1. Define the tensor product \( \phi \otimes \psi \in T^{k+\ell}V \) of two tensors \( \phi \in T^kV \) and \( \psi \in T^\ell V \).

2. Let \( SL(2) = \{ A \in M_{2\times2} \mid \det A = 1 \} \). Is it a manifold? Either prove that it is or prove that it isn’t. If it is, what is its dimension? Extra credit: same question for \( SL(n) \).

3. Recall that the spherical coordinates \((\rho, \phi, \theta)\) of a vector \( \vec{v} \in \mathbb{R}^3 \) are determined by the formula
   \[
   (x, y, z) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi).
   \]
   Here \( \rho \) is the length of \( \vec{v} \); \( \phi \) is its angle with the vertical; and \( \theta \) is the angle that its projection onto the horizontal makes with the positive \( x \)-axis.
   
   Fix intervals \((\rho_-, \rho_+) \subset [0, \infty), (\phi_-, \phi_+) \subset [0, \pi], \) and \((\theta_-, \theta_+) \subset [0, 2\pi]\). Let \( S \subset \mathbb{R}^3 \) be the bounded open set consisting of points \( \vec{v} \) whose spherical coordinates \((\rho, \phi, \theta)\) lie in these three intervals, respectively. Use the change of variables formula to compute the volume of \( S \), that is, \( \int_S 1 \). Hint: the Jacobian determinant simplifies dramatically.

4. (a) For \( U \subset \mathbb{R}^n \) open, \( \omega \in \Omega^p(U) \), and \( \eta \in \Omega^q(U) \), prove that
   \[
   d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.
   \]
   (b) If \( \nu \in \Omega^r(U) \) as well, state and prove a similar formula for \( d(\omega \wedge \eta \wedge \nu) \).

5. If \( M \subset \mathbb{R}^m \) and \( N \subset \mathbb{R}^n \) are compact oriented manifolds of dimensions \( k \) and \( \ell \) respectively, \( \mu \in \Omega^k(\mathbb{R}^m) \), and \( \nu \in \Omega^\ell(\mathbb{R}^n) \), prove that
   \[
   \int_{M \times N} \pi_1^* \mu \wedge \pi_2^* \nu = \int_M \mu \int_N \nu.
   \]

6. (a) If \( U \subset \mathbb{R}^n \) is open and \( M \subset U \) is a compact oriented manifold of dimension \( k \), prove that the linear functional \( \int_M : H^k(U) \to \mathbb{R} \) is well defined.
   (b) If \( M = \partial N \) for some compact oriented \( N \), prove that this functional is zero.

7. Suppose \( U \subset \mathbb{R}^3 \) is any open set such that whenever \((x, y, z) \in U \) and \( s, t \in \mathbb{R} \), then \((x, y + s, z + t) \) is also in \( U \). Compute the de Rham cohomology \( H^2(U) \).

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8. Suppose $M$ and $N$ are manifolds, and suppose a smooth map $h : M \to N$ may be expressed as $h = g \circ f$ for some smooth $f : M \to \mathbb{R}^k$ and $g : \mathbb{R}^k \to N$. Prove that if $\eta \in \Omega^i(N)$ is closed for $i > 0$, then $h^* \eta$ is exact.

9. We proved that there exists $\eta \in \Omega^n(S^n)$ such that $\int_{S^n} \eta \neq 0$. If $M$ is a compact oriented manifold of dimension $n > 0$ and $h : M \to S^n$ is smooth but not surjective, prove that $\int_M h^* \eta = 0$.

10. Say $\omega, \eta \in \Omega^1(\mathbb{R}^n)$ satisfy $d\omega = d\eta$, and $C, D$ are compact connected parametric curves (i.e. 1-dimensional manifolds with boundary) having the same initial and terminal points. Show that

$$\int_C \omega + \int_D \eta = \int_D \omega + \int_C \eta.$$