k^\times$ of D_+ and D_- minus their zero sections are quasi-weighted projective bundles over $\mathbb{V}_0\phi$. The group S acts naturally on these bundles, and the stabilizer $k^\times \subset S$ of the base acts trivially; they therefore descend to quasi-weighted projective bundles B_+ and B_- over $\mathbb{V}\phi/\!\!/v_0$.

We need to introduce one more notion before stating our main theorem. Suppose first that P is a simplicial cone, so that $\mathbb{V}\phi$ is smooth and affine. Let $k[\phi]^i$ be the ith weight space for the action of the above-mentioned stabilizer $k^\times$. Define the qth *weighted ideal* $W^q\Im$ to be

$$\sum_{\substack{i,j\in\mathbb{Z}\\ j\geq q}} k[\phi]^i \cdot k[\phi]^{i+j}$$

and the qth *weighted ideal sheaf*, also called $W^q\Im$, to be the corresponding ideal sheaf on $\mathbb{V}\phi$; it is supported on $\mathbb{V}_0\phi$. Equivalently, the weights of the $k^\times$-action define a homomorphism $\rho : \Lambda \to \mathbb{Z}$. Define another map $\tau : \Lambda \to \mathbb{N}$ by

$$\tau(v) = \max\{\rho(v_1) - \rho(v_2) \,|\, v_1, v_2 \in P \cap \Lambda, v_1 + v_2 = v\}.$$

Then τ is also a homomorphism, and $W^q\Im$ is the free vector space on $\{v \in P \cap \Lambda \,|\, \tau(v) \geq q\}$.

For an arbitrary quasi-smooth $\mathbb{V}\phi$, $W^q\Im$ is defined like a quasi-bundle: on each affine in a toric cover, it is the descent of $W^{pq}\Im$ from a smooth finite toric covering, where p is the degree of the induced cyclic covering of $k^\times$.

Since $W^q\Im$ is supported on $\mathbb{V}_0\phi$, it descends to an ideal sheaf supported on $\mathbb{V}_0\phi/\!\!/v_0 \subset \mathbb{V}\phi/\!\!/v_0$. By **(3.11)** the pullbacks of this ideal sheaf to $\mathbb{V}\phi/\!\!/v_+$ and $\mathbb{V}\phi/\!\!/v_-$ are supported on $\mathbb{V}_+\phi/\!\!/v_+$ and $\mathbb{V}_-\phi/\!\!/v_-$; call them simply $W^q\Im_+$ and $W^q\Im_-$, respectively.

(4.4) Lemma. *Let $\phi = (V, \Lambda, P)$ for P a polyhedral cone, so that $\mathbb{V}\phi$ is affine, and let a 1-parameter subgroup $S \subset T$ act. Then for q sufficiently divisible, $(W^q\Im_\pm)^p = W^{pq}\Im_\pm$ for all $p \in \mathbb{N}$.*

Here by *sufficiently divisible* we mean that q has some desired set of factors. This will be true, for example, for the factorial of any sufficiently large number.

Proof. In this case W is one-dimensional, and there is only one wall, at 0. The descent of $W^q\Im$ in the rational map $\mathbb{V}\phi \dashrightarrow \mathbb{V}\phi/\!\!/0$ is the free vector space on $\{v \in P/\!\!/0 \cap \Lambda \,|\, \tau(v) \geq q\}$. Since $W^q\Im_\pm$ are the pullbacks of this, it suffices to show that, for q sufficiently divisible, every $v \in P/\!\!/0 \cap \Lambda$ with $\tau(v) \geq pq$ is the sum of p elements $w_i \in P/\!\!/0 \cap \Lambda$ with $\tau(w_i) \geq q$.

