An introduction to the topology of the moduli space of stable bundles on a Riemann surface

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The moduli spaces of stable bundles on a Riemann surface have been so exhaustively studied and discussed in recent years that one cannot help wondering what is new to say about them. However, the present paper will seek, not to present new results, but to illuminate old ones from a slightly new angle. It closely follows expository lectures given at the 1995 summer school on Geometry and Physics in Odense, which attempted to explain not only the main results on the topology of the moduli spaces, but also the simplest and least technical proofs of those results. To make this feasible in a course of four lectures, the rank was assumed to be 2, and the construction of the moduli spaces, as well as the Narasimhan-Seshadri theorem, were taken for granted. This allowed each result to be proved from either the holomorphic or the symplectic point of view, depending on convenience, which greatly simplified the presentation. In this context, however, it often turned out that the simplest proof of a theorem is not the most familiar one. In particular, the notion of a connection is not necessary and is never mentioned. The reader should therefore think of the present approach as complementing, in a very small way, the magisterial work of Atiyah and Bott [4], where connections play a leading role.

The paper begins by introducing the theory of stable bundles from the point of view of algebraic geometry. The rigors of the construction of the moduli space are omitted, but two major results are fully proved by algebraic methods: Grothendieck's classification of vector bundles on $\mathbb{CP}^1$, and Atiyah and Bott's theorem giving generators for the cohomology ring of the moduli space.

However, the reader unfamiliar with algebraic geometry is urged to persevere, since the remaining sections are mostly symplectic and topological, and use only a few key facts from the early sections. The link between the algebraic and topological points of view is provided by the Narasimhan-Seshadri theorem, which allows stable bundles to be identified, in a certain sense, with representations of the fundamental group of the Riemann surface.

This theorem is stated in general, but afterwards the paper focuses on the topology of the moduli space in the special case of rank 2 and degree 1. In fact, two moduli spaces become involved: the original moduli space M^g, and the subspace N^g consisting of bundles with fixed determinant, which in some respects is more fundamental. Using the simplest methods, and proving as much as possible, formulas are derived for the Betti numbers, the cohomology pairings, and the Hilbert polynomials of these moduli spaces—the latter being the rank 2, degree 1 case of the celebrated Verlinde formula. The paper concludes by stating the presentation of the cohomology ring that has recently been derived using methods akin to those described here.

The notation in this paper is straightforward and should cause no confusion, with the possible exception of cohomology H^*. This is used to denote three different things. First, if Z is a space and R a ring, $H^*(Z, R)$ denotes cohomology with coefficients in R. If no ring is named, $\mathbb{Q}$ is understood. Second, if G is a group and ρ a module, $H^*(G, \rho)$, or just $H^*(\rho)$ for short, denotes group cohomology with coefficients in ρ. Finally, if M is a complex manifold and $\mathcal{S}$ a sheaf, $H^*(M, \mathcal{S})$, or $H^*(\mathcal{S})$ for short, denotes holomorphic cohomology with coefficients in $\mathcal{S}$. Ideally, no confusion should result despite the abuse of notation.

One more notational remark only: recall that the Pauli spin matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Basic observations

Fix a compact Riemann surface X of genus g. To any complex vector bundle E over X is associated an integer, the *degree* $\deg E = c_1(E)[X]$. This integer actually gives a complete topological classification of complex vector bundles on X.

Proposition. *Topological vector bundles over X are classified up to isomorphism by their rank and degree.*

Proof. Isomorphism classes of rank r bundles correspond to homotopy classes of maps from X to the classifying space $\mathrm{BGL}(r, \mathbb{C}) = \mathrm{BU}(r)$, which is an infinite complex Grassmannian. By the cellular approximation theorem [51, p. 404], such maps and their homotopies are homotopic to maps and homotopies with image in the 3-skeleton of $\mathrm{BU}(r)$, which is $\mathbb{CP}^1 \subset \mathbb{CP}^r$. But maps $X \to \mathbb{CP}^1$ are determined up to homotopy by their degree. $\square$

The topological classification of bundles is thus accomplished with a discrete invariant. However, the classification of holomorphic bundles is not so simple, and continuous parameters are involved. For example, it was known classically that isomorphism classes of degree 0 holomorphic line bundles are parametrized by the *Jacobian torus* $\mathrm{Jac}\, X = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$. The complex structure on the Jacobian is the right one in the sense that, given a complex manifold T and a holomorphic line bundle over $T \times X$ whose restriction to $t \times X$ has degree 0 for $t \in T$, the induced map $T \to \mathrm{Jac}\, X$ is holomorphic. The Jacobian is therefore said to be a *moduli space* of degree 0 line bundles on X, in the holomorphic category. (This rough definition is adequate for the purposes of gauge theory, but a proper algebro-geometric definition is slightly more delicate; see [47, 49].) With this elegant example to whet the appetite, one is tempted to look for moduli spaces of holomorphic vector bundles of higher rank.

The jump phenomenon

It is clear from the outset, however, that the general picture will not be so simple. More precisely, there is in general no Hausdorff moduli space of all holomorphic vector bundles of a given rank and degree, for the following reason.

Let L be a line bundle over X of positive degree. By Riemann-Roch, $H^1(X, L^{-1}) \neq 0$, so let $V \subset H^1(X, L^{-1})$ be any 1-dimensional subspace. Certainly $V^* \subset H^0(V, \mathcal{O})$, so the identity determines a natural class

$$I \in V^* \otimes V \subset H^0(V, \mathcal{O}) \otimes H^1(X, L^{-1}) \subset H^1(V \times X, \pi_2^* L^{-1}) = \operatorname{Ext}^1(V \times X; \mathcal{O}, \pi_2^* L),$$

where π_2 is the projection on X. There is hence a natural extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \pi_2^* L \longrightarrow 0$$

of line bundles on $V \times X$ such that $\mathcal{E}|_{t \times X}$ is the extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E_t \longrightarrow L \longrightarrow 0$$

determined by $t \in V$. For $t, t' \neq 0$, $E_t \cong E_{t'}$, but certainly $E_t \not\cong E_0 = \mathcal{O} \oplus L$ in general. For example, there is a tautological sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

on $\mathbb{CP}^1$, but $\mathcal{O}(-1) \oplus \mathcal{O}(1) \not\cong \mathcal{O} \oplus \mathcal{O}$ because there is no holomorphic map $\mathcal{O}(1) \to \mathcal{O}$.

So there is a family $\{E_t \,|\, t \in V\}$ of rank 2 bundles over X which for nonzero t are all isomorphic, but different from E_0. This is the so-called *jump phenomenon*; it reveals that, in any moduli space of bundles, E_0 must be in the closure of E_t, so the moduli space cannot be Hausdorff.

Stable bundles

There are at least two ways around the problem. One is to replace the notion of a moduli space with that of a moduli *stack*; this is a far-reaching generalization of an algebraic variety, in the spirit of schemes or algebraic spaces. The stack of vector bundles can in some sense be shown to be smooth, and much of the machinery of algebraic geometry can be generalized to stacks, but the subject is inevitably very technical and much better known to, say, number theorists than to gauge theorists; see Vistoli [56] or Faltings and Chai [20] for this point of view. A second approach, historically the older of the two, and more compatible with gauge theory, is to exclude a few *unstable* bundles having bad properties.

Definition. A holomorphic bundle E over X is *stable* (resp. *semistable*) if for all proper holomorphic subbundles $F \subset E$,

$$\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E} \quad (\text{resp.} \leq).$$

For example, the bundle E_0 of the previous section is destabilized by L.

Lemma ("Stable implies simple"). *If E, E' are stable of the same rank and degree, then $H^0(\operatorname{Hom}(E, E')) = \mathbb{C}$ if $E \cong E'$, 0 otherwise.*

Proof. If $E \not\cong E'$ and $0 \neq f \in H^0(\operatorname{Hom}(E, E'))$, then $\ker f$ and $\operatorname{im} f$ are proper coherent subsheaves of E and E', and either $\deg \ker f / \operatorname{rank} \ker f$ or $\deg \operatorname{im} f / \operatorname{rank} \operatorname{im} f$ is greater than or equal to $\deg E/\operatorname{rank} E$. The minimal subbundles of E and E' containing these subsheaves have the same rank and at least the same degree; hence one of them is destabilizing. Similarly, if $g : E \to E'$ is an isomorphism, the same argument applied to $f - \lambda g$ shows that it must always have maximal or zero rank, so $f = \lambda g$ for some $\lambda \in \mathbb{C}$. $\square$

Analogous to the Jordan-Hölder decomposition in group theory, a semistable bundle has a *composition series* or *Harder-Narasimhan filtration*

$$E = E_0 \supset E_1 \supset E_2 \supset \cdots \supset 0$$

of destabilizing bundles of maximal rank. Like the Jordan-Hölder decomposition, it is unique up to reordering, that is, $\operatorname{Gr} E = \bigoplus_i E_i/E_{i+1}$ is well-defined up to isomorphism. Two semistable bundles E, E' are said to be *S-equivalent* if $\operatorname{Gr} E \cong \operatorname{Gr} E'$. Note that if E is stable, then $\operatorname{Gr} E \cong E$; hence stable bundles are S-equivalent if and only if $E \cong E'$.

It turns out that discarding the unstable bundles and identifying the S-equivalent semistable bundles is enough to overcome problems like the jump phenomenon and produce a good moduli space.

Theorem. *For fixed X, r, d, there exists a connected moduli space M^g of S-equivalence classes of rank r, degree d semistable bundles over X, which is a complex projective variety, having dimension $r^2(g-1)+1$ when $g \geq 2$.*

This theorem, proved by Mumford [40] in the early 1960's, is the foundation not only of these lectures but of a vast amount of work on bundles over Riemann surfaces. However, the proof is technical and relies on several deep results and methods in algebraic geometry, so we content ourselves with the following.

Outline of proof. Let $d >> 0$. Then every semistable E of degree d has $H^1(E) = 0$, since for any nonzero $f \in H^0(\operatorname{Hom}(E, K_X))$, the minimal bundle containing $\ker f$ is destabilizing. Hence by Riemann-Roch, $\dim H^0(E) = \chi(E) = d + r(1-g)$. By general results of Grothendieck [25], there exists a projective variety, the so-called *Quot scheme*, parametrizing surjections $\mathcal{O}^\chi \to E \to 0$, where E is a coherent sheaf over X of rank r and degree d. The group $\mathrm{GL}(\chi, \mathbb{C})$ acts naturally on $\mathcal{O}^\chi$ and hence on Quot. Since this is a reductive group, the fundamental theorem of geometric invariant theory [42, 47] implies that there are Zariski open *stable* and *semistable* subsets $\mathrm{Quot}^s \subset \mathrm{Quot}^{ss} \subset \mathrm{Quot}$ such that $\mathrm{Quot}^{ss}/\mathrm{GL}(\chi, \mathbb{C})$ is a polarized projective variety and $\mathrm{Quot}^s/\mathrm{GL}(\chi, \mathbb{C}) \subset \mathrm{Quot}^{ss}/\mathrm{GL}(\chi, \mathbb{C})$ is an orbit space. The Hilbert-Mumford numerical criterion [42, 47] allows the stable and semistable subsets to be computed explicitly, and they turn out to coincide exactly with the loci where E is a stable or semistable bundle.

The above account begs the question why the Quot scheme exists, since it can itself be viewed as a moduli space, parametrizing pairs consisting of a sheaf over X together with a surjection from $\mathcal{O}^\chi$. Very roughly, there exists yet another variety, the *Hilbert scheme*, parametrizing curves of genus g and degree d in the Grassmannian $\mathrm{Gr}_{\chi-r}(\mathbb{C}^\chi)$, and the Quot scheme is the closure of the locus of curves isomorphic to X. Certainly if the curve is isomorphic to X, the pull-back of the tautological sequence on the Grassmannian gives a surjection $\mathcal{O}^\chi \to E$ of the desired kind. We must leave all further details on Quot schemes to Grothendieck [25], and on the existence of Hilbert schemes to Mumford [41]. A readable sketch of the latter topic can be found in Harris [28]. $\square$

Observations and remarks

(1) If E is stable (resp. semistable), then so is $E \otimes L$ for any line bundle L. Hence tensoring by L induces an isomorphism between the moduli spaces in degree d and $d + r \deg L$. Consequently, the moduli space M^g depends only on the residue class of d modulo r.

