Mathematics W4061y
Differentiable Manifolds

Answers to Practice Final

May 12, 2014

1. The tensor product is defined by $\phi \otimes \psi(v_1, \ldots, v_{k+\ell}) = \phi(v_1, \ldots, v_k)\, \psi(v_{k+1}, \ldots, v_{k+\ell})$.

2. Yes. We proved in class that if $f : \mathbf{R}^m \to \mathbf{R}^n$ is smooth and $Df(x)$ is surjective for all $x \in f^{-1}(c)$, then $f^{-1}(c)$ is a manifold of dimension $m - n$. In this case, $f(A) = \det A = ad - bc$ with the usual coordinates. The Jacobian matrix is therefore $(d \;\; -c \;\; -b \;\; a)$, which cannot be 0 if $ad - bc = 1$. It therefore defines a surjection $\mathbf{R}^4 \to \mathbf{R}$, so $SL(2)$ is a manifold of dimension 3.

Extra credit: expanding $\det A$ along the ith row and differentiating with respect to $a_{i,j}$ shows that $\frac{\partial \det}{\partial a_{i,j}}(A)$ equals $\pm$ the i, j minor of A. By Cramer's rule, these cannot all vanish for A invertible, so the Jacobian again defines a surjection $\mathbf{R}^{n^2} \to \mathbf{R}$, making $SL(n)$ a manifold of dimension $n^2 - 1$.

3. Consider the smooth map $f : (\rho_-, \rho_+) \times (\phi_-, \phi_+) \times (\theta_-, \theta_+) \to S$ given by $f(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$. Since its Jacobian determinant is

$$\begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix} = \rho^2 \sin\phi > 0,$$

f is a local diffeomorphism by the inverse function theorem. Because of the descriptions of ρ, ϕ, θ, it is also injective, hence a diffeomorphism. By change of variables and Fubini,

$$\begin{aligned} \int_S 1 &= \int_{\rho_-}^{\rho_+} \int_{\phi_-}^{\phi_+} \int_{\theta_-}^{\theta_+} |\det Df|\, d\rho\, d\phi\, d\theta \\ &= \int_{\rho_-}^{\rho_+} \rho^2 d\rho \int_{\phi_-}^{\phi_+} \sin\phi\, d\phi \int_{\theta_-}^{\theta_+} d\theta \\ &= (\rho_+^3 - \rho_-^3)(\cos\phi_- - \cos\phi_+)(\theta_+ - \theta_-)/3. \end{aligned}$$

4. (a) Both sides are bilinear in ω and η, so it suffices to take $\omega(x) = f(x)\, dx_I$ and $\eta(x) = g(x)\, dx_J$. Then

$$\begin{aligned} d(\omega \wedge \eta) &= \sum_{k=1}^n \frac{\partial (fg)}{\partial x_k} dx_k \wedge dx_I \wedge dx_J \\ &= \sum_{k=1}^n \left(\frac{\partial f}{\partial x_k} g + f \frac{\partial g}{\partial x_k} \right) dx_k \wedge dx_I \wedge dx_J \\ &= \sum_{k=1}^n \frac{\partial f}{\partial x_k} g\, dx_k \wedge dx_I \wedge dx_J + (-1)^p \sum_{k=1}^n f \frac{\partial g}{\partial x_k} dx_I \wedge dx_k \wedge dx_J \\ &= \left(\sum_{k=1}^n \frac{\partial f}{\partial x_k} dx_k \wedge dx_I \right) \wedge g\, dx_J + (-1)^p f\, dx_I \wedge \left(\sum_{k=1}^n \frac{\partial g}{\partial x_k} dx_k \wedge dx_J \right) \\ &= d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta. \end{aligned}$$

(b) Using (a) twice,

$$\begin{aligned} d(\omega \wedge \eta \wedge \nu) &= d(\omega \wedge \eta) \wedge \nu + (-1)^{p+q} \omega \wedge \eta \wedge d\nu \\ &= d\omega \wedge \eta \wedge \nu + (-1)^p \omega \wedge d\eta \wedge \nu + (-1)^{p+q} \omega \wedge \eta \wedge d\nu. \end{aligned}$$

