Mathematics W4061y
Differentiable Manifolds

Practice Final Exam
May 12, 2014

Hint: problems may be useful in solving other problems.

Manifolds are without boundary unless otherwise stated.

1. Define the *tensor product* $\phi \otimes \psi \in T^{k+\ell}V$ of two tensors $\phi \in T^k V$ and $\psi \in T^\ell V$.

2. Let $SL(2) = \{A \in M_{2\times 2} \mid \det A = 1\}$. Is it a manifold? Either prove that it is or prove that it isn't. If it is, what is its dimension? Extra credit: same question for $SL(n)$.

3. Recall that the *spherical coordinates* (ρ, ϕ, θ) of a vector $\vec{v} \in \mathbf{R}^3$ are determined by the formula

$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi).$$

Here ρ is the length of $\vec{v}$; ϕ is its angle with the vertical; and θ is the angle that its projection onto the horizontal makes with the positive x-axis.

Fix intervals $(\rho_-, \rho_+) \subset [0, \infty)$, $(\phi_-, \phi_+) \subset [0, \pi]$, and $(\theta_-, \theta_+) \subset [0, 2\pi]$. Let $S \subset \mathbf{R}^3$ be the bounded open set consisting of points $\vec{v}$ whose spherical coordinates (ρ, ϕ, θ) lie in these three intervals, respectively. Use the change of variables formula to compute the volume of S, that is, $\int_S 1$. Hint: the Jacobian determinant simplifies dramatically.

4. (a) For $U \subset \mathbf{R}^n$ open, $\omega \in \Omega^p(U)$, and $\eta \in \Omega^q(U)$, prove that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta.$$

(b) If $\nu \in \Omega^r(U)$ as well, state and prove a similar formula for $d(\omega \wedge \eta \wedge \nu)$.

5. If $M \subset \mathbf{R}^m$ and $N \subset \mathbf{R}^n$ are compact oriented manifolds of dimensions k and ℓ respectively, $\mu \in \Omega^k(\mathbf{R}^m)$, and $\nu \in \Omega^\ell(\mathbf{R}^n)$, prove that

$$\int_{M\times N} \pi_1^*\mu \wedge \pi_2^*\nu = \int_M \mu \int_N \nu.$$

6. (a) If $U \subset \mathbf{R}^n$ is open and $M \subset U$ is a compact oriented manifold of dimension k, prove that the linear functional $\int_M : H^k(U) \to \mathbf{R}$ is well defined.

(b) If $M = \partial N$ for some compact oriented N, prove that this functional is zero.

7. Suppose $U \subset \mathbf{R}^3$ is any open set such that whenever $(x, y, z) \in U$ and $s, t \in \mathbf{R}$, then $(x, y + s, z + t)$ is also in U. Compute the de Rham cohomology $H^2(U)$.

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8. Suppose M and N are manifolds, and suppose a smooth map $h : M \to N$ may be expressed as $h = g \circ f$ for some smooth $f : M \to \mathbf{R}^k$ and $g : \mathbf{R}^k \to N$. Prove that if $\eta \in \Omega^i(N)$ is closed for $i > 0$, then $h^*\eta$ is exact.

9. We proved that there exists $\eta \in \Omega^n(S^n)$ such that $\int_{S^n} \eta \neq 0$. If M is a compact oriented manifold of dimension $n > 0$ and $h : M \to S^n$ is smooth but *not* surjective, prove that $\int_M h^*\eta = 0$.

10. Say $\omega, \eta \in \Omega^1(\mathbf{R}^n)$ satisfy $d\omega = d\eta$, and C, D are compact connected parametric curves (i.e. 1-dimensional manifolds with boundary) having the same initial and terminal points. Show that

$$\int_C \omega + \int_D \eta = \int_D \omega + \int_C \eta.$$