(2) As suggested in the outline above, the open set in the Quot scheme where E is a bundle can be identified with the holomorphic maps of degree d from X to a Grassmannian. Hence for f in this open set, the tangent space $T_f \,\mathrm{Quot}$ is naturally isomorphic to $H^0(\mathrm{Hom}(\ker f, E))$. Now there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathrm{Hom}(E,E)) \longrightarrow H^0(\mathrm{Hom}(\mathcal{O}^\chi, E)) \longrightarrow H^0(\mathrm{Hom}(\ker f, E)) \\ \longrightarrow H^1(\mathrm{Hom}(E,E)) \longrightarrow H^1(\mathrm{Hom}(\mathcal{O}^\chi, E)) \longrightarrow \cdots. \end{aligned}$$

The last term is $\mathbb{C}^\chi \otimes H^1(E)$, which is 0 for $d >> 0$, the second is $\mathrm{Hom}(\mathbb{C}^\chi, \mathbb{C}^\chi) = \mathfrak{gl}(\chi, \mathbb{C})$, and the third is $T_f \,\mathrm{Quot}$. Moreover it is not hard to check that the map $\mathfrak{gl}(\chi, \mathbb{C}) \to T_f \,\mathrm{Quot}$ is the infinitesimal action. Hence for E stable (where the moduli space is an orbit space) the tangent space $T_E M^g$ is naturally isomorphic to $H^1(\mathrm{End}\, E)$. Since $H^0(\mathrm{End}\, E) = \mathbb{C}$ by the "stable implies simple" lemma, this means that M^g is smooth at the stable points, and the dimension of M^g as given in the theorem can be computed by Riemann-Roch. In fact, the smooth locus of M^g is exactly the stable locus except when $g = 2$, $r = 2$, and d is even; see Narasimhan and Ramanan [43].

(3) If r and d are coprime, then for obvious numerical reasons stability and semistability are equivalent. Hence in this case M^g is smooth as well as compact.

(4) For many purposes, it would be convenient if there were a holomorphic bundle U over $M^g \times X$ which was *universal* in the sense that $U|_{E \times X} \cong E$ for all E. When r and d are coprime, this is indeed the case; the construction goes roughly as follows. A universal bundle $\mathcal{U}$ certainly exists over $\mathrm{Quot}^s \times X$; it can be pulled back from the tautological bundle over the Grassmannian by the evaluation map, for example. If this bundle descended to the quotient, it would be the desired U, but it does not because the diagonal $\mathbb{C}^* \subset \mathrm{GL}(\chi, \mathbb{C})$ stabilizes Quot^s but acts with weight 1 on $\mathcal{U}$. However, if π_1 is projection on Quot^s, and $p \in X$ is a fixed point, then $\mathcal{U}|_{\mathrm{Quot}^s \times p}$ and the direct image $(R^1\pi_1)\mathcal{U}$ are vector bundles over Quot^s of rank r and $d + r(1-g)$ respectively, both acted on by $\mathbb{C}^*$ with weight 1. Their top exterior powers are therefore acted on with weights r and $d + r(1-g)$, so if r and d are coprime, there is a line bundle L over Quot^s such that $\pi_1^* L \otimes \mathcal{U}$ is acted on with weight 0. But Kempf's descent lemma [17] asserts that for a holomorphic bundle to descend to the quotient by a group action, it suffices for all stabilizers to act trivially.

On the other hand, when $(r,d) \neq 1$, there is no universal bundle on the stable locus, or even a Zariski open subset; see Ramanan [48].

Finally, note that even when it exists, the universal bundle U is not unique, since for any line bundle L over M^g, $\pi_1^* L \otimes U$ also has the universal property.

(5) There is a natural holomorphic map $\det : M^g \to \operatorname{Jac}_d X$, where $\operatorname{Jac}_d X$ is the torus parametrizing degree d line bundles, given by $\det E = \Lambda^r E$. For all $\Lambda, \Lambda' \in \operatorname{Jac}_d X$, $\det^{-1} \Lambda \cong \det^{-1} \Lambda'$ by the argument of remark 1. The space $N^g = \det^{-1}(\Lambda)$ is in some ways more fundamental than M^g: for example, it will later become clear that, in some sense, all the interesting cohomology of M^g is contained in N^g.

(6) The reader might enjoy verifying that all the above remarks apply in particular to $\operatorname{Jac}_d X$. For example, the universal bundle described in remark 4 is just the Poincaré line bundle on $\operatorname{Jac}_d X \times X$.

Bundles on the Riemann sphere

One might wonder why the dimension calculation in the theorem above includes the hypothesis that $g \geq 2$. This is because, when g is 0 or 1, there may be no stable bundles at all. The case $g = 0$, for example, is covered by the following theorem.

Theorem (Grothendieck). *Let E be a holomorphic vector bundle over $\mathbb{CP}^1$. Then there exist integers n_i, unique up to reordering, such that $E \cong \bigoplus_i \mathcal{O}(n_i)$.*

Consequently, there are no stable bundles on $\mathbb{CP}^1$, and the moduli space of semistable bundles is a point if r divides d, and empty otherwise. Although the proof of this theorem is something of a digression, we give it anyway, if only to show that one of Grothendieck's theorems has an easy proof.

Proof. The uniqueness of n_i follows easily from the vanishing of $H^0(\mathcal{O}(n))$ for $n < 0$.

To prove existence, note first that by Riemann-Roch, $H^0(E(-n)) \neq 0$ for n sufficiently small, so there is a nonzero map $\mathcal{O}(n) \to E$. Let D be the vanishing divisor of this map; then replacing n by $n + |D|$ gives a nowhere zero map $\mathcal{O}(n) \to E$. By induction on the rank, assume the quotient splits, so there is an exact sequence

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow E \longrightarrow \bigoplus_i \mathcal{O}(n_i) \longrightarrow 0$$

for some n_i. If there is a nowhere zero map $\mathcal{O}(k) \to E$, then either $\mathcal{O}(k) = \mathcal{O}(n)$ or there is a nonzero map $\mathcal{O}(k) \to \mathcal{O}(n_i)$ for some n_i. Therefore k is bounded above, so assume that n is maximal among such k.

We claim $n - n_i > -2$ for all i. If not, then $H^0(\mathcal{O}(n_i - n - 1)) \neq 0$ for some i, so there is a nonzero map $\mathcal{O}(n+1) \to \mathcal{O}(n_i) \to \bigoplus_i \mathcal{O}(n_i)$. Tensoring the exact sequence above and taking the long exact sequence yields

$$\cdots \longrightarrow H^0(\operatorname{Hom}(\mathcal{O}(n+1), E)) \longrightarrow H^0(\operatorname{Hom}(\mathcal{O}(n+1), \bigoplus_i \mathcal{O}(n_i))) \longrightarrow H^1(\mathcal{O}(-1)) \longrightarrow \cdots$$

The last term is zero, so our nonzero map comes from a nonzero map $\mathcal{O}(n+1) \to E$. Let D be the vanishing divisor of this map; then there is a nowhere zero map $\mathcal{O}(n+1+|D|) \to E$, contradicting the maximality of n. This proves the claim.

The extension class of the exact sequence is in $H^1(\bigoplus_i \mathcal{O}(n-n_i))$, but the claim implies this is zero. Hence the exact sequence splits, which completes the proof of the theorem. $\square$

The classification of vector bundles on an elliptic curve is a little more complicated, but simple enough to characterize the moduli space as a symmetric product of the curve. It was carried out by Atiyah [3]; for a modern account, see Tu [54].

Generators of the cohomology ring of M^g

Even at this early stage, we can already prove a profound theorem on the cohomology ring of M^g. Suppose that $(r,d)=1$, so that M^g is smooth and compact, and there exists a universal bundle U over $M^g \times X$. Let $\{a_i, b_i\}_{i=1}^g$ be the standard collection of loops on X forming a basis for $H_1(X,\mathbb{Z})$, so that the dual basis in $H^1(X,\mathbb{Z})$ satisfies $a^i a^j = 0$, $b^i b^j = 0$, and $a^i b^j = \delta^i_j x$, where x is the fundamental cohomology class of X. Decompose the Chern classes $c_k(U) \in H^{2k}(M^g \times X, \mathbb{Z})$ into their *Künneth components*: that is, write

$$c_k(U) = \alpha_k x + \sum_{i=1}^{g} (\psi_{i,k} a^i + \psi_{i+g,k} b^i) + \beta_k$$

where $\alpha_k \in H^{2k-2}(M^g,\mathbb{Z})$, $\psi_{i,k} \in H^{2k-1}(M^g,\mathbb{Z})$, and $\beta_k \in H^{2k}(M^g,\mathbb{Z})$.

Theorem (Atiyah and Bott). *The ring $H^*(M^g,\mathbb{Q})$ is generated over $\mathbb{Q}$ by these classes.*

Atiyah and Bott's original proof involved equivariant cohomology and the gauge group; we offer a deliciously simple alternative, due to Beauville [7], following Ellingsrud and Strømme [19].

Proof. For any compact manifold M, the diagonal $\Delta \subset M \times M$ has Poincaré dual $\sum_i \pi_1^* e^i \otimes \pi_2^* e_i$, where $\{e^i\}$ is an additive basis for $H^*(M,\mathbb{Q})$ and $\{e_i\}$ is the dual basis with respect to the intersection pairing $\langle a, b\rangle = (ab)[M]$. For any j, $\sum_i \pi_1^* e^i \otimes \pi_2^*(e_i e^j) = \pi_1^* e^j \otimes \pi_2^* m$, where m is the fundamental cohomology class of M. Hence if $\sum_i \pi_1^* e^i \otimes \pi_2^* e_i$ can be expressed in terms of pull-backs of Künneth components from both factors, then $\pi_1^* e^j \otimes \pi_2^* m$ can be expressed in terms of pull-backs of Künneth components from the first factor, so e^j can be expressed in terms of Künneth components on M, as desired. Therefore it suffices to express the Poincaré dual of $\Delta \subset M^g \times M^g$ in terms of the Künneth components.

Choose an effective divisor D on X with no multiple points such that for all stable E, F represented in M^g, $H^1(X, \mathrm{Hom}(E,F) \otimes \mathcal{O}(D)) = 0$. Let

$$\begin{array}{ccccc} & & M^g \times M^g \times X & & \\ & {}^{\pi_1}\swarrow & \downarrow{\scriptstyle \pi_2} & \searrow^{p} & \\ M^g \times X & & M^g \times X & & M^g \times M^g \end{array}$$

be the projections, and let $U_1 = \pi_1^* U$ and $U_2 = \pi_2^* U$ be the pull-backs of the universal bundle. Then tensoring

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}_D \longrightarrow 0$$

with $\mathrm{Hom}(U_1, U_2)$ and pushing down by p yields

$$0 \longrightarrow p_*(\mathrm{Hom}(U_1, U_2)) \longrightarrow p_*(\mathrm{Hom}(U_1, U_2) \otimes \mathcal{O}(D)) \longrightarrow p_*(\mathrm{Hom}(U_1, U_2) \otimes \mathcal{O}_D) \longrightarrow \cdots$$

The hypothesis on D implies that the last two terms above are vector bundles; call them $\mathcal{E}$ and $\mathcal{F}$, respectively. The induced map on the fibers at (E, F) is the right-hand map in the exact sequence

$$0 \longrightarrow H^0(\mathrm{Hom}(E, F)) \longrightarrow H^0(\mathrm{Hom}(E, F) \otimes \mathcal{O}(D)) \longrightarrow H^0(\mathrm{Hom}(E, F) \otimes \mathcal{O}_D).$$

Because stable bundles are simple, $H^0(\mathrm{Hom}(E, F))$ is nonzero if and only if $(E, F) \in \Delta$. So Δ is the degeneracy locus of the map $\mathcal{E} \to \mathcal{F}$, at least up to multiplicity. (In fact, the multiplicity is 1, but since everything is over $\mathbb{Q}$, this is not needed.) Notice that by Riemann-Roch, the expected codimension $\operatorname{rank}\mathcal{F} - \operatorname{rank}\mathcal{E} + 1$ of the degeneracy locus is $r^2(g-1)+1$, which is the actual codimension. This is precisely the setting in which Porteous's formula [2, p. 86] allows the Poincaré dual of the degeneracy locus to be expressed in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. So it remains to express the latter in terms of the Künneth components.

