Mathematics W4061y
Differentiable Manifolds

Answers to Practice Midterm

March 12, 2014

1. Suppose $W \subset \mathbf{R}^n \times \mathbf{R}^m$ is open, $f : W \to \mathbf{R}^m$ is C^1 in a neighborhood of (a, b), $f(a, b) = 0$, and the $m \times m$ matrix with i, j entry $(\partial f_i/\partial x_{n+j})(a, b)$ is invertible. Then there exist open $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$ and a C^1 function $g : U \to V$ such that $(a, b) \in U \times V \subset W$ and, for all $(x, y) \in U \times V$, $f(x, y) = 0$ if and only if $y = g(x)$.

2. $\operatorname{Alt} T(v_1, \ldots, v_n) = \dfrac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \; T(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$.

3. If $\psi_1 : V_1 \to M \cap U_1$ and $\psi_2 : V_2 \to M \cap U_2$ are two charts for M, the overlap map between them is $\psi_1^{-1} \circ \psi_2 : \psi_2^{-1}(U_1) \to \psi_1^{-1}(U_2)$. It is a smooth diffeomorphism of open sets in $\mathbf{R}^k$.

4. This is linear since $T_{A+B}(u, v) = u^t(A + B)v = u^tAv + u^tBv = T_A(u, v) + T_B(u, v)$ and $T_{xA} = u^t(xA)v = x(u^tAv) = xT_A(u, v)$ for $A, B \in M_{n\times n}$ and $x \in \mathbf{R}$. It is injective since $T_A = 0$ implies $0 = T_A(e_i, e_j) = e_i^tAe_j = A_{i,j}$ for all i, j, hence $A = 0$. Since domain and range both have dimension n^2, it is an isomorphism.

Extra credit: the skew-symmetric matrices correspond to alternating tensors, since $A = -A^t$ implies $T_A(v, u) = v^tAu = (v^tAu)^t = u^tA^tv = -u^tAv = T(u, v)$ and, conversely, if T_A is alternating, then $A_{j,i} = e_j^tAe_i = T_A(e_j, e_i) = -T_A(e_i, e_j) = -e_i^tAe_j = -A_{i,j}$.

5. Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be $f(x, y, z) = (x^2 + y^2 - 3z^2 - 2x, -x^2 - y^2 + z^2 - 1)$. Then the set in question is $M = f^{-1}(0, 0)$, and the Jacobian matrix is

$$f'(x, y, z) = \begin{pmatrix} 2x - 2 & 2y & -6z \\ -2x & -2y & 2z \end{pmatrix}.$$

It suffices to show that, for all $(x, y, z) \in M$, this has rank 2. If not, then all columns are multiples of a single vector, so the determinants of 2×2 minors vanish. Hence $0 = (2x - 2)(-2y) - (2y)(-2x) = 4y$, so $y = 0$, and $0 = (2x - 2)(2z) - (-6z)(-2x) = -4z - 8xz$, so $z = 0$ or $x = -1/2$. In the former case the right-hand equation becomes $-x^2 = 1$, which is impossible, and in the latter case the two equations sum to $-2z^2 = 0$, so $z = 0$, returning us to the former case.

6. If $f : M \to N$ is a diffeomorphism with inverse $g : N \to M$ and with $f(x) = y$, then $f \circ g = \mathrm{id}_N$, so by the chain rule for manifolds, $Df(x) \circ Dg(y) = D\mathrm{id}_N = \mathrm{id} : T_yN \to T_yN$. Similarly $Dg(y) \circ Df(x) = \mathrm{id} : T_xM \to T_xM$, so the two tangent spaces are isomorphic, so they have the same dimension. But a manifold has the same dimension as its tangent spaces.

7. A manifold of dimension 0 in $\mathbf{R}^n$ is discrete, and a manifold of dimension n in $\mathbf{R}^n$ is open. Since M is neither, it could only be a manifold of dimension 1. But the intersection of M with any open rectangular neighborhood of $(0, 0)$ fails both the vertical line test and the horizontal line test, so it isn't a graph of a function of either of the two variables. Hence M is not a manifold.