This is a variation on the proof of Gordan's lemma **(2.1)**. Choose a finite set $S \subset U$ such that $P/\!\!/0 = \{\sum x_i v_i \,|\, x_i \in \mathbb{Q}_{\geq 0}, v_i \in S\}$. Without loss of generality we may also suppose that $S \subset U \cap \Lambda$, and that $\tau(v_i) = 0$ or r for some fixed r. Let $q = r|S|$. Then for $\tau(\sum x_i v_i) \geq pq$, write $\sum x_i v_i = \sum(x_i - [x_i])v_i + \sum[x_i]v_i$; then $\tau(\sum(x_i - [x_i])v_i) < q$, so $\tau(\sum[x_i]v_i) > q(p-1)$. By our assumptions on the v_i, we may write $\sum[x_i]v_i = w_1 + \cdots + w_{p-1} + u$, where $\tau(w_1) = \cdots = \tau(w_{p_1}) = q$. Put $w_p = u + \sum(x_i - [x_i])v_i$; then $\tau(w_p) \geq q$, and $\sum x_i v_i = w_1 + \cdots + w_p$ as desired. □

We finally arrive at the main result of the paper.

(4.5) Theorem. (a) *There are natural isomorphisms $\mathbb{V}_+\phi/\!\!/v_+ \cong B_+$ and $\mathbb{V}_-\phi/\!\!/v_- \cong B_-$ compatible with the natural maps to $\mathbb{V}_0\phi/\!\!/v_0$;* (b) *if B_+ (say) has codimension* 1 *in $\mathbb{V}\phi/\!\!/v_+$, then $\mathbb{V}\phi/\!\!/v_- \to \mathbb{V}\phi/\!\!/v_0$ is an isomorphism and, for q sufficiently divisible, $\mathbb{V}\phi/\!\!/v_+ \to \mathbb{V}\phi/\!\!/v_0$ is the blow-up at $W^q\Im_-$;* (c) *if C_0 is an internal wall, then for q sufficiently divisible, the blow-up of $W^q\Im_-$ in $\mathbb{V}\phi/\!\!/v_-$ is naturally isomorphic to the blow-up of $W^q\Im_+$ in $\mathbb{V}\phi/\!\!/v_+$.*

Proof. We first make a series of reductions, then prove the theorem in the very simple case that results.

First, by **(3.9)** we may assume, without affecting the quotients, that $v_+ - v_0 = v_0 - v_-$. Let $L \subset W$ be the line generated by this element. This is complementary to the subspace of W generated by C_0; it therefore determines a subgroup $R \subset S$ complementary to the stabilizer $k^\times$ of $\mathbb{V}_0\phi$ mentioned above. The linearizations v_+, v_0, and v_- agree on the R-factor, so the corresponding S-quotients are all $k^\times$-quotients of a single variety $\mathbb{V}\phi/\!\!/R$. Similarly, every space and map mentioned in the theorem factors through $\mathbb{V}\phi/\!\!/R$; for example, $\mathbb{V}_\pm(\phi/\!\!/R) = (\mathbb{V}_\pm\phi)/\!\!/R$. The only exception is that $W^q\Im$ on the quotient is actually the descent of $W^{pq}\Im$, where $p = |R \cap k^\times|$, but this makes no difference to the statement. Hence we may replace S by $S/R = k^\times$, and so suppose that S is one-dimensional.

Second, by **(3.4)** and **(3.7)**, if $\operatorname{int} F - v_0$ meets U, then restriction to the affine $\mathbb{V}\phi_F$ preserves the stability and semistability conditions determined by $v_\pm$ and v_0; hence $\mathbb{V}_\pm\phi \cap \mathbb{V}\phi_F = \mathbb{V}_\pm\phi_F$ and $\mathbb{V}_0\phi \cap \mathbb{V}\phi_F = \mathbb{V}_0\phi_F$. Also by **(3.4)** and **(3.7)**, for F as above there exist natural embeddings $\mathbb{V}\phi_F/\!\!/v_\pm \hookrightarrow \mathbb{V}\phi/\!\!/v_\pm$ and $\mathbb{V}\phi_F/\!\!/v_0 \hookrightarrow \mathbb{V}\phi/\!\!/v_0$, and the images of the latter form the affine toric cover of $\mathbb{V}\phi/\!\!/v_0$. By naturality these embeddings commute with the projective contractions of **(3.11)**, so $\mathbb{V}\phi_F/\!\!/v_\pm$ are just the inverse images of $\mathbb{V}\phi_F/\!\!/v_0$ in the contractions $\mathbb{V}\phi/\!\!/v_\pm \to \mathbb{V}\phi/\!\!/v_0$. But the definitions of the weighted ideal sheaves and quasi-bundles, and hence the conclusions of the theorem, are purely local over the quotient $\mathbb{V}\phi/\!\!/v_0$. Hence we may suppose that $\mathbb{V}\phi$ is affine as well as quasi-smooth, so that P is a simple cone.

