Mathematics W4081y
Differentiable Manifolds

Assignment #1

Due Monday, February 3, 2013

1. Prove that if $W \subset V$ are finite-dimensional vector spaces of the same dimension, then $W = V$.

2. (Contraction to a basis) If $v_1, \ldots, v_n$ spans V, prove that it contains a basis.

3. If V is a vector space and $v_1, \ldots, v_n \in V$, prove that any two of the following imply the third: (i) $n = \dim V$; (ii) $v_1, \ldots, v_n$ span V; (iii) $v_1, \ldots, v_n$ are linearly independent.

4. Let V be a finite-dimensional vector space with an inner product. Show that $v \mapsto \langle\ ,v\rangle$ defines an isomorphism $V \to V^*$.

5. (a) Let U be the vector space consisting of all continuous functions $[0,1] \to \mathbf{R}$. Prove that $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ defines an inner product on U.

(b) Show that $f \mapsto f(1/2)$ defines an element of the dual space U^* that is not given by $f \mapsto \langle f, g\rangle$ for any $g \in U$.

(c) Show that the linear map $G : U \to U^*$ given by $g \mapsto \langle\ , g\rangle$ is injective but not surjective.

6. Let V be a vector space with bases u_1, u_2 and v_1, v_2 related by $u_1 = av_1 + cv_2$ and $u_2 = bv_1 + dv_2$. Show that the dual bases of V^* are related by $v^1 = au^1 + bu^2$ and $v^2 = cu^1 + du^2$. Show that the isomorphisms $V \to V^*$ given by $u_i \mapsto u^i$ and $v_i \mapsto v^i$ are distinct unless $A^T A = I$ where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and I is the identity matrix. In other words, the isomorphism $V \to V^*$ depends on the choice of basis.

7. Prove that $\det A = \det A^T$ using the formula stated in class:

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

(Here $\operatorname{sgn} \sigma$, the *sign* of a permutation σ, is defined to be 1 or -1 depending whether σ is a composition of an even or an odd number of transpositions. It is well-defined.)

8. (a) If $S, T : \mathbf{R}^n \to \mathbf{R}^n$ are linear maps with $S \circ T = \mathrm{id}$, prove that $T \circ S = \mathrm{id}$ too. Hint: use rank-nullity.

(b) If $A, B \in M_{n\times n}(\mathbf{R})$ are matrices with $AB = I$, then $BA = I$ too. Hint: composition of linear maps corresponds to matrix multiplication (which you may assume).

9. A sequence $v_1, \ldots, v_m \in \mathbf{R}^n$ is said to be *orthonormal* if $v_i \cdot v_j = \delta_{ij}$, the Kronecker delta. Prove that an orthonormal sequence of n vectors in $\mathbf{R}^n$ is a basis.

10. (a) If the rows of a matrix $A \in M_{n\times n}(\mathbf{R})$ form an orthonormal sequence, prove that the columns do too.

(b) Prove that such a matrix has determinant ± 1.