Mathematics W4081y
Differentiable Manifolds

Assignment #5

Due March 3, 2014

1. Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be C^1. Assume that $f(3, -1, 2) = 0$ and that

$$D_{(3,-1,2)}f = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(a) Show there is a C^1 function $g : U \to \mathbf{R}^2$ defined on an open $U \subset \mathbf{R}$ containing 3 such that $f(x, g_1(x), g_2(x)) = 0$ for all $x \in U$, and $g(3) = (-1, 2)$.

(b) Find $g'(3)$.

(c) Discuss the problem of solving $f(x, y, z) = 0$ for an arbitrary pair of the unknowns in terms of the third near the point $(3, -1, 2)$.

2. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be C^1 with $f(a) = 0$ and suppose that $D_a f$ is surjective. Show that if c is a point of $\mathbf{R}^m$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.

In the remaining exercises, all manifolds are assumed to be smooth, that is, C^∞.

3. Prove that open subsets $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$ are manifolds of dimension n and m, respectively, and that a map $f : U \to V$ is smooth in the classical sense if and only if it is smooth as a map of manifolds.

4. More generally, if $M \subset \mathbf{R}^n$ is a manifold of dimension k and $U \subset \mathbf{R}^n$ is open, prove that $M \cap U$ is a manifold of dimension k.

5. Prove that any manifold of dimension n in $\mathbf{R}^n$ is an open subset.

6. Prove that $\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n \mid x \cdot y = 0, |x|\,|y| = 1\}$ is a manifold. What is its dimension?

7. Exercise **5–6** in Spivak is incorrect as stated. Give a proof of one direction and a counterexample to the other.

8. Exercise **5–7** in Spivak is also incorrectly stated. Do the corrected version, as follows. Let $\mathbf{K}^n = \{x \in \mathbf{R}^n \mid x_1 = 0 \text{ and } x_2 > 0\}$. If $M \subset \mathbf{K}^n$ is a k-dimensional manifold and N is obtained by rotating M around the axis $x_1 = x_2 = 0$ while leaving other variables fixed, show that N is a $(k+1)$-dimensional manifold. Use this to give another proof that the torus is a 2-dimensional manifold.

9. Let $M \subset \mathbf{R}^n$ be a k-dimensional manifold. Prove that the *cone*

$$C = \{(t, tx) \in \mathbf{R}^{n+1} \mid t > 0, x \in M\}$$

is a $(k+1)$-dimensional manifold. Sketch an example where M is a curve in the plane. (Style tip: index the coordinates in $\mathbf{R}^{n+1}$ from 0 to n.)