Mathematics W4081y
Differentiable Manifolds

Assignment #11

Due April 28, 2014

1. Let $\psi_j : V_j \to M \cap U_j$ be a finite oriented atlas for a compact manifold M consisting of ℓ charts, the first k of which have disjoint images but almost cover M in the sense that, if $S = M \setminus \bigcup_{i=1}^k \psi_i(V_i)$, then $\psi_i^{-1}(S)$ has content zero for all i. Prove that for any form ω defined in a neighborhood of M,

$$\int_M \omega = \sum_{i=1}^k \int_{V_i} \psi^* \omega.$$

Hint: let χ_i equal 1 on U_i and 0 elsewhere, and consider $\sum_{i=1}^k \sum_{j=1}^\ell \int_{V_i} \psi_i^*(\chi_i\, \phi_j\, \omega)$ for a partition of unity ϕ_j.

2. Let B be the unit ball in $\mathbf{R}^n$ and let $f_\pm : B \to \mathbf{R}^{n+1}$ be given by $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \pm\sqrt{1 - x_1^2 - \cdots - x_n^2})$. Prove that

$$\int_{S^n} \omega = \int_B f_+^* \omega - \int_B f_-^* \omega.$$

3. Let $T \subset \mathbf{R}^3$ be the torus which is the image of $f : [0, 2\pi] \times [0, 2\pi] \to \mathbf{R}^3$ given by $f(\theta, \phi) = (2\cos\theta, 2\sin\theta, 0) + (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$. Prove that

$$\int_T \omega = \int_0^{2\pi}\int_0^{2\pi} f^* \omega.$$

4. Let $D \subset \mathbf{R}^2$ be the unit disk, $h : D \to \mathbf{R}^4$ be

$$h(u, v) = ((u^2 + v^2)^2 + v,\ (u^2 + v^2)^3 - u,\ (u^2 + v^2)^4 + v,\ (u^2 + v^2)^5 - u),$$

and let $\eta = z\, dw + y\, dx - x\, dy - w\, dz$ on $\mathbf{R}^4$. Compute $\int_{h(D)} d\eta$. Discuss the orientation.

5. Let R be a region (i.e. a 3-manifold with boundary) in $\mathbf{R}^3$. Show that the volume of R is

$$\frac{1}{3}\int_{\partial R} x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

and use this to compute the volume of the unit ball.

6. Let M be a compact oriented manifold without boundary of dimension k, and let ω be an exact form defined on an open set containing M. Show that $\int_M \omega = 0$.

7. Do problem **5–19** in Spivak.