5. If $\omega = f(x)\, dx_1 \cdots dx_m$ and $\eta = g(y)\, dy_1 \cdots dy_n$ are smooth forms supported in rectangles $A \subset \mathbf{R}^m$ and $B \subset \mathbf{R}^n$ respectively, then by Fubini

$$\int_{A \times B} \pi_1^* \omega \times \pi_2^* \eta = \int_{A \times B} f(x)\, g(y)\, dx\, dy = \int_A f(x)\, dx \int_B g(y)\, dy = \int_A \omega \int_B \eta.$$

Now choose oriented atlases ψ_α and $\tilde{\psi}_\beta$ for M and N respectively, and choose partitions of unity ϕ_α and $\tilde{\phi}_\beta$ subordinate to them. Then $\psi_\alpha \times \tilde{\psi}_\beta$ is an oriented atlas for $M \times N$ with subordinate partition of unity $\pi_1^* \phi_\alpha\, \pi_2^* \tilde{\phi}_\beta$, and

$$\begin{aligned} \int_{M \times N} \pi_1^* \mu \wedge \pi_2^* \nu &= \sum_{\alpha, \beta} \int_{V_\alpha \times \tilde{V}_\beta} (\psi_\alpha \times \tilde{\psi}_\beta)^* (\pi_1^* (\phi_\alpha \mu) \wedge \pi_2^* (\tilde{\phi}_\beta \nu)) \\ &= \left(\sum_\alpha \int_{V_\alpha} \psi_\alpha^* \phi_\alpha \mu \right) \left(\sum_\beta \int_{\tilde{V}_\beta} \tilde{\psi}_\beta^* \tilde{\phi}_\beta \nu \right) \\ &= \int_M \mu \int_N \nu. \end{aligned}$$

6. (a) It is well defined since if $[\omega] = [\tilde{\omega}] \in H^k(U)$, then $\omega - \tilde{\omega} = d\eta$ and $\int_M \omega - \int_M \tilde{\omega} = \int_M d\eta = \int_{\partial M} \eta = 0$ by Stokes.

(b) If $M = \partial N$, then by Stokes again $\int_M \omega = \int_{\partial N} \omega = \int_N d\omega = 0$ since ω is closed.

7. This means that $U = V \times \mathbf{R}^2$ where $V = U \cap (\mathbf{R} \times 0 \times 0)$ is an open set in $\mathbf{R}$. Hence $H^2(U) = H^2(V \times \mathbf{R}^2) \cong H^2(V) = 0$ since all 2-forms on an open set in $\mathbf{R}$ are zero.

8. Since $dg^* \eta = g^* d\eta = 0$, $g^* \eta$ is closed. By the Poincaré lemma, it is exact, say $g^* \eta = d\nu$. Then $h^* \eta = f^* g^* \eta = f^* d\nu = df^* \nu$ is exact too.

9. If h is not surjective, missing some $p \in S^n$, then its image is contained in $S^n \setminus p$ for some p. Now there is a diffeomorphism $g : \mathbf{R}^n \to S^n \setminus p$ given by rotation and stereographic projection; then $f = g^{-1} \circ h$ is also well defined and smooth, and $h = g \circ f$. Since any $\eta \in \Omega^n(S^n)$ is closed, $h^* \eta$ is exact by **8**. Then by Stokes $\int_M h^* \eta = 0$.

10. Since $d(\omega - \eta) = 0 \in \Omega^2(\mathbf{R}^n)$, by the Poincaré lemma $\omega - \eta = df$ for some $f \in \Omega^0(\mathbf{R}^n)$. If C and D have initial point $\vec{a}$ and terminal point $\vec{b}$, then by the fundamental theorem of line integrals (aka Stokes again),

$$\int_C \omega - \eta = f(\vec{b}) - f(\vec{a}) = \int_D \omega - \eta.$$

Another approach is to apply Stokes to a surface bounded by C and D: morally correct, but technically tricky since the surface may have corners at $\vec{a}$ and $\vec{b}$.