1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be $C^1$. Assume that $f(3, -1, 2) = 0$ and that
   \[ D_{(3,-1,2)}f = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}. \]
   (a) Show there is a $C^1$ function $g : U \to \mathbb{R}^2$ defined on an open $U \subset \mathbb{R}$ containing 3 such that $f(x, g_1(x), g_2(x)) = 0$ for all $x \in U$, and $g(3) = (-1, 2)$.
   (b) Find $g'(3)$.
   (c) Discuss the problem of solving $f(x, y, z) = 0$ for an arbitrary pair of the unknowns in terms of the third near the point $(3, -1, 2)$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be $C^1$ with $f(a) = 0$ and suppose that $D_a f$ is surjective. Show that if $c$ is a point of $\mathbb{R}^m$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.

   In the remaining exercises, all manifolds are assumed to be smooth, that is, $C^\infty$.

3. Prove that open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are manifolds of dimension $n$ and $m$, respectively, and that a map $f : U \to V$ is smooth in the classical sense if and only if it is smooth as a map of manifolds.

4. More generally, if $M \subset \mathbb{R}^n$ is a manifold of dimension $k$ and $U \subset \mathbb{R}^n$ is open, prove that $M \cap U$ is a manifold of dimension $k$.

5. Prove that any manifold of dimension $n$ in $\mathbb{R}^n$ is an open subset.

6. Prove that $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \cdot y = 0, \|x\| \|y\| = 1\}$ is a manifold. What is its dimension?

7. Exercise 5–6 in Spivak is incorrect as stated. Give a proof of one direction and a counterexample to the other.

8. Exercise 5–7 in Spivak is also incorrectly stated. Do the corrected version, as follows. Let $K^n = \{x \in \mathbb{R}^n \mid x_1 = 0 \text{ and } x_2 > 0\}$. If $M \subset K^n$ is a $k$-dimensional manifold and $N$ is obtained by rotating $M$ around the axis $x_1 = x_2 = 0$ while leaving other variables fixed, show that $N$ is a $(k+1)$-dimensional manifold. Use this to give another proof that the torus is a 2-dimensional manifold.

9. Let $M \subset \mathbb{R}^n$ be a $k$-dimensional manifold. Prove that the cone
   \[ C = \{(t, tx) \in \mathbb{R}^{n+1} \mid t > 0, x \in M\} \]
   is a $(k+1)$-dimensional manifold. Sketch an example where $M$ is a curve in the plane.
   (Style tip: index the coordinates in $\mathbb{R}^{n+1}$ from 0 to $n$.)