But

$$\mathcal{F} = \bigoplus_{p \in D} \mathrm{Hom}(U_1, U_2)|_{M^g \times M^g \times p},$$

so $c(\mathcal{F})$ can actually be expressed in terms of the β_k alone. On the other hand, by the Grothendieck-Riemann-Roch theorem [29, Appendix A],

$$\begin{aligned} \operatorname{ch}\mathcal{E} &= p_*(\operatorname{ch}\mathrm{Hom}(U_1, U_2) \otimes \mathcal{O}(D) \operatorname{td} X) \\ &= p_*(\operatorname{ch} U_1^* \operatorname{ch} U_2 (1 + (|D| + 1 - g)x)), \end{aligned}$$

which implies that $c(\mathcal{E})$ can also be expressed in terms of the Künneth components. □

The Narasimhan-Seshadri theorem

At this point we leave the realm of algebraic geometry and turn to another, more topological view of stable bundles.

Recall that, in terms of the loops $\{a_i, b_i\}$ introduced in the last section, the fundamental group of X is given by

$$\pi_1(X) = \langle a_i, b_i \rangle_{i=1}^g / (\prod_i [a_i, b_i]).$$

This is because X may be viewed topologically as a quotient of a $4g$-gon by an equivalence relation on the boundary. The generators a_i and b_i are the images of the sides in the quotient, and the relation involving commutators reflects the contractibility of the boundary regarded as a loop on the $4g$-gon.

Now let $\rho : \pi_1(X) \to \mathrm{U}(r)$ be any representation, and consider the diagonal action induced by ρ of $\pi_1(X)$ on $\tilde{X} \times \mathbb{C}^r$, where $\tilde{X}$ is the universal cover of X. The quotient $E_\rho = (\tilde{X} \times \mathbb{C}^r)/\pi_1(X)$ is naturally a rank r vector bundle over X. Moreover, its transition functions are locally constant, so it carries a natural holomorphic structure. Notice that, since the induced representation $\det\rho : \pi_1(X) \to \mathrm{U}(1)$ is homotopic to the trivial representation, the line bundle $\Lambda^r E_\rho = E_{\det\rho}$ is topologically trivial, so $\deg E_\rho = 0$.

We are now in the position to state the following deep theorem.

Theorem (Narasimhan and Seshadri). (1) *For any ρ, $E_\rho \cong \mathrm{Gr}\, E$ for some semistable E;* (2) *ρ is irreducible if and only if E is stable;* (3) *for any semistable E of degree 0, there exists ρ, unique up to conjugation, such that $E_\rho \cong \mathrm{Gr}\, E$.*

There are two proofs of the theorem; the original proof of Narasimhan and Seshadri [44] used algebraic methods to show that the bundles coming from representations are open and closed in the stable bundles, while a more recent proof, due to Donaldson [14], constructs the representation as an absolute minimum of the Yang-Mills functional on connections on the bundle.

In fact, the theorem as stated by Narasimhan and Seshadri applies to bundles of any degree, not just 0. The statement above can be generalized to arbitrary degree in the following way.

Let $D \subset X$ be an embedded closed unit disc, let $\frac{1}{2}D$ be the disc of radius $\frac{1}{2}$ inside it, and let $Y = X \backslash \frac{1}{2}D$. Removing a disc from X destroys the contractibility of the boundary of the $4g$-gon, so the fundamental group $\pi_1(Y)$ is just the free group $\langle a_i, b_i \rangle$ on $2g$ generators. Let $\xi = e^{2\pi i/r}$, and let $\rho : \pi_1(Y) \to \mathrm{U}(r)$ be a representation such that $\rho(\partial D) = \xi^d$ for some $d \in \mathbb{Z}$. As before, the quotient $(\tilde{Y} \times \mathbb{C}^r)/\pi_1(Y)$ is a holomorphic vector bundle over Y. Give it the obvious trivialization on a chart wrapping $1\frac{1}{2}$ times around the annulus $Y \cap D$; then the transition function on the self-overlap of this chart is ξ^d. Then glue in a trivialization on D having transition function $z^{d/r}$ to the self-overlapping chart. This yields a well-defined holomorphic vector bundle E_ρ over all of X, with $\deg E_\rho = d$. The obvious analogue of the theorem stated above then applies to this situation.

So let $\mu_g : \mathrm{U}(r)^{2g} \to \mathrm{U}(r)$ be given by $(A_i, B_i) \mapsto \prod_i [A_i, B_i]$; then the Narasimhan-Seshadri theorem induces a bijection $M^g \to \mu_g^{-1}(\xi^d)/\mathrm{U}(r)$, where $\mathrm{U}(r)$ acts on $\mu_g^{-1}(\xi^d)$ by conjugation on all factors. We will take for granted not only the Narasimhan-Seshadri theorem, but a mild strengthening of it which is widely used but somewhat cumbersome to prove.

Proposition. *The bijection $M^g \to \mu_g^{-1}(\xi^d)/\mathrm{U}(r)$ is a homeomorphism, and restricts to a diffeomorphism on the stable locus.*

Thus even though M^g is highly dependent on the complex structure of X a priori, it can be identified with a space independent of the complex structure.

The results of this section apply equally to the fixed-determinant space N^g introduced in remark 5, if $\mathrm{U}(r)$ is replaced with $\mathrm{SU}(r)$.

Topological construction of End U

When the rank and degree are coprime, the construction of the section above can be performed globally on M^g. By this we mean the following. The group $\pi_1(Y)$ acts on $\mu_g^{-1}(\xi^d) \times \tilde{Y} \times \mathbb{C}^r$, the action on $\mathbb{C}^r$ being determined by the coordinate in $\mu_g^{-1}(\xi^d)$. The quotient is a vector bundle over $\mu_g^{-1}(\xi^d) \times Y$, to which the conjugation action of $\mathrm{U}(r)$ on $\mu_g^{-1}(\xi^d)$ lifts naturally. Since the gluing construction of the last section was canonical given the disc D, one may glue this bundle to a trivial bundle over $\mu_g^{-1}(\xi^d) \times D$ to yield a bundle over $\mu_g^{-1}(\xi^d) \times X$, to which the $\mathrm{U}(r)$-action again lifts naturally. The diagonal subgroup $\mathrm{U}(1) \subset \mathrm{U}(r)$ acts nontrivially on this bundle, but trivially on its endomorphism bundle. The latter bundle is therefore the pull-back of a bundle $\operatorname{End} U$ over $M^g \times X$ provided that the stabilizer of every point is the diagonal $\mathrm{U}(1)$, which is the case precisely when $(r,d) = 1$.

The bundle $\operatorname{End} U$ is in some sense universal: after all, $\operatorname{End} U|_{\rho \times X}$ naturally carries the holomorphic structure of E_ρ, which varies continuously in ρ, so $\operatorname{End} U$ is a family of stable holomorphic bundles. Because stable implies simple, if E and F are isomorphic stable bundles, then $\operatorname{End} E \cong \operatorname{End} F$ canonically. Hence $\operatorname{End} U$ may be canonically identified with the bundle of the same name constructed earlier by algebro-geometric methods.

This new construction, however, used only representations of $\pi_1(Y)$. Hence any orientation-preserving homeomorphism $f : X \to X$ fixing D induces a map $\hat{f} : M^g \to M^g$ which lifts to a natural isomorphism $(\hat{f} \times f)^* \operatorname{End} U \cong \operatorname{End} U$. Moreover, $\hat{f}$ depends only on the action of f on $\pi_1(Y)$, so any homeomorphism isotopic to the identity acts trivially. Thus M^g and $\operatorname{End} U$ are acted upon by the *mapping class group* Γ_g, that is, the group of orientation-preserving homeomorphisms of X modulo those isotopic to the identity.

All the results of this section hold for N^g as well, if $\mathrm{U}(r)$ is replaced by $\mathrm{SU}(r)$.

The symplectic structure

We have seen that the moduli space M^g and the universal endomorphism bundle $\operatorname{End} U$ can be constructed without using the complex structure on X. Next we will see that the same is true of the Kähler form on M^g.

Since M^g can be constructed as a smooth manifold by representation-theoretic methods, it is natural to look for a representation-theoretic analogue of the deformation theory described in remark 2 above. This does indeed exist; it identifies the tangent space $T_\rho M^g$ with the group cohomology $H^1(\pi_1(X), \operatorname{ad} \rho)$. This makes sense, because $\operatorname{ad} \rho$ descends to a representation of $\pi_1(X)$. Explicitly, it is just the first cohomology of the complex

$$\mathfrak{u}(r) \longrightarrow \mathfrak{u}(r)^{2g} \longrightarrow \mathfrak{u}(r)$$

where the former map is the derivative of conjugation at (A_i, B_i), and the latter is the derivative of μ_g. So from the representation-theoretic point of view, the smoothness of M^g when $(r,d) = 1$ follows from the surjectivity of the second map, or equivalently, the fact that ξ^d is a regular value of μ_g; this is proved, for example, by Igusa [30].

Since X is an Eilenberg-Mac Lane space, $H^2(\pi_1(X), \mathbb{R}) = H^2(X, \mathbb{R}) = \mathbb{R}$. Combining the cup product and the symmetric form $\langle A, B \rangle = \frac{1}{4\pi^2} \operatorname{tr} AB$ gives a non-degenerate antisymmetric map

$$H^1(\operatorname{ad}\rho) \otimes H^1(\operatorname{ad}\rho) \xrightarrow{\cup} H^2(\operatorname{ad}\rho \otimes \operatorname{ad}\rho) \xrightarrow{\langle\,,\,\rangle} H^2(\mathbb{R}) = \mathbb{R},$$

which determines a non-degenerate 2-form ω on M^g.

Theorem (Goldman). *The form ω is closed, and coincides with the Kähler form arising from any complex structure on X.*

The theorem shows that the symplectic structure on M^g, like the universal bundle, is essentially independent of the complex structure. It is surprisingly difficult: Goldman's original proof [22] used infinite-dimensional quotients in the style of Atiyah and Bott, and it was only quite recently that a purely finite-dimensional proof was provided by Karshon [34].

All the results of this section hold for N^g as well, if $\mathfrak{u}(r)$ is replaced by $\mathfrak{su}(r)$.

The Betti numbers of N^g

From now on, we will assume that $r = 2$ and $d = 1$. This is the simplest interesting case where r and d are coprime. Considerably more is known here than in general, and the proofs are simpler.

Our goal will be to gather as much topological information as possible on M^g and N^g, focusing on the latter space. We will start off by computing its Betti numbers, in the form of the *Poincaré polynomial* $P_t(N^g) = \sum_i t^i \dim H^i(N^g, \mathbb{Q})$.

Let $f : N^g \to [-1, 1]$ be given by $(A_i, B_i) \mapsto \frac{1}{2} \operatorname{tr} A_g$, which is well-defined since the trace is conjugation-invariant. Then $\mathrm{U}(1)$ acts on $f^{-1}(-1, 1)$ as follows. If $A_g \neq \pm I$, then there is a unique homomorphism $\phi : \mathrm{U}(1) \to \mathrm{SU}(2)$ such that $A_g \in \phi(\{\operatorname{Im} z > 0\})$. Let $\lambda \cdot (A_i, B_i) = (A_1, B_1, \ldots, A_g, B_g \cdot \phi(\lambda))$.

Proposition (Goldman). *This action preserves the symplectic form ω, and it has moment map $\frac{i}{\pi} \arccos f$.*

Proof. See Goldman [23] and Jeffrey and Weitsman [32]. □

For definitions and properties of moment maps, see the paper by Jeffrey in the present volume. The key point is that, for a global symplectic $\mathrm{U}(1)$-action on a compact manifold, the moment map (times $-i$) is a Bott-Morse function which is *perfect*, meaning that the Morse inequalities are equalities [4, 36]. In the present case, however, the $\mathrm{U}(1)$-action does not extend over $f^{-1}(\pm 1)$, and $\frac{1}{\pi} \arccos f$ is not even differentiable there. Nevertheless, the following is true.

Theorem. *The map f is a perfect Bott-Morse function on N^g.*

It was first noticed that f had to be perfect by Jeffrey and Weitsman [33], simply by comparing the left-hand side of its Morse inequalities with the known formula for the Poincaré polynomial. In the next section we shall outline a direct proof, to be given in full in [53]. Before doing so, however, let us identify the critical submanifolds and compute the Poincaré polynomial using the theorem.

