Mathematics W4081y
Differentiable Manifolds

Assignment #2

Due February 10, 2014

Let V be a vector space with basis $e_1, \dots, e_n$, and let V^* have the dual basis $e^1, \dots, e^n$.

1. (20 pts) A 2-tensor $f \in \bigotimes^2 V^*$ is said to be *symmetric* if for all $x, y \in V$, $f(x, y) = f(y, x)$.

(a) Show that the set S^2V^* of all symmetric tensors is a linear subspace of $\bigotimes^2 V^*$.

(b) What is a basis for S^2V^* in terms of a basis $e_1, \dots, e_n$ for V and the dual basis $e^1, \dots, e^n$ for V^*? What is $\dim S^2V^*$? Hint: try $V = \mathbf{R}^2$ to get a feeling.

(c) Define a linear map Sym: $\bigotimes^2 V^* \to S^2V^*$ similar to Alt and prove that its restriction to S^2V^* is the identity.

(d) Prove that Sym $\oplus$ Alt: $\bigotimes^2 V^* \to S^2V^* \oplus \Lambda^2V^*$ is an isomorphism.

(e) Define a linear map $M_{n\times n} \to \bigotimes^2 \mathbf{R}^n$, show it is an isomorphism, and show that it takes the symmetric and anti-symmetric matrices to S^2V^* and Λ^2V^*, respectively.

(f) Note that any inner product on V is an element of S^2V^*. Show that the set $C \subset S^2V^*$ consisting of inner products is a *cone* in the sense that $f, g \in C$, $a, b \in \mathbf{R}$, and $a, b > 0$ imply $af + bg \in C$.

(g) Sketch this cone in the case $V = \mathbf{R}^2$.

2. If $i_1, \dots, i_k \in \{1, \dots, n\}$ are distinct, show that

$$\operatorname{Alt}(e^{i_1} \otimes \dots e^{i_k})(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 0 & \text{if } j_1, \dots, j_k \text{ is not a reordering of } i_1, \dots, i_k \\ \frac{1}{k!} \operatorname{sgn} \sigma & \text{if each } j_\ell = i_{\sigma(\ell)} \text{ for some } \sigma \in S_k \end{cases}$$

3. If $1 \le i_1 < i_2 < \dots < i_k \le n$, prove that for any k-tuple $u_1, \dots, u_k$ of vectors in V, with $u_j = \sum u_{ij} e_i$, the number $e^{i_1} \wedge \dots \wedge e^{i_k}(u_1, \dots, u_k)$ is the determinant of the $k \times k$ matrix obtained by selecting rows $i_1, \dots, i_k$ from the $n \times k$ matrix $U = (u_{ij})$.

4. More generally, if $\alpha_1, \dots, \alpha_k \in V^*$, prove that

$$\alpha_1 \wedge \dots \wedge \alpha_k(u_1, \dots, u_k) = \det(\alpha_i(u_j)).$$

5. Prove that $\{e^{i_1} \wedge \dots \wedge e^{i_k} \,|\, 1 \le i_1 < i_2 < \dots < i_k \le n\}$ is a linearly independent set in $\Lambda^k V^*$.

6. Prove that the wedge product is the *unique* binary operation $\Lambda^k V^* \times \Lambda^\ell V^* \to \Lambda^{k+\ell} V^*$ satisfying the following properties:

(i) Linearity: $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$ and $(t\omega) \wedge \eta = t(\omega \wedge \eta)$

(ii) Anti-commutativity: $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$

(iii) Associativity: $(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi)$

(iv) If $\alpha_1, \dots, \alpha_k \in \Lambda^1 V^* = V^*$ and $u_1, \dots, u_k \in V$, then

$$\alpha_1 \wedge \dots \wedge \alpha_k(u_1, \dots, u_k) = \det(\alpha_i(u_j)).$$