Third, by **(3.1)** and **(3.3)**, dividing by a finite group $G \subset T$ preserves the stability and semistability conditions determined by $v_\pm$ and v_0, so $\mathbb{V}_\pm(\phi/G) = (\mathbb{V}_\pm\phi)/G$ and $\mathbb{V}_0(\phi/G) = (\mathbb{V}_0\phi)/G$. But the definitions of the weighted ideal sheaves and quasi-bundles, and the conclusions of the theorem, since they are defined in terms of arbitrary smooth finite toric covers, are invariant under such quotients, except again that $W^q\Im$ may be the descent of $W^{pq}\Im$. In particular, this implies that, for q sufficiently divisible, blowing up $W^q\Im_\pm$ commutes with dividing by G, since by **(4.4)** the blow-up is $\operatorname{Proj}\sum_p (W^q\Im_\pm)^p = \operatorname{Proj}\sum_p W^{pq}\Im_\pm$. Hence, passing to a finite cover, we may suppose that $\mathbb{V}\phi$ is actually smooth, so isomorphic to $k^{\times\, n-p} \times k^p$ for some p, and that the S-action corresponds to a diagonal action of $k^\times$ on $k^{\times\, n-p} \times k^p$. By passing to a further cover, we may also suppose that the weights of the $k^\times$-action are all -1, 0, or 1.

Fourth, if any factor isomorphic to $k^\times$ is acted on with weight ± 1, then U is not transverse to the maximal subspace $Q \subset P$, so the polyhedra $P/\!\!/v$ for different v are just translates of one another, and all the quotients $\mathbb{V}\phi/\!\!/k^\times$ are naturally isomorphic; the theorem is hence vacuous. On the other hand, if any factors, say $k^{\times q} \times k^r$, are acted on with weight 0, then for any splitting $k^{\times\, n-p} \times k^p \cong (k^{\times q} \times k^r) \times (k^{\times\, n-p-q} \times k^{p-r})$, the quotients with respect to any linearization satisfy $k^{\times\, n-p} \times k^p/\!\!/k^\times \cong (k^{\times\, n-p-q} \times k^{p-r}/\!\!/k^\times) \times (k^{\times q} \times k^r)$. Moreover, any two splittings induce the same splitting of the quotient. Similarly, every space and map mentioned in the theorem splits off a trivial factor of $k^{\times q} \times k^r$. Hence we may suppose that $\mathbb{V}\phi \cong k^n$, and that the S-action corresponds to a diagonal $k^\times$-action with weights ± 1.