First, let $S_+ = f^{-1}(1)$. As the absolute maximum of f, S_+ is of course a critical submanifold. It is exactly the locus where $A_g = I$; hence B_g may be arbitrary, and the product of the first $g - 1$ commutators must be $-I$, so $S_+ = (\mu_{g-1}^{-1}(-I) \times \mathrm{SU}(2))/\mathrm{SU}(2)$, which is an SU(2)-bundle over N^{g-1}. This is an adjoint, not a principal bundle, so it may have a section without being trivial. Indeed, $B_g = I$ determines such a section; hence the Euler class vanishes and so by the Gysin sequence $P_t(S_+) = (1+t^3)P_t(N^{g-1})$. Since $\dim S_+ = 6g-9$ and $\dim N^g = 6g-6$, S_+ has index 3.

Exactly the same is true of $S_- = f^{-1}(-1)$, except that it is the absolute minimum of f, the locus where $A_g = -I$, and hence has index 0.

Within $f^{-1}(-1,1)$, on the other hand, the critical points of f are the critical points of $\frac{1}{\pi}\arccos f$, which is the moment map of the U(1)-action. They are therefore exactly the fixed points of that action, and hence are represented by $2g$-tuples $(A_i, B_i) \in \mathrm{SU}(2)^{2g}$ that are conjugate to $(A_1, B_1, \ldots, A_g, B_g \cdot \phi(\lambda))$ for all $\lambda \in \mathrm{U}(1)$. It is straightforward to check that these are all conjugate to $2g$-tuples such that $A_g = i\sigma_3$, $B_g = i\sigma_2$ (where σ_j are the Pauli spin matrices: see introduction) and the remaining A_i and B_i are diagonal. Hence the only other critical value is $\frac{1}{2}\operatorname{tr} i\sigma_3$, which is 0, and S_0 is a $2g-2$-torus. Because there is an involution $A_g \mapsto -A_g$ on N^g changing the sign of f, the upward and downward flows from S_0 must have the same dimension, so S_0 has index $2g-2$.

The main consequence of the theorem is then the following.

Corollary. *The Poincaré polynomial of N^g is*

$$P_t(N^g) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Proof. Since N^1 is a point, $P_t(N^1) = 1$ as desired. It follows from the theorem and the discussion above that

$$P_t(N^g) = (1+t^3)^2 P_t(N^{g-1}) + t^{2g-2}(1+t)^{2g-2}.$$

This gives a recursion for $P_t(N^g)$, which is satisfied by the formula. □

This is the *Harder-Narasimhan formula*, first computed by other means by Newstead [45], Harder and Narasimhan [27] and Atiyah and Bott [4]. It implies in particular that the Euler characteristic of N^g vanishes, since it equals $P_{-1}(N^g)$.

Proof of the above theorem

To begin the proof of the theorem, we give two lemmas describing the structure of N^g near $S_\pm$. The results are stronger than we need here, but they will also be used later in computing the characteristic numbers of N^g.

Lemma. *The map $\mu_g^{-1}(-I) \to \mathrm{SU}(2)\times\mathrm{SU}(2)$ given by $(A_i, B_i) \mapsto (A_g, B_g)$ is a submersion at the locus where $A_g = \pm I$.*

Proof. This means that the infinitesimal map $\ker d\mu_g \to \mathfrak{su}(2)\times\mathfrak{su}(2)$ induced by projection of $\mathfrak{su}(2)^{2g}$ on the last two factors is surjective when $A_g = \pm I$. A computation shows that in this case,

$$d\mu_g(a_i, b_i) = d\mu_{g-1}(a_1, \ldots, b_{g-1}) + a_g - B_g a_g B_g^{-1}.$$

But as mentioned before, $-I$ is a regular value of μ_{g-1}. Hence $d\mu_{g-1}$ is surjective and so a_g and b_g can take any values for $(a_i, b_i) \in \ker d\mu_g$. $\square$

Lemma ("Local model"). *There is an* $\mathrm{SU}(2)$*-equivariant diffeomorphism between a neighborhood of* $S_\pm \subset \mu_g^{-1}(-I)$ *and a neighborhood of* $\mu_{g-1}^{-1}(-I) \times \{I\} \times \mathrm{SU}(2) \subset \mu_{g-1}^{-1}(-I) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$*, identifying the map of the previous lemma with the projection on the last two factors.*

Proof. This follows immediately from the previous lemma using the equivariant version of the tubular neighborhood theorem for submanifolds [5, Thm. 2.2.1]. $\square$

At last we proceed to sketch a proof of the theorem.

Outline of proof. In Witten's approach to Morse theory [10, 57], one defines a chain complex with a basis in one-to-one correspondence with the critical points of a Morse function on a manifold M, and a d operator given by counting the number of flow lines between them. The cohomology of this complex is then isomorphic to $H^*(M, \mathbb{Q})$. Suppose, however, that one has not a Morse function, but only a *Bott-Morse function.* This means that the critical points, instead of being isolated, form a union of submanifolds, and the Hessian is non-degenerate on the normal bundles. In this case, the chain complex gets replaced by a spectral sequence whose $E_{-\infty}^{ij}$ term is $H^j(C_i, \mathbb{Q})$, where C_i is the critical submanifold of index i; see for example Fukaya [21]. The spectral sequence starts at $-\infty$ for ease of indexing, but only finitely many differentials are nonzero. This spectral sequence then abuts to $H^*(M, \mathbb{Q})$, meaning that $H^k(M, \mathbb{Q}) \cong \bigoplus_{i+j=k} E_\infty^{ij}$.

The differentials in the spectral sequence are given roughly as follows. Let C_{jk} be the space of downward flow lines from C_j to C_k, and let $p : C_{jk} \to C_j$ and $q : C_{jk} \to C_k$ be the flows at time $\pm\infty$. Then $d_{j-k} = p_* q^*$. This is only a rough definition for two reasons. First, C_{jk} is not generally compact, so as usual, a judicious compactification must be chosen to get the right answer. Second, before acting with d_k, one takes cohomology with respect to d_j for all $j < k$, so it must be checked that d_k descends to the cohomology.

A Bott-Morse function is perfect if and only if all differentials vanish. The moment map of a circle action, for example, is perfect because $\mathrm{U}(1)$ acts freely on C_{jk}, and p and q factor through the quotient map r:

$$\begin{tikzcd} & C_j \\ C_{jk} \arrow[ur, "p"] \arrow[r, "r"] \arrow[dr, "q"'] & C_{jk}/\mathrm{U}(1) \arrow[u] \arrow[d] \\ & C_k \end{tikzcd}$$

The differential d_{j-k} then vanishes because in a $\mathrm{U}(1)$-bundle, $r_* r^* = 0$.

In the present case, $C_0 = S_-$, $C_3 = S_+$, and $C_{2g-2} = S_0$. The only differentials which can possibly be nonzero are those corresponding to downward flows, namely d_{5-2g}, d_3, and d_{2g-2}.

Consider first d_{2g-2}. Here the maps are $p : C_{2g-2,0} \to S_0$ and $q : C_{2g-2,0} \to S_-$. But the argument from the moment map case no longer works, since the fibers of the principal

$\mathrm{U}(1)$-bundle are not collapsed by q. Rather, they go to the fibers of the Hopf fibration on the $\mathrm{SU}(2)$ fibers of the bundle $\pi : S_- \to N^{g-1}$. This follows easily from the "local model" lemma. The composition πq, however, does therefore collapse the $\mathrm{U}(1)$ fibers, so by the moment map argument $p_* q^* \pi^* = 0$. By the Gysin sequence, $H^*(S_-) = \pi^* H^*(N^{g-1}) \oplus \sigma\pi^* H^*(N^{g-1})$ for a certain class $\sigma \in H^3(S_-)$, and $\ker \pi_* = \pi^* H^*(N^{g-1})$. So it suffices to show that $p_* q^*$ annihilates $\sigma\pi^* H^*(N^{g-1})$.

There is a symplectomorphism of N^g induced by a half-twist of the gth handle of the Riemann surface X. Explicitly, it is given by $A_g \mapsto A_g B_g A_g^{-1} B_g^{-1} A_g^{-1}$ and $B_g \mapsto A_g B_g^{-1} A_g^{-1}$. This fixes f and S_0, acts trivially on $\pi^* H^*(N^{g-1})$, and is compatible with $p_* q^*$, but it changes the sign of σ. Hence for reasons of parity $\sigma\pi^* H^*(N^{g-1})$ must be annihilated by $p_* q^*$, as desired.

Next consider d_{5-2g}. Now the maps are $p : C_{3,2g-2} \to S_+$ and $q : C_{3,2g-2} \to S_0$. Because of the symmetry $A_g \mapsto -A_g$, the space of flows is isomorphic to that considered before, but the pull-backs and push-forwards go the opposite way. The moment map argument now shows that $\pi_* p_* q^* = 0$. It therefore suffices to show that $p_* q^*$ has no component in $\sigma\pi^* H^*(N^{g-1})$. This again follows from a parity argument, once one notices that the half-twist symplectomorphism reverses the orientation of S_+, but not the space of flows.

Finally, consider d_3. Now the maps are $p : C_{3,0} \to S_+$ and $q : C_{3,0} \to S_-$. Since $H^*(S_\pm)$ both split in two, $p_* q^*$ has four components. Three of these vanish by the moment map and parity arguments above; all that remains is the component $\sigma\pi^* H^*(N^{g-1}) \to \sigma\pi^* H^*(N^{g-1})$.

It is not too hard to show that $\sigma \in H^3(S_+)$ and $\sigma \in H^3(S_-)$ are both restrictions of the same global class $\sigma \in H^3(N^g)$. In fact, σ is the restriction to N^g of the class $\psi_{2g,2} \in H^3(M^g)$. (It is what will be called ψ_{2g} later in the paper.) Since p and q are homotopy equivalent as maps $C_{3,0} \to N^g$, this implies that $p^*\sigma = q^*\sigma$. Then by the push-pull formula, for any $\eta \in H^*(N^{g-1})$,

$$p_* q^*(\sigma\pi^*\eta) = p_*(q^*\sigma)(q^*\pi^*\eta) = p_*(p^*\sigma)(q^*\pi^*\eta) = \sigma(p_* q^* \pi^* \eta);$$

but $p_* q^* \pi^* \eta$ was already shown to vanish. □

Remarks on the proof

In fact, additional information about the cohomology can be obtained by the same argument. For example, consider the action of $(\mathbb{Z}/2)^{2g}$ on N^g given by $(\delta_i, \epsilon_i) \cdot (A_i, B_i) = ((-1)^{\delta_i} A_i, (-1)^{\epsilon_i} B_i)$.

Lemma. *The induced action of $(\mathbb{Z}/2)^{2g}$ on $H^*(N^g, \mathbb{Q})$ is trivial.*

Proof. The action of $(\mathbb{Z}/2)^{2g-2}$ on the first $2g-2$ factors preserves the $\mathrm{U}(1)$-action, the map f and so on. The whole Morse theory argument of the last section can therefore be $(\mathbb{Z}/2)^{2g-2}$-graded. But since $(\mathbb{Z}/2)^{2g-2}$ acts trivially on σ and on $H^*(S_0)$, by induction the whole grading is trivial. Hence $(\mathbb{Z}/2)^{2g-2}$ acts trivially on $H^*(N^g)$.

The last two factors are really no harder. After all, the choice of an ordering on the handles of X was arbitrary. For example, the whole Morse theory argument still works if f is replaced by $\frac{1}{2} \operatorname{tr} A_1$, and so on. □

The next result explains why, for cohomological purposes, it is reasonable to concentrate on the fixed-determinant space N^g rather than M^g.

Theorem. *As rings,* $H^*(M^g,\mathbb{Q}) \cong H^*(T^{2g},\mathbb{Q}) \otimes H^*(N^g,\mathbb{Q})$.

Proof. The torus T^{2g} is essentially the Jacobian: it is the moduli space M^g with $r = 1$, or equivalently, $H^1(X, \mathrm{U}(1))$. There is a natural map $\kappa : T^{2g} \times N^g \to M^g$ given by the tensor product. Indeed, $M^g = (T^{2g} \times N^g)/(\mathbb{Z}/2)^{2g}$, where $(\mathbb{Z}/2)^{2g}$ acts diagonally on T^{2g} and N^g as above. The induced action on $H^*(T^{2g},\mathbb{Q})$ is certainly trivial, and the induced action on $H^*(N^g,\mathbb{Q})$ is trivial by the lemma above. But the rational cohomology of a quotient by a finite group is the invariant part of the rational cohomology (see Grothendieck [24]), so this completes the proof. $\square$

This result actually holds for arbitrary coprime rank and degree; see Harder and Narasimhan [27] or Atiyah and Bott [4].

