Mathematics W4061y
Differentiable Manifolds

Practice Midterm Exam

March 12, 2014

PART I: Statements and definitions (10 pts each).

1. Briefly and precisely state the *implicit function theorem*.

2. Define Alt $T(v_1, \ldots, v_n)$, where $T : V \times \cdots \times V \to \mathbf{R}$ is an n-tensor.

3. Carefully define the *overlap map* between two charts of a manifold, and name a significant property that it satisfies.

PART II: Proofs and calculations (15 pts each).

4. Let $M_{n\times n}$ denote the vector space of $n \times n$ matrices with real entries.

For $A \in M_{n\times n}$, let $T_A \in \bigotimes^2(\mathbf{R}^n)^*$ be defined by $T_A(u, v) = u^t Av$, where u^t is the transpose of u.

Show that the map taking A to T_A is a linear isomorphism $M_{n\times n} \to \bigotimes^2(\mathbf{R}^n)^*$.

[Extra credit: what subspace of $M_{n\times n}$ corresponds to alternating tensors?]

5. Show that the set of points in $\mathbf{R}^3$ satisfying the equations

$$x^2 + y^2 - 3z^2 = 2x \quad \text{and} \quad -x^2 - y^2 + z^2 = 1$$

is a 1-manifold.

6. Prove that if two manifolds are diffeomorphic, then they have the same dimension.

7. Let M be the union of the x-axis and the y-axis in $\mathbf{R}^2$. Is it a manifold? Either prove that it is or prove that it isn't. If it is, what is its dimension?