Let $V$ be a vector space with basis $e_1, \ldots, e_n$, and let $V^*$ have the dual basis $e^1, \ldots, e^n$.

1. (20 pts) A 2-tensor $f \in \bigotimes^2 V^*$ is said to be symmetric if for all $x, y \in V$, $f(x, y) = f(y, x)$.

   (a) Show that the set $S^2 V^*$ of all symmetric tensors is a linear subspace of $\bigotimes^2 V^*$.
   (b) What is a basis for $S^2 V^*$ in terms of a basis $e_1, \ldots, e_n$ for $V$ and the dual basis $e^1, \ldots, e^n$ for $V^*$? What is dim $S^2 V^*$? Hint: try $V = \mathbb{R}^2$ to get a feeling.
   (c) Define a linear map $\text{Sym}: \bigotimes^2 V^* \to S^2 V^*$ similar to $\text{Alt}$ and prove that its restriction to $S^2 V^*$ is the identity.
   (d) Prove that $\text{Sym} \oplus \text{Alt}: \bigotimes^2 V^* \to S^2 V^* \oplus \Lambda^2 V^*$ is an isomorphism.
   (e) Define a linear map $M_{n \times n} \to \bigotimes^2 \mathbb{R}^n$, show it is an isomorphism, and show that it takes the symmetric and anti-symmetric matrices to $S^2 V^*$ and $\Lambda^2 V^*$, respectively.
   (f) Note that any inner product on $V$ is an element of $S^2 V^*$. Show that the set $C \subset S^2 V^*$ consisting of inner products is a cone in the sense that $f, g \in C$, $a, b \in \mathbb{R}$, and $a, b > 0$ imply $af + bg \in C$.
   (g) Sketch this cone in the case $V = \mathbb{R}^2$.

2. If $i_1, \ldots, i_k \in \{1, \ldots, n\}$ are distinct, show that

   \[
   \text{Alt}(e^{i_1} \otimes \ldots \otimes e^{i_k})(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 
   0 & \text{if } j_1, \ldots, j_k \text{ is not a reordering of } i_1, \ldots, i_k \\
   \frac{1}{k!} \text{sgn } \sigma & \text{if each } j_\ell = i_{\sigma(\ell)} \text{ for some } \sigma \in S_k
   \end{cases}
   \]

3. If $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, prove that for any $k$-tuple $u_1, \ldots, u_k$ of vectors in $V$, with $u_j = \sum u_{ij} e_i$, the number $e^{i_1} \wedge \cdots \wedge e^{i_k}(u_1, \ldots, u_k)$ is the determinant of the $k \times k$ matrix obtained by selecting rows $i_1, \ldots, i_k$ from the $n \times k$ matrix $U = (u_{ij})$.

4. More generally, if $\alpha_1, \ldots, \alpha_k \in V^*$, prove that

   \[
   \alpha_1 \wedge \cdots \wedge \alpha_k(u_1, \ldots, u_k) = \text{det}(\alpha_1(u_j)).
   \]

5. Prove that $\{e^{i_1} \wedge \cdots \wedge e^{i_k} | 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is a linearly independent set in $\Lambda^k V^*$.

6. Prove that the wedge product is the unique binary operation $\Lambda^k V^* \times \Lambda^\ell V^* \to \Lambda^{k+\ell} V^*$ satisfying the following properties:

   (i) Linearity: $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$ and $(t\omega) \wedge \eta = t(\omega \wedge \eta)$
   (ii) Anti-commutativity: $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$
   (iii) Associativity: $(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi)$
   (iv) If $\alpha_1, \ldots, \alpha_k \in \Lambda^1 V^* = V^*$ and $u_1, \ldots, u_k \in V$, then

   \[
   \alpha_1 \wedge \cdots \wedge \alpha_k(u_1, \ldots, u_k) = \text{det}(\alpha_i(u_j)).
   \]