Our methods can also be used to prove the following useful fact.

Proposition. *The space N^g is simply connected.*

Proof. Since N^1 is a point, certainly $\pi_1(N^1) = 1$. So assume by induction that $g > 1$ and $\pi_1(N^{g-1}) = 1$. The absolute maximum S_+ of f then has codimension 3, and the locus of points flowing down to S_0 has codimension $2g-2 \geq 2$, so any loop in N^g may be perturbed to miss these loci. The downward Morse flow of f then takes the loop to S_-, which, as an $\mathrm{SU}(2)$-bundle over N^{g-1}, is simply connected by the exact homotopy sequence. $\square$

Finally, let us double-check the example $g = 2$. In this case, the Harder-Narasimhan formula says $P_t(N^2) = 1 + t^2 + 4t^3 + t^4 + t^6$. On the other hand, N^2 can be described explicitly via algebraic geometry. Any Riemann surface of genus 2 can be expressed as a double cover of the Riemann sphere, branched at 6 points. Rotate the sphere so that all 6 points are $\neq \infty$, and call them $x_1, \ldots, x_6 \in \mathbb{C}$.

Theorem (Newstead). *In this setting,* $N^2 \cong Q_1 \cap Q_2$, *where*

$$Q_1 = \{[z_i] \in \mathbb{CP}^5 \mid \sum_i z_i^2 = 0\}, \quad Q_2 = \{[z_i] \in \mathbb{CP}^5 \mid \sum_i x_i z_i^2 = 0\}.$$

Proof. See [46]. $\square$

Applying the Lefschetz hyperplane theorem [26, p. 156] first to Q_1, then to $Q_1 \cap Q_2$ implies that $P_t(N^2) = 1+t^2+mt^3+t^4+t^6$ for some m. Then a calculation using the adjunction formula [26] implies that $c_3(N^2)[N^2] = 0$. Since $c_3(N^2)[N^2]$ is the Euler characteristic of N^2, this implies that $m = 4$, as desired.

Incidentally, a generalization of this theorem, due to Desale and Ramanan [12], gives an explicit description of N^g for a Riemann surface of arbitrary genus which is *hyperelliptic*, that is, a double branched cover of the Riemann sphere.

Generators of the cohomology ring of N^g

As we have already done for $H^*(M^g, \mathbb{Q})$, let us now look for generators of $H^*(N^g)$ in terms of Künneth components. Let U be a universal bundle over $N^g \times X$, and let $2\alpha \in H^2(N^g)$, $4\psi_i \in H^3(N^g)$, and $-\beta \in H^4(N^g)$ be the Künneth components of $c_2(\operatorname{End} U)$. (The scalars are inserted to agree with the conventions of Newstead [46].) Using $\operatorname{End} U$ is preferable to U itself, because it is uniquely defined, so these classes are canonical. The following is then an analogue—indeed, historically a predecessor—of the theorem of Atiyah and Bott stated earlier.

Theorem (Newstead). *The ring $H^*(N^g, \mathbb{Q})$ is generated by α, β, and the ψ_i.*

Proof. Since $H^*(N^g)$ is a quotient ring of $H^*(M^g)$ as seen above, it is generated as a ring by the Künneth components of $c_1(U)$ and $c_2(U)$. The Harder-Narasimhan formula implies that $P_t(N^g) = 1 + t^2 + O(t^3)$, that is, $H^1(N^g) = 0$ and $H^2(N^g)$ is 1-dimensional. Since $H^1 = 0$, the Chern classes may be written as $c_1(U) = x + \beta_1$ and $c_2(U) = \alpha_2 x + \sum_{i=1}^g \psi_i a^i + \psi_{i+g} b^i + \beta_2$; it is then easy to check $\alpha = 2\alpha_2 - \beta_1$ and $\beta = \beta_1^2 - 4\beta_2$. Since H^2 is 1-dimensional, β_1 and α_2 are dependent; hence to prove the theorem, it suffices to show $\alpha \neq 0$ for $g \geq 2$, which is accomplished by the lemma below. $\square$

Lemma. *For $g \geq 2$, there is a holomorphic map $f : \mathbb{CP}^1 \to N^g$ such that $f^*\alpha$ is the fundamental class of $\mathbb{CP}^1$.*

Proof. Recall that $N^g = \det^{-1}(\Lambda)$ for some line bundle Λ of degree 1. By Riemann-Roch, $\dim H^1(X, \Lambda^{-1}) = g$; let $V \subset H^1(X, \Lambda^{-1})$ be a 2-dimensional subspace. As in the example of the jump phenomenon, the identity determines a natural class

$$\begin{aligned} I \in V^* \otimes V &\subset H^0(\mathbb{P}V, \mathcal{O}(1)) \otimes H^1(X, \Lambda^{-1}) \\ &\subset H^1(\mathbb{P}V \times X, \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\Lambda^{-1}) = \operatorname{Ext}^1(\mathbb{P}V \times X; \pi_1^*\mathcal{O}(1), \pi_2^*\Lambda), \end{aligned}$$

which gives an extension

$$0 \longrightarrow \pi_1^*\mathcal{O}(1) \longrightarrow F \longrightarrow \pi_2^*\Lambda \longrightarrow 0.$$

Regard this as a holomorphic family of extensions over X whose extension class at $[t] \in \mathbb{P}V$ is t. All of these extensions are stable bundles of determinant Λ. After all, any proper subbundle except $\mathcal{O}$ must have a nonzero map to Λ, which cannot be an isomorphism as that would split the extension; hence its degree must be ≤ 0.

So there is a holomorphic map $f : \mathbb{P}V \to N^g$ such that $f^* \operatorname{End} U \cong \operatorname{End} F$. Hence $f^*\alpha$ is half the Künneth component of $\operatorname{End} F$ in $H^2(\mathbb{P}V, \mathbb{Q})$. From the extension above, the Chern roots of F are the fundamental classes v and x of $\mathbb{P}V$ and X respectively; hence $c_2(\operatorname{End} F) = -(v + x)^2$ and the result follows. $\square$

Notice that because $\alpha = 2\alpha_2 - \beta_1$ as stated in the proof, α is actually an integral class. And because α restricts to the fundamental class of $\mathbb{P}V$, it is indivisible in $H^2(N^g, \mathbb{Z})$. But since N^g is simply connected, by [39, A.1] $H^2(N^g, \mathbb{Z})$ is torsion-free. Since its rank is 1, it follows that α generates $H^2(N^g, \mathbb{Z})$.

There are alternative interpretations of both α and β. For example, β can be defined as $-c_2(E)$, where E is the rank 4 vector bundle over N^g defined by $(\mu_g^{-1}(-I) \times \mathrm{End}\,\mathbb{C}^2)/\mathrm{SU}(2)$. Indeed, the topological construction of $\mathrm{End}\,U$ shows that E is isomorphic to $\mathrm{End}\,U|_{N^g \times p}$ for any $p \in X$. Likewise, α can be defined as twice the cohomology class of the symplectic form ω. Certainly by the previous theorem, α is some nonzero multiple of $[\omega]$. We leave it as an exercise to show that the scalar is 2.

To conclude the section, here is a consequence of the above theorem which turns out to be useful in evaluating Casson's invariant [1, p. 130].

Since the group of automorphisms of the ring $H^*(X,\mathbb{Z})$ is the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$, the action of the mapping class group Γ_g on $H^*(X,\mathbb{Z})$ induces a homomorphism $\Gamma_g \to \mathrm{Sp}(2g,\mathbb{Z})$. Its kernel is known as the *Torelli group*. Now recall that Γ_g acts naturally on N^g, and hence on $H^*(N^g,\mathbb{Q})$.

Theorem. *The Torelli group acts trivially on* $H^*(N^g,\mathbb{Q})$.

Proof. Let $f : X \to X$ be a homeomorphism representing an element of the Torelli group, and let $\hat{f} : N^g \to N^g$ be the induced map. By the topological construction of $\mathrm{End}\,U$, $(\hat{f} \times f)^* \,\mathrm{End}\,U \cong \mathrm{End}\,U$, so the Chern classes of $\mathrm{End}\,U$ are fixed by $(\hat{f} \times f)^*$. Hence α, β, and the ψ_i are fixed by f, and the result follows by Newstead's theorem. □

Consequently, the Γ_g-action on $H^*(N^g,\mathbb{Q})$ descends to an $\mathrm{Sp}(2g,\mathbb{Z})$-action.

Characteristic numbers

Continuing our study of the ring structure of $H^*(N^g,\mathbb{Q})$, we will now seek to evaluate the *characteristic numbers* or *cohomology pairings*. These are the numbers defined by expressions of the form $(\alpha^m \beta^n \prod_i \psi_i^{p_i})[N^g]$ where $2m+4n+3\sum_i p_i = 6g-6$. They are interesting for two reasons. First, they are the analogues of many gauge-theory invariants defined as cohomology pairings on moduli spaces, such as the Donaldson, Gromov, and Seiberg-Witten invariants. Second, since Poincaré duality means that a cohomology class $\zeta \in H^k(N^g)$ is 0 if and only if $(\zeta\eta)[N^g] = 0$ for all $\eta \in H^{6g-6-k}(N^g)$, the characteristic numbers in principle determine the entire structure of the ring. Indeed, a complete set of relations has been worked out in this fashion by Zagier [59].

Our strategy will be to evaluate first those pairings for which $p_i = 0$, that is, those of the form $(\alpha^m\beta^n)[N^g]$. We will follow an approach, essentially due to Donaldson [14], that fits in well with our other arguments. This will occupy the next three sections. The pairings for $p_i \neq 0$ will be derived relatively easily from these in the penultimate section of the paper.

For $\tau \in (0,1)$, let S^2_τ be the 2-sphere of radius $\frac{1}{2} - \frac{1}{\pi}\arccos\tau$ in $\mathbb{R}^3$, centered at $(0,0,\frac{1}{2})$, with the standard symplectic form. The circle group $\mathrm{U}(1)$ acts on S^2_τ by rotation, with moment map given by the third coordinate. Let N^g_τ be the symplectic reduction of $f^{-1}(-1,1) \times S^2_\tau$ by $\mathrm{U}(1)$. This is a compact symplectic manifold with a natural symplectic $\mathrm{U}(1)$-action; as a topological space, it is $f^{-1}(-\tau,\tau)$ with the $\mathrm{U}(1)$-orbits in $f^{-1}(\pm\tau)$ collapsed. It is a simple example of a *symplectic cut* [38].

Let ω_τ be the symplectic form on N^g_τ, and let α_τ denote twice the cohomology class of ω_τ. The map f descends to a map $f_\tau : N^g_\tau \to \mathbb{R}$ with image $[-\tau, \tau]$, and the moment map for the U(1)-action is $\frac{1}{\pi} \arccos f_\tau$. Also, the rank 4 bundle E of the previous section is acted on naturally by U(1) over $f^{-1}(-1,1)$ and therefore descends to a bundle E_τ over N^g_τ. Let $\beta_\tau = -c_2(E_\tau)$.

