Mathematics W4081y
Differentiable Manifolds

Assignment #9

Due April 14, 2014

In Spivak, do problem **4–19**ab. Also do the following.

1. Compute the exterior derivative of the following forms. Remember, a hat means that the term is omitted.

(a) $e^{xyz}dx$

(b) $\sum_{i=1}^{n} x_i^2 dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

(c) $\|x\|^p \sum_{i=1}^{n} (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$, where p is a real constant. Be sure to simplify your answer.

2. Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be $f(r, \phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi)$, the map taking spherical coordinates to rectangular coordinates (x, y, z). Compute the pullback $f^*\alpha$ for the following forms α: (a) dx, (b) dy, (c) dz, (d) $dy \wedge dz$, (e) $dx \wedge dy \wedge dz$.

3. Let B_ϵ^n denote the n-dimensional ball $B_\epsilon(0) \subset \mathbf{R}^n$.

(a) Let $\lambda_n = \text{vol } B_1^n$; use change of variables to prove that vol $B_\epsilon^n = \epsilon^n \lambda_n$.

(b) Compute λ_1 and λ_2.

(c) Compute λ_n in terms of λ_{n-2}.

(d) Obtain a general formula for λ_n. Hint: divide into two cases according as n is even or odd.

4. Let $\eta = v\, du - u\, dv \in \Omega^1(\mathbf{R}^2)$ and let $\omega = 3u\, du \wedge dv \in \Omega^2(\mathbf{R}^2)$.
Also let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be $f(x, y, z) = (x^2 y, x^2 z)$.

(a) By an explicit computation of both sides, verify directly that $f^* d\eta = d\, f^*\eta$.

(b) By an explicit computation of both sides, verify directly that $f^*(\eta \wedge \omega) = f^*\eta \wedge f^*\omega$.

5. Let $U \subset \mathbf{R}^n$ be open and let $\Omega^*(U) = \bigoplus_{p=0}^{\infty} \Omega^p(U)$. Prove that the exterior derivative is the *unique* linear map $d : \Omega^*(U) \to \Omega^*(U)$ satisfying the following properties:

(i) $d(\Omega^p(U)) \subset \Omega^{p+1}(U)$;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$ for $\omega \in \Omega^p(U)$ and $\eta \in \Omega^q(U)$;

(iii) $df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$ for $f \in \Omega^0(U)$;

(iv) $d \circ d = 0$.