Mathematics W4061y
Differentiable Manifolds

Answers to Midterm Exam

March 12, 2014

1. A *tensor* is a map $F : V^k \to \mathbf{R}$ such that, for each i between 1 and k,

$$F(v_1, \ldots, v_{i-1}, tv_i + t'v_i', v_{i+1}, \ldots, v_k) = tF(v_1, \ldots, v_i, \ldots, v_k) + t'F(v_1, \ldots, v_i', \ldots, v_k).$$

It is *alternating* if for all i between 1 and $k-1$,

$$F(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -F(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k).$$

2. Let $B = \{x \in \mathbf{R}^n \,|\, |x| \le 1\}$, let $K \in [0, 1)$, and suppose $f : B \to B$ satisfies $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in B$. Then there exists a unique $x \in B$ such that $f(x) = x$.

3. For any chart $\psi : V \to M \cap U$ with $p \in M \cap U$, the *tangent space* at p is im $D_{\psi^{-1}(p)}\psi$. (This does not depend on ψ.)

4. Let $e^1, \ldots, e^n$ be a basis for V^*. Then $\omega = \sum_{i=1}^n a_i\, e^1 \wedge \cdots \wedge \hat{e}^i \wedge \cdots \wedge e^n$ with not all $a_i = 0$. Choose j so that $a_j \ne 0$ and let $\mu = e^j$; then $\omega \wedge \mu = a_j\, e^1 \wedge \cdots \wedge \hat{e}^j \wedge \cdots \wedge e^n \wedge e^j = (-1)^{n-j} a_j\, e^1 \wedge \cdots \wedge e^n \ne 0$.

5. It cannot. If there were an inverse k, then $\mathrm{id} = k \circ g \circ h$ and hence, by the chain rule for manifolds, $\mathrm{id} = D_{f(p)}k \circ D_{h(p)}g \circ D_p h : T_pM \to T_pM$. But $D_p h$, as a linear map from a 7-dimensional tangent space to a 5-dimensional tangent space, cannot be injective, so it cannot be invertible.

6. Let P be a partition of A and let S be a subrectangle of P. Since for all $x \in S$ we have $f(x) \le g(x)$, any lower bound for $f(S)$ is a lower bound for $g(S)$, so $\inf f(S) \le \inf g(S)$. Hence $\underline{I}(f, P) = \sum_S \inf f(S) \mathrm{vol}\, S \le \sum_S \inf g(S) \mathrm{vol}\, S = \underline{I}(g, P)$; taking suprema over all partitions P (and reasoning as before with upper bounds), we find $\int_A f = \underline{I}(f) = \sup\{\underline{I}(f, P) \,|\, P \text{ partition}\} \le \sup\{\underline{I}(g, P) \,|\, P \text{ partition}\} = \underline{I}(g) = \int_A g$.

7. The smooth map F_1 taking $(x, y) \mapsto (r, \theta) = (\sqrt{x^2 + y^2}, \arctan(y/x))$ maps $U \to (0, \infty) \times (0, \pi/2)$ with smooth inverse $(r, \theta) \mapsto (r \cos\theta, r \sin\theta)$, and the smooth map F_2 taking $(r, \theta) \mapsto (r - g(\theta), \theta)$ maps $(0, \infty) \times (0, \pi/2) \to V = \{(r, \theta) \in (0, \infty) \times (0, \pi/2) \,|\, r > -g(\theta)\}$ with smooth inverse $(r, \theta) \mapsto (r + g(\theta), \theta)$. Then $F_2 \circ F_1 : U \to V$ is a diffeomorphism of open sets and $S = (F_2 \circ F_1)^{-1}(0 \times \mathbf{R})$.