Proposition. *For $2m + 4n = 6g - 6$,*

$$\lim_{\tau \to 1} \alpha^m_\tau \beta^n_\tau [N^g_\tau] = \alpha^m \beta^n [N^g].$$

Proof. The limit in the upward Morse flow on the original space N^g induces a homotopy equivalence $f^{-1}(\frac{1}{2}, 1] \to S_+$. Hence the restriction $E|_{f^{-1}(\frac{1}{2},1]}$ is pulled back from $E|_{S_+}$. But $E|_{S_+} = (\mu^{-1}_{g-1}(-I) \times \mathrm{SU}(2) \times \mathrm{End}\, \mathbb{C}^2)/\mathrm{SU}(2)$, so it is pulled back from N^{g-1}. Similarly, $E|_{f^{-1}[-1,-\frac{1}{2})}$ is pulled back from N^{g-1} via the downward Morse flow. It follows from Chern-Weil theory [39, Appendix C] that $\beta = -c_2(E)$ is represented by a 4-form ϕ which on $f^{-1}[-1, -\frac{1}{2})$ and $f^{-1}(\frac{1}{2}, 1]$ is pulled back from N^{g-1}. As mentioned once before, the local model lemma implies that the Morse flow takes U(1)-orbits to fibers of the Hopf fibration in the SU(2)-fibers of $S_\pm$ over N^{g-1}. Hence the compositions $f^{-1}(\frac{1}{2}, 1] \to S_+ \to N^{g-1}$ and $f^{-1}[-1, -\frac{1}{2}) \to S_- \to N^{g-1}$ collapse the U(1)-orbits in their domains. The form ϕ therefore vanishes along those orbits, so for $\tau > \frac{1}{2}$, it descends after averaging to a closed 4-form ϕ_τ representing β_τ on the symplectic cut N^g_τ. On the other hand, ω is already U(1)-invariant, so

$$\alpha^m_\tau \beta^n_\tau [N^g_\tau] = \int_{N^g_\tau} \omega^m_\tau \phi^n_\tau = \int_{f^{-1}(-\tau,\tau)} \omega^m \phi^n.$$

As $\tau \to 1$, this clearly approaches $\int_{N^g} \omega^m \phi^n = \alpha^m \beta^n [N^g]$. □

The localization formula

Things are now in good shape, because we can work on N^g_τ, which has a global U(1)-action. In particular, we may apply the *localization formula* of Berline and Vergne [4, 8]. This states that, if a torus acts on a compact manifold M with fixed-point set having components F_i, and η is an equivariant cohomology class, then

$$\eta[M] = \sum_i \frac{\eta|_{F_i}}{e(\nu_{F_i/M})}[F_i],$$

where $e(\nu_{F_i/M})$ denotes the equivariant Euler class of the normal bundle, oriented compatibly with chosen orientations on M and F_i.

The fixed-point set of the U(1)-action on N^g_τ has three components: the absolute maxima and minima of f_τ, denoted F_+ and F_-, and a component $F_0 \subset f^{-1}_\tau(0)$, which for all practical purposes may be identified with S_0. Both α_τ and β_τ extend to U(1)-equivariant cohomology classes, say $\tilde{\alpha}_\tau$ and $\tilde{\beta}_\tau$. Indeed, α_τ is twice the class of the symplectic form ω_τ, which becomes equivariantly closed when the moment map is added; so $\tilde{\alpha}_\tau = \alpha_\tau + \frac{2}{\pi} \arccos f$.

And $\beta_\tau = -c_2(E_\tau)$; since E_τ admits a lifting of the U(1)-action, $\tilde{\beta}_\tau$ is its equivariant second Chern class.

Since $F_\pm$ and F_0 are fixed by $\mathrm{U}(1)$, the equivariant cohomology is naturally a polynomial ring: $H^*_{\mathrm{U}(1)}(F_\pm) \cong H^*(F_\pm)[u]$ and $H^*_{\mathrm{U}(1)}(F_0) \cong H^*(F_0)[u]$. The classes on the right-hand side of the localization formula are therefore Laurent series in u. The evaluation is performed by taking the constant term and applying it to the fundamental class in the usual way; see for example Atiyah and Bott [4]. We now set out to compute the right-hand side explicitly. Let N^g_τ and the fixed-point components F_i be oriented so that ω_τ is positive definite on each.

Lemma. (a) *The restriction of ω_τ to F_0 is $2\theta \in H^2(F_0)$, where θ is a class such that $\theta^{g-1}[F_0] = (g-1)!$;* (b) *the normal bundle ν_{F_0/N^g} has equivariant Euler class $(-1)^{g-1}u^{2g-2}$.*

Proof. For any $\rho \in F_0$, the representation $\operatorname{ad}\rho$ splits as $\zeta \oplus \xi$, corresponding to the splitting $\mathfrak{su}(2) \cong \mathbb{R} \oplus \mathbb{C}$ into diagonal and off-diagonal matrices. It is straightforward to check that $TF_0 \subset TN^g$ is $H^1(\zeta) \subset H^1(\operatorname{ad}\rho)$, so that $\nu_{F_0/N^g} = H^1(\xi)$, and that the U(1)-action is induced from scalar multiplication on ξ. Moreover, in the complex

$$\mathfrak{su}(2) \xrightarrow{d_1} \mathfrak{su}(2)^{2g} \xrightarrow{d_2} \mathfrak{su}(2)$$

computing $H^1(\operatorname{ad}\rho)$, a computation shows that the differentials restrict on ζ to $d_1(t) = (0, \ldots, 0, 2t)$ and $d_2(a_i, b_i) = 2a_g$. Hence the $2g-2$ coordinates $a_1, b_1, \ldots, a_{g-1}, b_{g-1}$ descend to a basis of $H^1(\zeta)$. Since ζ is trivial on these factors, the cup product is just the standard symplectic form on $\mathbb{R}^{2g-2}$. Also, the form $\langle A, B\rangle = \frac{1}{4\pi^2}\operatorname{tr} AB$ on diagonal matrices is $\langle a\sigma_3, b\sigma_3\rangle = \frac{1}{2\pi^2}ab$. Hence ω_τ restricts to $\frac{1}{2\pi^2}\sum_i da_i \wedge db_i$ on F_0. In other words, F_0 splits as a product of 2-tori, and $\omega_\tau = 2\theta$, where θ is the sum of the unit volume forms on the 2-tori. Then certainly $\theta^{g-1}[F_0] = (g-1)!$, so the proof of (a) is complete. (By the way, the diligent reader might find this a good point to solve the exercise posed in the last section but one.)

The restrictions of the differentials to ξ are more complicated, but the last two components are given by $d_1(t) = (\ldots, -2t, \bar{t} - t)$ and $d_2(0, \ldots, 0, a_g, b_g) = a_g - \bar{a}_g + 2\bar{b}_g$, where t, a_g, and b_g are now complex numbers. Hence if $a_g = 0$ and a_i, b_i are arbitrary for $i < g$, then there is a unique b_g such that $d_2(a_i, b_i) = 0$. This fixes an isomorphism $\mathbb{C}^{2g-2} \cong H^1(\xi)$ which depends continuously on ρ, with U(1) acting with weight 1 on each coordinate. This shows that ν_{F_0/N^g} is trivial, and acted on with weight 1 relative to the trivialization.

The cup product $H^1(\xi) \otimes H^1(\xi) \to H^2(\xi \otimes \xi)$ is linear over $\mathbb{C}$, but the form $\langle A, B\rangle = \frac{1}{4\pi^2}\operatorname{tr} AB$ on off-diagonal matrices is

$$\left\langle \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \right\rangle = -\frac{1}{2\pi^2}\operatorname{Re} a\bar{b}.$$

Hence the orientation on ν_{F_0/N^g} coming from $\mathbb{C}^{2g-2}$ differs from that induced by the symplectic orientations on F_0 and N^g by a factor of $(-1)^{g-1}$. So the equivariant Euler class with respect to the symplectic orientation is $(-1)^{g-1}u^{2g-2}$, as claimed in (b). $\square$

Lemma. (a) *Both F_+ and F_- are diffeomorphic to the fibred product with itself of the S^2-bundle $(\mu_{g-1}^{-1}(-I) \times S^2)/\mathrm{SU}(2)$ over N^{g-1};* (b) *as rings,*

$$H^*(F_\pm) \cong H^*(N^{g-1})[h_1, h_2]/(h_1^2 - \beta/4, h_2^2 - \beta/4);$$

(c) *regarded as the total space of a* $\mathrm{U}(1)$*-bundle over $F_\pm$, $f^{-1}(\pm\tau)$ has first Chern class $h_1 + h_2$;* (d) *the cohomology class of $\omega_\tau|_{F_\pm}$ is $\alpha/2 + (1-\tau)(h_1 + h_2)$.*

Proof. Parts (a) and (c) follow straightforwardly from the local model lemma: the point is that, on $\mu_{g-1}^{-1}(-I) \times (\mathrm{SU}(2) \backslash \{I\}) \times \mathrm{SU}(2)$, the relevant $\mathrm{U}(1)$-action involves only the last two factors and is completely explicit.

The S^2-bundle over N^{g-1} is isomorphic to $\mathbb{P}U|_{N^{g-1} \times p}$ for $p \in X$, where U is the universal bundle; this follows from the topological construction of $\operatorname{End} U$. Since

$$(c_1^2 - 4c_2)(U|_{N^{g-1} \times p}) = -c_2(\operatorname{End} U|_{N^{g-1} \times p}) = \beta,$$

part (b) follows from standard facts on the cohomology rings of projective bundles [11, §20].

It is easy to check that on N^g, the restriction of ω to $S_\pm$ is $\pi^*\omega$, where $\pi : S_\pm \to N^{g-1}$ is the projection. Hence as τ approaches 1, the cohomology class $[\omega_\tau]|_{F_\pm}$ approaches $\alpha/2$. On the other hand, the first theorem of Duistermaat and Heckman [18, Thm. 1.1], applied to $N^g_{\tau'}$ for $\tau' > \tau$, asserts that $[\omega_\tau]|_{F_\pm}$ is an affine function of τ with derivative the first Chern class given in part (c). This implies part (d). □

Proposition. (1) $\tilde{\alpha}_\tau|_{F_\pm} = \alpha + 2(1-\tau)(h_1 + h_2) + \frac{2}{\pi}\arccos(\pm\tau)u$*;* (2) $\tilde{\alpha}_\tau|_{F_0} = 4\theta + u$*;* (3) $\tilde{\beta}_\tau|_{F_\pm} = \beta$*;* (4) $\tilde{\beta}_\tau|_{F_0} = u^2$.

Proof. Since $\tilde{\alpha}_\tau = \alpha_\tau + \frac{2}{\pi}\arccos f$, (1) follows from part (d) of the last lemma, and (2) follows from part (a) of the one before that.

Since E_τ restricted to $\mu_g^{-1}(\pm\tau)/\mathrm{U}(1)$ descends from $\mu_g^{-1}(\pm\tau)$, certainly $\tilde{\beta}_\tau|_{F_\pm} = \beta_\tau|_{F_\pm}$. But this equals the pull-back from N^{g-1} of β by, for example, the proof of the proposition in the last section.

Finally, to prove (4), it suffices to calculate the equivariant second Chern class of the bundle E_τ restricted to F_0, or equivalently, of E restricted to S_0. Now the explicit formula $A_g = i\sigma_3$, $B_g = i\sigma_2$, A_i, B_i diagonal for $i < g$ gives a lifting of S_0 to $\mu_g^{-1}(-I)$. Because of this lifting, $E|_{S_0}$ is the trivial bundle $S_0 \times \operatorname{End}\mathbb{C}^2$. But it is not equivariantly trivial. Indeed, since A_g is diagonal, $\lambda \in \mathrm{U}(1)$ acts by $B_g \mapsto B_g \operatorname{diag}(\lambda, \lambda^{-1})$. To restore $B_g = i\sigma_2$, one must conjugate by a square root $\operatorname{diag}(\lambda^{1/2}, \lambda^{-1/2})$; this acts on $\operatorname{End}\mathbb{C}^2$ with weights 0, 0, 1, and -1, so the equivariant Chern roots of the bundle are 0,0, u and $-u$, and hence $\tilde{\beta}_\tau|_{F_0} = -u^2$. □

This leaves only the normal bundles to $F_\pm$ to be determined. But it follows from part (c) of the last lemma that they are $h \mp u$, where h is short for $h_1 + h_2$.

