PART I: Statements and definitions (10 pts each).

1. Briefly and precisely state the *implicit function theorem*.

2. Define $\text{Alt} T(v_1, \ldots, v_n)$, where $T : V \times \cdots \times V \rightarrow \mathbb{R}$ is an $n$-tensor.

3. Carefully define the *overlap map* between two charts of a manifold, and name a significant property that it satisfies.

PART II: Proofs and calculations (15 pts each).

4. Let $M_{n \times n}$ denote the vector space of $n \times n$ matrices with real entries.
   
   For $A \in M_{n \times n}$, let $T_A \in \bigotimes^2(\mathbb{R}^n)^*$ be defined by $T_A(u, v) = u^t A v$, where $u^t$ is the transpose of $u$.
   
   Show that the map taking $A$ to $T_A$ is a linear isomorphism $M_{n \times n} \rightarrow \bigotimes^2(\mathbb{R}^n)^*$.
   
   [Extra credit: what subspace of $M_{n \times n}$ corresponds to alternating tensors?]

5. Show that the set of points in $\mathbb{R}^3$ satisfying the equations

   \[ x^2 + y^2 - 3z^2 = 2x \text{ and } -x^2 - y^2 + z^2 = 1 \]

   is a 1-manifold.

6. Prove that if two manifolds are diffeomorphic, then they have the same dimension.

7. Let $M$ be the union of the $x$-axis and the $y$-axis in $\mathbb{R}^2$. Is it a manifold? Either prove that it is or prove that it isn’t. If it is, what is its dimension?