At last we are in the simple case that was promised in the beginning of the proof: $(V, \Lambda, P) \cong (\mathbb{Q}^n, \mathbb{Z}^n, \mathbb{Q}^n_{\geq 0})$, so that $\mathbb{V}\phi \cong k^n$, and $M : V \to \mathbb{Q}$ given with respect to the standard basis $\{e_i\}_{i=1}^n$ for $\mathbb{Q}^n$ by $M(e_i) = 1$ if $1 \leq i \leq p$, $M(e_i) = -1$ if $p < i \leq n$. As a right inverse for M, take $1 \mapsto e_n$. Since $M(\mathrm{sk}_0 P) = 0$, the chambers are simply $(-\infty, 0]$ and $[0, \infty)$; let $v_- = -1$, $v_0 = 0$, and $v_+ = 1$. By **(3.10)**, $\mathbb{V}_0\phi$ is the single point $\mathrm{orb}\, 0$ corresponding to the vertex of P, $\mathbb{V}_+\phi$ is the toric subvariety isomorphic to k^p corresponding to the face P_+ spanned by e_i for $1 \leq i \leq p$, and $\mathbb{V}_-\phi$ is the toric subvariety isomorphic to k^{n-p} corresponding to the face P_- spanned by e_i for $p < i \leq n$. Notice that $\mathbb{V}_+\phi$ and $\mathbb{V}_-\phi$ meet transversely at $\mathbb{V}_0\phi$, and that $\mathbb{V}_\pm\phi$ is acted on homogeneously by $S \cong k^\times$ with weight ± 1. Hence $\mathbb{V}_\pm\phi \cong D_\pm$ naturally up to scalars. But $\mathbb{V}_+\phi /\!\!/ 1 \subset \mathbb{V}\phi /\!\!/ 1$ corresponds to the face $P_+ /\!\!/ 1$, which is a standard simplex spanned by $\{e_i - e_n \,|\, 1 \leq i \leq p\}$, so $\mathbb{V}_+\phi /\!\!/ 1 \cong \mathbb{P}^{p-1}$: it is naturally the quotient B_+ of $\mathbb{V}_+\phi \backslash \mathbb{V}_0\phi \cong D_+ \backslash 0$ by the homogeneous $S \cong k^\times$-action. Similarly $\mathbb{V}_-\phi /\!\!/ -1 \cong \mathbb{P}^{n-p-1}$ is the quotient B_- of $\mathbb{V}_-\phi \backslash \mathbb{V}_0\phi \cong D_- \backslash 0$ by $S \cong k^\times$. This proves (a).

Note that B_+ has codimension 1 if and only if $p = n-1$; then $P /\!\!/ -1$ and $P /\!\!/ 0$ are both the standard simplicial cone generated by $\{e_i - e_n \,|\, 1 \leq i \leq n\}$, so $\mathbb{V}\phi /\!\!/ -1 \cong \mathbb{V}\phi /\!\!/ 0$. Moreover, B_- is a single point, corresponding to $0 \in P /\!\!/ -1$, and $P /\!\!/ 1$ is the polyhedron in U generated by $(P /\!\!/ 0) \cap \Lambda \backslash 0$. Hence we are in the situation of **(2.11)**, and $\mathbb{V}\phi /\!\!/ 1$ is the blow-up of $\mathbb{V}\phi /\!\!/ -1$ at B_-; this proves (b).

Now let $\phi' = (\mathbb{Q} \times V, \mathbb{Z} \times \Lambda, P')$ with $P' = \mathbb{Q}_{\geq 0} \times P$, and let $M' : \mathbb{Q} \times V \to \mathbb{Q}^2$ be given with respect to the standard basis $\{e_i\}_{i=0}^n$ for $\mathbb{Q} \times \mathbb{Q}^n$ by $M'(e_0) = (1, 0)$, $M'(e_i) = (-1, M(e_i))$ for $i > 0$. Then $\phi' /\!\!/ (-1, \pm 1) = \phi /\!\!/ \pm 1$, but $P' /\!\!/ (-2, \pm 1)$ are the polyhedra generated by $P /\!\!/ \pm 1 \setminus P_\pm /\!\!/ \pm 1$. Since $M'(\mathrm{sk}_1 P')$ consists of the three rays from the origin through $(1, 0)$, $(-1, 1)$, and $(-1, -1)$, the points $(-2, 1)$ and $(-2, -1)$ lie in the same chamber, so by **(3.9)** $\mathbb{V}\phi' /\!\!/ (-2, 1) = \mathbb{V}\phi' /\!\!/ (-2, -1)$. On the other hand, by **(3.11)** there are contractions $\mathbb{V}\phi' /\!\!/ (-2, \pm 1) \to \mathbb{V}\phi /\!\!/ \pm 1$. We will show that these are ordinary blow-ups at $\mathbb{V}_\pm\phi /\!\!/ \pm 1$.