Cohomological calculations

At last we are ready to apply the localization formula. Putting together all the results of the last two sections,

$$\begin{aligned}
\alpha^m\beta^n[N^g] &= \lim_{\tau\to 1}\alpha_\tau^m\beta_\tau^n[N_\tau^g] \\
&= \lim_{\tau\to 1}\sum_i \frac{\alpha_\tau^m\beta_\tau^n|_{F_i}}{e(\nu_{F_i/N_\tau^g})}\,[F_i] \\
&= \lim_{\tau\to 1}\frac{(\alpha+2(1-\tau)h+\frac{2}{\pi}\arccos(-\tau)u)^m\beta^n}{h+u}\,[F_-] \\
&\quad +\lim_{\tau\to 1}\frac{(\alpha+2(1-\tau)h+\frac{2}{\pi}\arccos(\tau)u)^m\beta^n}{h-u}\,[F_+] \\
&\quad +\lim_{\tau\to 1}\frac{(4\theta+u)^m u^{2n}}{(-1)^{g-1}u^{2g-2}}\,[F_0].
\end{aligned}$$

When $\tau\to 1$, there are no terms in u left in the numerator of the class on F_+, so the constant term in its Laurent expansion is 0. Hence

$$\begin{aligned}
\alpha^m\beta^n[N^g] &= \frac{(\alpha+2u)^m\beta^n}{h+u}\,[F_-]+\frac{(4\theta+u)^m u^{2n}}{(-1)^{g-1}u^{2g-2}}\,[F_0] \\
&= \frac{1}{u}\,(\alpha+2u)^m\beta^n\sum_{i=0}^{\infty}\left(\frac{-h}{u}\right)^i[F_-]+\frac{(4\theta+u)^m u^{2n}}{(-1)^{g-1}u^{2g-2}}\,[F_0] \\
&= \sum_{i=0}^{\infty}(-1)^i\binom{m}{i+1}2^{i+1}\,\alpha^{m-i-1}\beta^n h^i\,[F_-]+(-1)^{g-1}\binom{m}{g-1}(4\theta)^{g-1}\,[F_0].
\end{aligned}$$

The last term is easily evaluated using $\theta^{g-1}[F_0]=(g-1)!$. The terms in the sum, on the other hand, can be evaluated with the help of part (b) of the last lemma. Indeed, this implies that for $j>0$, $h^{2j}=\frac{1}{2}\beta^j+2\beta^{j-1}h_1h_2$ and $h^{2j-1}=\beta^{j-1}h$, but also that for any $\eta\in H^*(N^{g-1})$, $\eta[F_\pm]=0$, $\eta h[F_\pm]=0$, and $\eta h_1h_2[F_\pm]=\eta[N^{g-1}]$. Hence in the sum, only those terms where i is even make a nonzero contribution; so

$$\alpha^m\beta^n[N^g]=\sum_{j=1}^{\left[\frac{m-1}{2}\right]}\binom{m}{2j+1}2^{2j+2}\,\alpha^{m-2j-1}\beta^{n+j}[N^{g-1}]+(-1)^{g-1}\,2^{2g-2}\,\frac{m!}{(m-g+1)!},$$

unless $n\geq g$, or equivalently $m<g-1$, for then $\binom{m}{g-1}=0$ and hence the last term disappears. Since $n\geq g$ certainly implies $n+j\geq g-1$ for all positive j, applying the equation recursively then shows that $\alpha^m\beta^n[N^g]=0$ whenever $n\geq g$.

To deduce the remaining pairings, first repackage them as follows. Let $k=g-1-n$ and note that $2k=m-g+1$; then let

$$I_k^g=\frac{(-1)^g}{2^{2g-2}m!}\,\alpha^m\beta^n[N^g].$$

The recursion above then becomes

$$I_k^g+\sum_{j=1}^{k}\frac{2^{2j}}{(2j+1)!}\,I_{k-j}^{g-1}=\frac{1}{(2k)!},$$

the sum only needing to run up to k because of the vanishing deduced above. Now $2^{2j}/(2j+1)! = \operatorname{Coeff}_{u^{2j}} \sinh(2u)/(2u)$ and $1/(2k)! = \operatorname{Coeff}_{u^{2k}} \cosh u$. Hence the solution to the recursion is

$$I_k^g = \operatorname*{Coeff}_{u^{2k}} \frac{2u \cosh u}{\sinh(2u)} = \operatorname*{Coeff}_{u^{2k}} \frac{u}{\sinh u},$$

the first term playing the role of $j = 0$ in the sum. These coefficients can be expressed in terms of Bernoulli numbers: indeed, $I_k^g = (2^{2k} - 2)B_{2k}/(2k)!$ (see for example Milnor and Stasheff [39, Appendix B], but beware of their conflicting notation for the Bernoulli numbers). Hence for $m \geq g - 1$

$$\begin{aligned} \alpha^m \beta^n [N^g] &= (-1)^g \, 2^{2g-2} \, m! \operatorname*{Coeff}_{u^{m-g+1}} \frac{u}{\sinh u} \\ &= (-1)^g \, 2^{2g-2} \frac{m!}{(m-g+1)!} (2^{m-g+1} - 2) B_{m-g+1}. \end{aligned}$$

The SU(2) Verlinde formula

As an application of the above formula, we revisit the holomorphic category and study the spaces of holomorphic sections of line bundles on N^g. The Picard group of N^g can be identified with $H^1(N^g, \mathcal{O}^\times)$, where $\mathcal{O}^\times$ is the sheaf of nowhere zero holomorphic functions. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow H^1(N^g, \mathcal{O}) \longrightarrow H^1(N^g, \mathcal{O}^\times) \xrightarrow{c_1} H^2(N^g, \mathbb{Z}) \longrightarrow H^2(N^g, \mathcal{O}) \longrightarrow \cdots$$

We saw earlier that $H^2(N^g, \mathbb{Z}) \cong \mathbb{Z}$, with generator α. Let $\mathcal{L}$ be a holomorphic line bundle with $c_1(\mathcal{L}) = \alpha$. A calculation like that used to show $\alpha \neq 0$ implies that $c_1(K_{N^g}) = -2\alpha$. Kodaira vanishing then implies that for all $k \geq 0$, $H^i(N^g, \mathcal{L}^k) = 0$ for all $i > 0$, so that $\dim H^0(N^g, \mathcal{L}^k) = \chi(N^g, \mathcal{L}^k)$. In particular, $H^1(N^g, \mathcal{O}) = H^2(N^g, \mathcal{O}) = 0$, so $\operatorname{Pic} N^g \cong \mathbb{Z}$, generated by $\mathcal{L}$.

For any positive line bundle on a projective variety, the function $\chi(N^g, \mathcal{L}^k)$ is a polynomial, the so-called *Hilbert polynomial*. The Hirzebruch-Riemann-Roch theorem [29, Appendix A] states that

$$\begin{aligned} \chi(N^g, \mathcal{L}^k) &= (\operatorname{ch} \mathcal{L}^k \operatorname{td} N^g)[N^g] \\ &= (\exp(k\alpha) \operatorname{td} N^g)[N^g]. \end{aligned}$$

But the Todd class is defined in terms of the Chern roots as

$$\begin{aligned} \operatorname{td} &= \prod_{\substack{\text{Chern} \\ \text{roots } \xi}} \frac{\xi}{1 - e^{-\xi}} \\ &= \prod_{\xi} \exp(\xi/2) \frac{\xi/2}{\sinh \xi/2}. \end{aligned}$$

Since $(\xi/2)/(\sinh\xi/2)$ is an even function of ξ, the second half of the product can be expressed formally in terms of the Pontrjagin roots, while the first half can be expressed in terms of c_1:

$$\mathrm{td} = \exp(c_1/2) \prod_{\substack{\text{Pontrjagin}\\ \text{roots } \zeta}} \frac{\sqrt{\zeta}/2}{\sinh\sqrt{\zeta}/2}.$$

As was explained long ago in remark 2, the tangent space to M^g at E is $H^1(X, \operatorname{End} E)$. The splitting $\operatorname{End} E = \mathcal{O} \oplus \operatorname{End}_0 E$ into scalar and trace-free parts corresponds to the splitting $TM^g = \det^* T \operatorname{Jac} X \oplus TN^g$. Hence the tangent bundle to N^g is the direct image $(R^1\pi_1)\operatorname{End}_0 U$, where U is the universal bundle and $\pi_1 : N^g \times X \to N^g$ is the projection. The Grothendieck-Riemann-Roch theorem can then be used to show that

$$\mathrm{ch}_{2n}(TN^g) = (2n)!\, 2(g-1)\beta^n,$$

and consequently,

$$\mathrm{ch}_{2n}(TN^g \oplus T^*N^g) = 4(g-1)\beta^n,$$
$$c(TN^g \oplus T^*N^g) = (1-\beta)^{2g-2},$$

and

$$p(TN^g) = (1+\beta)^{2g-2};$$

see Newstead [46] for details. Hence $2g-2$ of the Pontrjagin roots are β and the rest are 0. The Hirzebruch-Riemann-Roch theorem and the formula for the characteristic numbers derived in the last section, after a little calculation, give

$$\begin{aligned}
\chi(N^g, \mathcal{L}^k) &= \exp(k+1)\alpha \left(\frac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\right)^{2g-2} [N^g] \\
&= (-1)^g \operatorname*{Coeff}_{x^{3g-3}} \left[((k+1)x)^{g-1} \left(\frac{x}{\sinh x}\right)^{2g-2} \frac{(2k+2)x}{\sinh(2k+2)x}\right] \\
&= (-1)^g (4(k+1))^{g-1} \operatorname*{Res}_{x=0} \left[\left(\frac{1}{2\sinh x}\right)^{2g-2} \frac{(2k+2)x}{\sinh(2k+2)x}\, dx\right].
\end{aligned}$$

First substituting $z = e^{2x}$, then applying the residue theorem, yields

$$\begin{aligned}
\chi(N^g, \mathcal{L}^k) &= (-1)^g (4(k+1))^{g-1} \operatorname*{Res}_{z=1} \left[\frac{z^{g-1}}{(z-1)^{2g-2}} \frac{(2k+2)z^{k+1}}{z^{2k+2}-1} \frac{dz}{z}\right] \\
&= (-4(k+1))^{g-1} \sum_{\substack{\lambda^{2k+2}=1\\ \lambda\neq 1}} \operatorname*{Res}_{z=\lambda} \left[\frac{z^{g-1}}{(z-1)^{2g-2}} \frac{(2k+2)z^{k+1}}{z^{2k+2}-1} \frac{dz}{z}\right] \\
&= (-4(k+1))^{g-1} \sum_{\substack{\lambda^{2k+2}=1\\ \lambda\neq 1}} -\frac{\lambda^{g-1}\lambda^{k+1}}{(\lambda-1)^{2g-2}} \\
&= (k+1)^{g-1} \sum_{j=1}^{2k+1} \frac{(-1)^{j+1}}{\left(\sin\frac{j\pi}{2k+2}\right)^{2g-2}},
\end{aligned}$$

which is the SU(2) *Verlinde formula* in the degree 1 case [55, 52].

A formula like this one was originally obtained by Verlinde [55] in the degree 0 case. He used a very different method, namely a "factorization" or "gluing" principle, arising from conformal field theory and giving a recursion in the genus. The calculation above is essentially due to Zagier [52, 59]. However, similar formulas were obtained earlier by Dowker [15, 16] in another context.

The U(2) Verlinde formula

From this formula, a corresponding formula for M^g follows easily, as pointed out by Donagi and Tu [13]. Indeed, let $\kappa : \operatorname{Jac} X \times N^g \to M^g$ be the map given by the tensor product. This is a holomorphic principal bundle over M^g with structure group $(\mathbb{Z}/2)^{2g}$, as identified with the square roots of unity in $\operatorname{Jac} X$. Since $H^1(N^g, \mathcal{O}) = 0$, by [29, III Ex. 12.6] $\operatorname{Pic}(\operatorname{Jac} X \times N^g) = \operatorname{Pic} \operatorname{Jac} X \times \operatorname{Pic} N^g$. So if Θ is a fixed theta-divisor on $\operatorname{Jac} X$, any $\mathcal{P} \in \operatorname{Pic} M^g$ pulls back to $\pi_1^*(\mathcal{O}(j\Theta) \otimes \xi) \otimes \pi_2^* \mathcal{L}^k$ for some j, k, and $\xi \in \operatorname{Pic}_0 \operatorname{Jac} X$. We will compute the holomorphic cohomology of $\mathcal{P}$ in the case where j and k are positive.

By Kodaira vanishing and the Künneth formula, $\dim H^i(\operatorname{Jac} X \times N^g, \kappa^* \mathcal{P}) = 0$ for $i > 0$, while for $i = 0$ it is $\chi(\operatorname{Jac} X, \mathcal{O}(j\Theta) \otimes \xi)\, \chi(N^g, \mathcal{L}^k)$. By Riemann-Roch applied to $\operatorname{Jac} X$,

$$\chi(\operatorname{Jac} X, \mathcal{O}(j\Theta) \otimes \xi) = \operatorname{ch}(\mathcal{O}(j\Theta) \otimes \xi)\ \operatorname{td} \operatorname{Jac} X\, [\operatorname{Jac} X] = \exp(j\Theta)[\operatorname{Jac} X] = j^g.$$

On the other hand, because $(\mathbb{Z}/2)^{2g}$ is discrete, $\kappa_* \kappa^* \mathcal{P}$ is a rank 2^{2g} vector bundle and satisfies $H^i(M^g, \kappa_* \kappa^* \mathcal{P}) \cong H^i(\operatorname{Jac} X \times N^g, \kappa^* \mathcal{P})$. Moreover, by the push-pull formula, $\kappa_* \kappa^* \mathcal{P} = \mathcal{P} \otimes \kappa_* \kappa^* \mathcal{O}$.

