Mathematics W4081y
Differentiable Manifolds

Assignment #10
Due April 21, 2014

1. If $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ are orientable manifolds, prove that $X \times Y \subset \mathbf{R}^{m+n}$ is too.

2. Prove that the cylinder $\{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 = 1, -1 < z < 1\}$ is orientable.

3. (a) If a k-dimensional manifold $M \subset \mathbf{R}^n$ is oriented, prove that there exist an open $U \supset M$ and a form $\omega \in \Omega^k(U)$ such that for every oriented chart $\psi_\alpha : V_\alpha \to U_\alpha \cap M$, we have $\psi_\alpha^* \omega = f(x)\, dx_1 \wedge \cdots \wedge dx_k$ with $f(x) > 0$ for all $x \in V_\alpha$.
Hint: show that, after shrinking U_α and V_α if necessary, each chart may be extended to a diffeomorphism $V_\alpha \times W_\alpha \to U_\alpha$; let $U = \bigcup_\alpha U_\alpha$ and add up suitable forms using a partition of unity.

(b) If a k-dimensional manifold $M \subset \mathbf{R}^n$ is oriented, prove that there exist an open $U \supset M$ and a form $\omega \in \Omega^k(U)$ such that for every chart $\psi_\alpha : V_\alpha \to U_\alpha \cap M$ whatsoever, we have $\psi_\alpha^* \omega = f(x)\, dx_1 \wedge \cdots \wedge dx_k$ with $f(x) \neq 0$ for all $x \in V_\alpha$.

4. (a) Explain why the Möbius strip M is the image of $f : \mathbf{R} \times (-1, 1) \to \mathbf{R}^3$ given by

$$f(t, u) = (2 \cos 2t, 2 \sin 2t, 0) + u(\cos 2t \cos t, \sin 2t \cos t, \sin t).$$

(b) Describe an atlas for M given by restricting f to open subsets.

5. Prove that the Möbius strip is not orientable.
Hint: pull back a suitable form by f.

6. Let $M \subset \mathbf{R}^m$ and $N \subset \mathbf{R}^n$ be compact oriented manifolds of dimensions k and ℓ, respectively, and let $\omega \in \Omega^k(\mathbf{R}^m)$ and $\eta \in \Omega^\ell(\mathbf{R}^n)$ be differential forms. If $\pi_1 : \mathbf{R}^{m+n} \to \mathbf{R}^m$ and $\pi_2 : \mathbf{R}^{m+n} \to \mathbf{R}^n$ are projection on the first and last factors, prove that

$$\int_{M \times N} \pi_1^* \omega \wedge \pi_2^* \eta = \int_M \omega \int_N \eta$$

when $M \times N$ is suitably oriented. Also show that, if ω and η are forms of any other degree, then the left-hand side vanishes.

7. If $f : M \to N$ is an oriented diffeomorphism of oriented n-dimensional manifolds, and if ω is a differential n-form defined on an open neighborhood of N, prove that

$$\int_M f^* \omega = \int_N \omega.$$