Any face $F \subset P_+$ is spanned by elements of the form $e_i - e_n$; fix one such element $e_j - e_n$. Let $X \subset U$ be the subspace generated by F. Also let Q, R be the standard simplicial cones spanned respectively by $\{e_i - e_j \,|\, 1 \leq i \leq p, e_i - e_n \notin F\}$ and $\{e_i + e_j \,|\, p < i \leq n\}$, and let $Y, Z \subset U$ be the subspaces they generate. Then up to translation by $e_j - e_n$,

$$\phi_F /\!\!/ 1 = (X, X \cap \Lambda, X) \times (Y, Y \cap \Lambda, Q) \times (Z, Z \cap \Lambda, R).$$

With respect to this splitting, $(P_+)_F$ corresponds to $X \times Q \times 0$, so the inverse image of $\mathbb{V}\phi_F /\!\!/ 1$ in the contraction $\mathbb{V}\phi' /\!\!/ (-2, 1) \to \mathbb{V}\phi /\!\!/ 1$ is $\mathbb{V}\psi$ for

$$\psi = (X, X \cap \Lambda, X) \times (Y, Y \cap \Lambda, Q) \times (Z, Z \cap \Lambda, S),$$

where S is the polyhedron in Z generated by $R \cap \Lambda \backslash 0$. Since R is a standard simplicial cone, this is just the example of **(2.11)** crossed with $(X, X \cap \Lambda, X) \times (Y, Y \cap \Lambda, Q)$. So over $\mathbb{V}\phi_F /\!\!/ 1$, the contraction is the blow-up at $\mathbb{V}_+\phi_F /\!\!/ 1$. Hence globally the contraction is the blow-up at $\mathbb{V}_+\phi /\!\!/ 1$. Similarly, the contraction $\mathbb{V}\phi' /\!\!/ (-2, -1) \to \mathbb{V}\phi /\!\!/ -1$ is the blow-up at $\mathbb{V}_-\phi /\!\!/ -1$.

But the qth weighted ideal on P is the free vector space on $\{v \in P \cap \Lambda \,|\, \tau(v) \geq q\}$ where, if v corresponds to $(v_i) \in \mathbb{Z}^n$, $\tau(v) = \sum_i v_i$. So with respect to the above splitting of $\phi_F /\!\!/ 1$, τ is independent of the first two factors, and is given on the third by $\tau(e_i + e_j) = 1$ (not 2 because of the translation by $(e_j - e_n)$); hence $W^q \Im_+$ is pulled back from the qth power of the ordinary ideal sheaf on the third factor. Hence the blow-up of $W^q\Im_+$ is just the ordinary blow-up at $\mathbb{V}_+\phi/\!\!/1$. Similarly, the blow-up of $W^q\Im_-$ is the ordinary blow-up at $\mathbb{V}_-\phi/\!\!/-1$. This completes the proof of (c). □

(4.6) Corollary. *If C_0 is an external wall of a chamber C_+, then $\mathbb{V}\phi/\!\!/v_+ = B_+$.*

Proof. In this case $\mathbb{V}_-\phi = \varnothing$ and $\mathbb{V}_+\phi$ contains the v_+-semistable set, so $\mathbb{V}\phi/\!\!/v_+ = \mathbb{V}_+\phi/\!\!/v_+$. □

Hence for a chamber having an external wall, the corresponding quotient has the structure of a quasi-weighted projective bundle.

(4.7) Example. To conclude, we shall work out an entertaining example which was alluded to at the end of [13]. Let $\phi = (\mathbb{Q}^n, \mathbb{Z}^n, [0,1]^n)$, so that $\mathbb{V}\phi = (\mathbb{P}^1)^n$. Let $M : \mathbb{Q}^n \to \mathbb{Q}$ be given by $(x_i) \mapsto \sum x_i$, determining a diagonal action of $k^\times$ on $(\mathbb{P}^1)^n$.

