Mathematics W4081y
Differentiable Manifolds

Assignment #6
Due March 24, 2014

1. *The orthogonal group.* (20 pts) Let $M(n) = M_{n\times n}$ denote the set of $n \times n$ matrices with real entries; let $S(n)$ denote the subset consisting of symmetric matrices, that is, $S(n) = \{A \in M(n) \mid A^T = A\}$; and let $O(n)$ denote the subset consisting of orthogonal matrices, that is, $O(n) = \{A \in M(n) \mid A^T A = I\}$. This exercise will outline a proof that $O(n)$ is a manifold of dimension $n(n-1)/2$.

(a) Note that $M(n)$ is a vector space of dimension n^2, hence isomorphic to $\mathbf{R}^{n^2}$. Show that $S(n)$ is a linear subspace of dimension $1 + 2 + 3 + \cdots + n = n(n+1)/2$.

(b) Define $f : M(n) \to S(n)$ by $f(A) = A^T A$. Show that f is smooth and $O(n) = f^{-1}(I)$.

(c) Given $A, B \in M(n)$, prove that $Df(A)(B) = B^T A + A^T B \in S(n)$. Hint: consider a suitable parametric curve through A and use the chain rule.

(d) Show that, if $A \in O(n)$, then $Df(A)$ is surjective. Hint: to solve $X^T A + A^T X = C$ when C is symmetric, write $C = \frac{1}{2}C + \frac{1}{2}C^T$.

(e) Prove that $O(n)$ is a manifold of dimension $n(n-1)/2$.

2. Prove that the tangent space to $O(n)$ at the identity matrix I is the vector space $K(n)$ of all skew-symmetric $n \times n$ real matrices, that is, $K(n) = \{A \in M(n) \mid A^T = -A\}$.

3. Let M be a smooth manifold. For any tangent vector $v \in T_pM$, show there exists $\epsilon > 0$ and a regular parametric $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma'(0) = v$.

4. Recall that a topological space M is *connected* if the only subsets that are both closed and open are M and $\varnothing$. Prove that if a manifold M is connected, then any two points $p, q \in M$ are joined by a piecewise regular parametric curve: that is, there exist regular parametric curves $\gamma_i : [a_i, b_i] \to M$ with $\gamma_1(a_1) = p$, $\gamma_r(b_r) = q$, and $\gamma_i(b_i) = \gamma_{i+1}(a_i)$. Hint: fix p and consider the set of q joined to p by a prpc.

5. State a "slice" definition of a k-dimensional manifold with boundary, that is, a definition analogous to the definition of a manifold as a subset $M \subset \mathbf{R}^n$ such that every $p \in M$ has a neighborhood $U \subset \mathbf{R}^n$ and a diffeomorphism of open sets $F : U \to V \subset \mathbf{R}^n$ satisfying $F^{-1}(\mathbf{R}^k \times 0) = M \cap U$. Prove that it is equivalent to the "chart" definition of a manifold with boundary given in class.

6. Prove that the closed unit ball $B = \{x \in \mathbf{R}^n \mid |x| \leq 1\}$ is a manifold with boundary.

7. If $\gamma : [a, b] \to \mathbf{R}^n$ is smooth and injective with $\gamma'(t) \neq 0$ for all $t \in [a, b]$, prove that its image is a 1-dimensional manifold with boundary. (So no 6-figures in this case.)