Let $g : \operatorname{Jac} X \to \operatorname{Jac}_d X$ be given by $g(L) = L^2 \otimes \Lambda$, where Λ is the line bundle such that $N^g = \det^{-1}(\Lambda)$. Then the square

$$\begin{array}{ccc} \operatorname{Jac} X \times N^g & \xrightarrow{\kappa} & M^g \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \det} \\ \operatorname{Jac} X & \xrightarrow{g} & \operatorname{Jac}_d X \end{array}$$

commutes, so $\kappa_* \mathcal{O} = \det^* g_* \mathcal{O}$. But the natural $(\mathbb{Z}/2)^{2g}$-action on $g_* \mathcal{O}$ decomposes it as a sum of 1-dimensional weight spaces:

$$g_* \mathcal{O} = \bigoplus_{\substack{\zeta \in \operatorname{Pic}_0 \operatorname{Jac} X \\ \zeta^2 \cong \mathcal{O}}} \zeta.$$

Since $\mathcal{P} \otimes \det^* \zeta$ is topologically equivalent to $\mathcal{P}$, and the higher cohomology again vanishes,

$$\begin{aligned} \dim H^0(M^g, \kappa_* \kappa^* \mathcal{P}) &= \sum_{\zeta} \dim H^0(M^g, \mathcal{P} \otimes \det^* \zeta) \\ &= 2^{2g} \dim H^0(M^g, \mathcal{P}). \end{aligned}$$

Consequently,

$$\dim H^0(M^g, \mathcal{P}) = (j/4)^g \dim H^0(N^g, \mathcal{L}^k).$$

Combining this with the formula at the end of the last section yields the U(2) Verlinde formula in the degree 1 case.

It is worth remarking that Verlinde and the other physicists who worked on these spaces of holomorphic sections obtained much more than a formula for their dimension. If the surface X is fixed as a smooth manifold up to diffeotopy (that is, diffeomorphic isotopy), then its complex structures are parametrized by Teichmüller space. This is therefore the base of a vector bundle whose fiber is the space of holomorphic sections on the moduli space, with respect to the given complex structure. The physicists found a canonical *projectively flat connection* on this bundle, or equivalently, since Teichmüller space is contractible, a canonical trivialization of its projectivization. So the spaces of holomorphic sections coming from different complex structures can be identified with one another, up to a scalar. Since these spaces are regarded in physics as the "quantizations" of the moduli spaces, one may, somewhat fancifully, think of the projectively flat connection as a quantum analogue of the Narasimhan-Seshadri theorem.

Characteristic numbers (reprise)

So far, we have computed those characteristic numbers involving only α and β. We shall now explain how to include the ψ_i as well. The material in this section is drawn from [52].

Proposition. *The cohomology pairing* $(\alpha^m\beta^n\prod_i\psi_i^{p_i})[N^g]$ *is 0 unless* $p_i = p_{i+g} \leq 1$ *for* $1 \leq i \leq g$.

Proof. Since $\psi_i, \psi_{i+g} \in H^3(N^g)$, certainly $\psi_i^2 = \psi_{i+g}^2 = 0$, so $p_i, p_{i+g} \leq 1$ is necessary to get a nonzero pairing. Now any homeomorphism $f : X \to X$ induces a symplectomorphism $\hat{f} : N^g \to N^g$, so

$$\hat{f}^*(\alpha^m\beta^n\prod_i\psi_i^{p_i})[N^g] = (\alpha^m\beta^n\prod_i\psi_i^{p_i})[N^g].$$

But topologically, $(\hat{f}\times f)^*\operatorname{End}U \cong \operatorname{End}U$, so $\hat{f}^*\alpha = \alpha$, $\hat{f}^*\beta = \beta$, and

$$\sum_i \hat{f}^*\psi_i f^*a^i + \hat{f}^*\psi_{i+g}f^*b^i = \sum_i \psi_i a^i + \psi_{i+g}b^i.$$

If f is a half twist of the jth handle, then $f^*a^j = -a^j$ and $f^*b^j = -b^j$, but the other a^i and b^i are fixed; so $\hat{f}^*\psi_j = -\psi_j$, $\hat{f}^*\psi_{j+g} = -\psi_{j+g}$, and $\hat{f}^*\psi_i = \psi_i$ otherwise. Hence $p_j = 1$, $p_{j+g} = 0$ implies

$$(\alpha^m\beta^n\prod_i\psi_i^{p_i})[N^g] = -(\alpha^m\beta^n\prod_i\psi_i^{p_i})[N^g] = 0,$$

and similarly if $p_j = 0$, $p_{j+g} = 1$. $\square$

Every appearance of ψ_i in a nonzero pairing must therefore be matched by an appearance of ψ_{i+g}; so let $\gamma_i = \psi_i\psi_{i+g}$ for $1 \leq i \leq g$.

Proposition. *If* $i_1, \ldots, i_p$ *are distinct, then* $(\alpha^m\beta^n\gamma_{i_1}\cdots\gamma_{i_p})[N^g]$ *is independent of the choice of* $i_1, \ldots, i_p$.

Proof. Similar to the previous proposition, but using a diffeomorphism interchanging the relevant handles. □

In the following proposition, the notation $\langle A_i \rangle^*$ denotes the Poincaré dual of the locus where $A_i = I$.

Proposition. *For $1 \leq i \leq g$, $\psi_i = \langle A_i \rangle^*$ and $\psi_{i+g} = \langle B_i \rangle^*$.*

Proof. By definition, if U is a universal bundle, then the Künneth component of $c_2(\operatorname{End} U)$ in $H^3(N^g) \otimes H^1(X)$ is $4\sum_{i=1}^g \psi_i a^i + \psi_{i+g} b^i$. If a_i and b_i are regarded as loops on X, this means that $c_2(\operatorname{End} U|_{N^g \times a_i}) = 4\psi_i a^i$, and similarly for b_i.

The universal cover of a_i is $\mathbb{R}$, so the topological construction of $\operatorname{End} U|_{N^g \times a_i}$ amounts to the following. There is a vector bundle $\mathcal{U}_i$ over $\mu_g^{-1}(-I) \times a_i$ defined as the quotient $(\mu_g^{-1}(-I) \times \mathbb{R} \times \mathbb{C}^2)/\mathbb{Z}$, where the action is given by $n \cdot (\rho, t, v) = (\rho, t+n, \rho(a_i)^n v)$. The action of $\mathrm{SU}(2)$ on $\mu_g^{-1}(-I)$ lifts naturally to $\mathcal{U}_i$, and $\operatorname{End} \mathcal{U}_i$ descends to $U|_{N^g \times a_i}$. But by its construction, $\mathcal{U}_i$ is pulled back from a vector bundle over $\mathrm{SU}(2) \times a_i$, defined as a quotient $(\mathrm{SU}(2) \times \mathbb{R} \times \mathbb{C}^2)/\mathbb{Z}$. It is easy to check that the second Chern class of this latter bundle is the fundamental class, which of course equals the Poincaré dual of $\{I\} \subset \mathrm{SU}(2)$ times a^i. Hence $c_2(\mathcal{U}_i) = \langle A_i \rangle^* a^i$. Since $\mathcal{U}_i$ has structure group $\mathrm{SU}(2)$, $c_2(\operatorname{End} \mathcal{U}_i) = 4c_2(\mathcal{U}_i) = 4\langle A_i \rangle^* a^i$. Since pull-back is Poincaré dual to inverse image, $c_2(\operatorname{End} U|_{N^g \times a_i}) = 4\langle A_i \rangle^* a^i$ also. The proof for b^i is of course similar. □

Since the loci where $A_i = I$ and $B_i = I$ intersect transversely, this proposition implies that γ_i is Poincaré dual to the locus where $A_i = I = B_i$. This locus can be identified with N^{g-1}, and α and β on N^g restrict to their counterparts on N^{g-1}. Thus the proposition implies the following corollary, which completes the computation of the characteristic numbers of N^g.

Corollary. *For $2m + 4n + 6\sum_i p_i = 6g - 6$,*

$$(\alpha^m \beta^n \prod_i \gamma_i^{p_i})[N^g] = (\alpha^m \beta^n)[N^{g-p}],$$

where $p = \sum p_i$. In particular for $2m + 4n + 6p = 6g - 6$,

$$(\alpha^m \beta^n \gamma^p)[N^g] = \frac{2^p g!}{(g-p)!}(\alpha^m \beta^n)[N^{g-p}],$$

where $\gamma = 2\sum_i \gamma_i$.

In particular, for any $\eta \in H^{2g-6}(N^g)$, $\beta^g \eta[N^g] = 0$. It follows from Poincaré duality that $\beta^g = 0$. This confirms an old conjecture of Newstead [46].

In a remarkable work, Witten [58] has given general formulas for the characteristic numbers in arbitrary rank, which include all the rank 2 formulas stated in this paper. A rigorous proof of Witten's formulas has recently been announced by Jeffrey and Kirwan [31].

The relations in the cohomology ring

In this final section, rather than try to prove anything more, we will merely state a few of the recent results on the relations in the cohomology ring of N^g.

The earliest conjectures on the relations between the generators of $H^*(N^g)$ go back to unpublished work of Mumford [4], who pointed out the following. Let K_X be the canonical bundle of X and P be the Poincaré line bundle over $\operatorname{Jac} X \times X$, and consider the bundle $\mathcal{E} = (\pi_1 \times \pi_3)^* U \otimes (\pi_2 \times \pi_3)^* P \otimes \pi_3^* K_X$ over $N^g \times \operatorname{Jac} X \times X$, where π_i represent the projections on the various factors. Then by Riemann-Roch, $\mathcal{F} = (\pi_1 \times \pi_2)_* \mathcal{E}$ is a rank $2g-1$ bundle over $N^g \times \operatorname{Jac} X$. For any $\eta \in H^*(\operatorname{Jac} X)$ and any $i \geq 0$, $(\pi_1)_*(c_i(\mathcal{F}) \cdot \eta)$ can be evaluated using the Grothendieck-Riemann-Roch theorem as a polynomial in the generators α, β, and the ψ_i. For $i > 2g-1$, this must vanish, giving relations between the generators.

Mumford conjectured that these generated all the relations in the cohomology ring. This indeed turns out to be true, as was proved much later by Kirwan [37]. But there is considerable redundancy among the Mumford relations. In fact, the shape of the Harder-Narasimhan formula suggests that the cohomology ring is isomorphic to the quotient of $\mathbb{Q}[\alpha,\beta] \otimes \Lambda^*(\psi_i)$, whose Poincaré polynomial is given by the first term of the formula, by the ideal freely generated over $\mathbb{Q}[\alpha,\beta]$ by the Mumford relations for $i = 2g$ only, whose Poincaré polynomial would be exactly the second term of the formula. This question remains open.

More recently, the following very explicit characterization of the ring has been obtained by several authors [6, 35, 50, 59].

Theorem. *As an* $\mathrm{Sp}(2g,\mathbb{Z})$*-algebra,*

$$H^*(N^g,\mathbb{Q}) \cong \bigoplus_{k=0}^{g} \Lambda_0^k H^3(N^g) \otimes \mathbb{Q}[\alpha,\beta,\gamma]/I_{g-k}$$

where

$$\Lambda_0^k = \ker \gamma^{g-k+1} : \Lambda^k H^3(N^g) \to \Lambda^{2g-k+2} H^3(N^g)$$

and I_k *is generated by the 3 nonzero elements of*

$$\prod_{i=1}^{k} \begin{pmatrix} \alpha & (k-i)^2 & 0 \\ \beta & 0 & \frac{2(k-i)}{k-i+1} \\ \gamma & 0 & 0 \end{pmatrix}.$$

Theoretical physicists have recently introduced the notion of *quantum cohomology*, which is a deformation of the ring structure of the cohomology of a smooth complex variety with negative canonical bundle, depending on a parameter q. Recent physical work of Bershadsky et al. [9] indicates that the quantum cohomology of N^g is characterized by the theorem above, but with the matrix entry β replaced with $\beta + (-1)^{k-i+1} 8q$. A rigorous mathematical proof of this fact has been announced by Siebert and Tian.

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