Since $\mathrm{sk}_0\, P = \{0,1\}^n$, $M(\mathrm{sk}_0\, P) = \{0, 1, \ldots, n\}$. Choose v_0 in this set; then $\mathbb{V}_0\phi = \cup\{\mathrm{orb}(x_i) \mid (x_i) \in \{0,1\}^n, \sum x_i = v_0\}$, and so consists of the $\binom{n}{v_0}$ points in $(\mathbb{P}^1)^n$ with v_0 coordinates equal to 0 and the rest equal to ∞. The weight spaces $D_\pm$ therefore have pure weight ± 1 and dimension $n - v_0$ and v_0, respectively. So the weighted projective bundles $B_\pm$ are just $\binom{n}{v_0}$ disjoint copies of $\mathbb{P}^{n-v_0-1}$ and $\mathbb{P}^{v_0-1}$, respectively. For reasons like those in the simple example of the proof, the weighted ideal sheaves are just powers of the ordinary ideal sheaves of $\mathbb{V}_\pm\phi/\!\!/v_\pm$.

The theorem therefore tells us the following. First, taking $v_0 = 0$ or n, and using **(4.6)**, we find that $\mathbb{V}\phi/\!\!/1/2 \cong \mathbb{P}^{n-1}$ and $\mathbb{V}\phi/\!\!/(n-1/2) \cong \mathbb{P}^{n-1}$. Second, taking $v_0 = 1$ or $n-1$, we find that $\mathbb{V}\phi/\!\!/3/2$ is the blow-up of $\mathbb{V}\phi/\!\!/1/2$ at n points, and similarly for $\mathbb{V}\phi/\!\!/(n-3/2)$ and $\mathbb{V}\phi/\!\!/(n-1/2)$. Third, taking n between 2 and $n-2$, we find that the blow-up of $\mathbb{V}\phi/\!\!/(v_0 - 1/2)$ along B_- is isomorphic to the blow-up of $\mathbb{V}\phi/\!\!/(v_0 + 1/2)$ along B_+.

In fact, B_- is just the proper transform of the (v_0-1)-dimensional orbits in $\mathbb{V}\phi/\!\!/1/2 \cong \mathbb{P}^{n-1}$. This follows directly from **(3.11)**, because both come from the v_0-dimensional faces of P containing 0. A similar statement holds for B_+ and the $(n-v_0-1)$-dimensional orbits in $\mathbb{V}\phi/\!\!/(n-1/2)$. It follows that the contractions in these flips do not always reduce Picard numbers by 1. For example, when $n = 4$, consider the flip at $v_0 = 2$. The quotients $\mathbb{V}\phi/\!\!/3/2$ and $\mathbb{V}\phi/\!\!/5/2$ both have Picard number 5, since they are blow-ups of $\mathbb{P}^4$ at 4 points. But using the description just given of B_-, it is easy to write down the condition imposed by each of the 6 components for a line bundle to descend from $\mathbb{V}\phi/\!\!/3/2$ to $\mathbb{V}\phi/\!\!/2$, and to show that 4 of them are independent. Hence $\mathbb{V}\phi/\!\!/2$ has Picard number 1.

A little more work would prove one last thing: that the birational map $\mathbb{V}\phi/\!\!/1/2 \cong \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1} \cong \mathbb{V}\phi/\!\!/(n-1/2)$ can be given in coordinates as $[z_i] \mapsto [1/z_i]$. Consequently, our sequence of flips factors this Cremona transformation, generalizing the famous factorization of the Cremona transformation on $\mathbb{P}^2$ [5, V 4.2.3]. We leave this to the reader. Hint: with respect to the basis $\{e_j - e_n \mid j < n\}$ for U, $P/\!\!/1/2$ is a translate of the standard simplex of **(2.10)**; with respect to $\{e_n - e_j \mid j < n\}$, the same is true of $P/\!\!/(n-1/2)$.

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