Mathematics W4081y
Differentiable Manifolds

Assignment #12

Due May 5, 2014

1. If M, N are compact oriented manifolds, $f, g : M \to N$ are (smoothly) homotopic, and $\omega \in \Omega^p(N)$ is closed, show that

$$\int_M f^*\omega = \int_M g^*\omega.$$

2. Use de Rham cohomology to prove that $\mathbf{R}^2 \setminus 0$ is not diffeomorphic to a contractible open subset of $\mathbf{R}^2$.

3. Let $M \subset \mathbf{R}^n$ be a manifold, $\{\psi_\alpha : V_\alpha \to M \cap U_\alpha \mid \alpha \in A\}$ an atlas, $V_{\alpha\beta} = \psi_\alpha^{-1}(U_\beta)$, and $f_{\alpha\beta} = \psi_\alpha^{-1} \circ \psi_\beta \ : V_{\beta\alpha} \to V_{\alpha\beta}$. Recall that $\Omega^p(M)$ was defined in class as the subset of $\prod_{\alpha \in A} \Omega^p(V_\alpha)$ consisting of forms ω_α such that $\omega_\alpha|_{V_{\alpha\beta}} = f^*_{\alpha\beta}\omega_\beta$.

(a) Let $S = \{\omega \in \Omega^p(\mathbf{R}^n) \mid \text{ for all } x \in M \text{ and } v_1, \ldots, v_p \in T_xM,\ \omega(v_1, \ldots, v_p) = 0\}$. Prove that there is a natural isomorphism $\Omega^p(\mathbf{R}^n)/S \to \Omega^p(M)$. Hint: find a natural linear map $\Omega^p(\mathbf{R}^n) \to \Omega^p(M)$ and use a partition of unity to show that it is surjective.

(b) Show that the definition of $\Omega^p(M)$ given in class does not depend on the atlas.

4. Let $F : \mathbf{R} \to S^1 \subset \mathbf{R}^2$ be $F(t) = (\cos t, \sin t)$.

(a) Carefully show that the image of $F^* : \Omega^0(S^1) \to \Omega^0(\mathbf{R})$ is

$$\{f(t) \in \Omega^0(\mathbf{R}) \mid f(t + 2\pi) = f(t)\},$$

and that the image of $F^* : \Omega^1(S^1) \to \Omega^1(\mathbf{R})$ is

$$\{g(t)\, dt \in \Omega^1(\mathbf{R}) \mid g(t + 2\pi) = g(t)\}.$$

Hint: use the previous problem and a judicious choice of atlas on S^1.

(b) Use this to prove that $H^1(S^1) \cong \mathbf{R}$.

5. *Stereographic projection.*
Let $\mathbf{R}^{n+1}$ have coordinates $(x_0, \ldots, x_n)$, and let $S^n = \{\vec{x} \in \mathbf{R}^{n+1} \mid \|\vec{x}\| = 1\}$.

(a) For any $\vec{y} \neq \vec{e}_0 \in S^n$, find a formula for the unique point $\vec{z}$ on the line through $\vec{y}$ and $\vec{e}_0$ and on the plane $x_0 = 0$. Draw a picture.

(b) Carefully show that $S^n \setminus \{\vec{e}_0\}$ is diffeomorphic to $\mathbf{R}^n$.

(c) Show also that $S^n \setminus \{\pm\vec{e}_0\}$ is diffeomorphic to $S^{n-1} \times \mathbf{R}$.